Notes

Viscosity

- In reality, nearby molecules travelling at different velocities occasionally bump into each other, transferring energy
  - Differences in velocity reduced (damping)
  - Measure this by strain rate (time derivative of strain, or how far velocity field is from rigid motion)
  - Add terms to our constitutive law

Strain rate

- At any instant in time, measure how fast chunk of material is deforming from its current state
  - **Not** from its original state
  - So we're looking at infinitesimal, incremental strain updates
  - Can use linear Cauchy strain!
- So the strain rate tensor is

$$\dot{\varepsilon}_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Viscous stress

- As with linear elasticity, end up with two parameters if we want isotropy:

$$\sigma_{ij}^{viscous} = 2\mu\dot{\varepsilon}_{ij} + \lambda\dot{\varepsilon}_{kk}\delta_{ij}$$

  - μ and λ are coefficients of viscosity (first and second)
  - These are not the Lame coefficients! Just use the same symbols
  - λ damps only compression/expansion
- Usually $\lambda \approx -2/3\mu$ (exact for monatomic gases)
- So end up with $\sigma_{ij}^{viscous} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)$

Navier-Stokes

- Navier-Stokes equations include the viscous stress
- Incompressible version:

$$u_t + u\cdot\nabla u + \tfrac{1}{\rho}\nabla p = g + \tfrac{1}{\rho}\nabla\cdot\mu\left(\nabla u + \nabla u^T\right)$$

$$\nabla\cdot u = 0$$

- Often (but **not** always) viscosity μ is constant, and this reduces to

$$u_t + u\cdot\nabla u + \tfrac{1}{\rho}\nabla p = g + \tfrac{\mu}{\rho}\nabla^2 u$$

  - Call $\nu=\mu/\rho$ the "kinematic viscosity"

Nondimensionalization

- Actually go even further
- Select a characteristic length L
  - e.g. the width of the domain,
- And a typical velocity U
  - e.g. the speed of the incoming flow
- Rescale terms
  - $x'=x/L$, $u'=u/U$, $t'=tU/L$, $p'=p/\rho U^2$ so they all are dimensionless

$$u'_t + u'\cdot\nabla u' + \nabla p' = \frac{Lg}{U^2} + \frac{\nu}{UL}\nabla^2 u'$$

Nondimensional parameters

- Re=UL/ν is the Reynold's number
  - The smaller it is, the more viscosity plays a role in the flow
  - High Reynold's numbers are hard to simulate
- Fr= $U/\sqrt{|g|L}$ is the Froude number
  - The smaller it is, the more gravity plays a role in the flow
  - Note: often can ignore gravity (pressure gradient cancels it out)

$$u_t + u \cdot \nabla u + \nabla p = \frac{(0,-1,0)}{\mathrm{Fr}^2} + \frac{1}{\mathrm{Re}} \nabla^2 u$$

Why nondimensionalize?

- Think of it as a user-interface issue
- It lets you focus on what parameters matter
  - If you scale your problem so nondimensional parameters stay constant, solution scales
- Code rot --- you may start off with code which has true dimensions, but as you hack around they lose meaning
  - If you're nondimensionalized, there should be only one or two parameters to play with
- Not always the way to go --- you can look up material constants, but not e.g. Reynolds numbers

Other quantities

- We may want to carry around auxiliary quantities
  - E.g. temperature, the type of fluid (if we have a mix), concentration of smoke, etc.
- Use material derivative as before
- E.g. if quantity doesn't change, just is transported ("advected") around:

$$\frac{Dq}{Dt} = q_t + \underbrace{u \cdot \nabla q}_{advection} = 0$$

Boundary conditions

- Inviscid flow:
  - Solid wall: u•n=0
  - Free surface: p=0 (or atmospheric pressure)
  - Moving solid wall: u•n=u_{wall}•n
    - Also enforced in-flow/out-flow
  - Between two fluids: u_1•n=u_2•n and $p_1=p_2+\sigma\kappa$
- Viscous flow:
  - No-slip wall: u=0
  - Other boundaries can involve coupling tangential components of stress tensor…
- Pressure/velocity coupling at boundary:
  - u•n modified by ∂p/∂n

What now?

- Can solve numerically the full equations
  - Will do this later
  - Not so simple, could be expensive (3D)
- Or make assumptions and simplify them, then solve numerically
  - Simplify flow (e.g. irrotational)
  - Simplify dimensionality (e.g. go to 2D)

Vorticity

- How do we measure rotation?
  - Vorticity of a vector field (velocity) is: $\omega = \nabla \times u$
  - Proportional (but not equal) to angular velocity of a rigid body - off by a factor of 2
- Vorticity is what makes smoke look interesting
  - Turbulence

Vorticity equation

- Start with N-S, constant viscosity and density

$$u_t + u\cdot\nabla u + \tfrac{1}{\rho}\nabla p = g + \nu\nabla^2 u$$

- Take curl of whole equation

$$\nabla\times u_t + \nabla\times(u\cdot\nabla u) + \nabla\times\tfrac{1}{\rho}\nabla p = \nabla\times g + \nabla\times\left(\nu\nabla^2 u\right)$$

- Lots of terms are zero:
  - g is constant (or the potential of some field)
  - With constant density, pressure term too

$$\nabla\times u_t + \nabla\times(u\cdot\nabla u) = \nu\nabla\times\nabla^2 u$$

- Then use vector identities to simplify…

$$\nabla\times u_t + \nabla\times\left((\nabla\times u)\times u + \tfrac{1}{2}\nabla u^2\right) = \nu\nabla^2(\nabla\times u)$$

$$\omega_t + \nabla\times(\omega\times u) = \nu\nabla^2\omega$$

Vorticity equation continued

- Simplify with more vector identities, and assume incompressible to get:

$$\frac{D\omega}{Dt} = \omega\cdot\nabla u + \nu\nabla^2\omega$$

- Important result: Kelvin Circulation Theorem
  - Roughly speaking: if $\omega=0$ initially, and there's no viscosisty, $\omega=0$ forever after (following a chunk of fluid)
- If fluid starts off irrotational, it will stay that way (in many circumstances)

Potential flow

- If velocity is irrotational:

$$\nabla\times u = 0$$

  - Which it often is in simple laminar flow
- Then there must be a stream function (potential) such that: $u = \nabla\phi$
- Solve for incompressibility:

$$\nabla\cdot\nabla\phi = 0$$

- If u•n is known at boundary, we can solve it

Potential in time

- What if we have a free surface?
- Use vector identity $u\cdot\nabla u=(\nabla\times u)\times u+\nabla|u|^2/2$
- Assume
  - incompressible ($\nabla\cdot u=0$), inviscid, irrotational ($\nabla\times u=0$)
  - constant density
  - thus potential flow ($u=\nabla\phi$, $\nabla^2\phi=0$)
- Then momentum equation simplifies (using G=-gy for gravitational potential)

$$u_t + (\nabla\times u)\times u + \tfrac{1}{2}\nabla|u|^2 + \tfrac{1}{\rho}\nabla p = g$$

$$\nabla\phi_t + \tfrac{1}{2}\nabla|u|^2 + \tfrac{1}{\rho}\nabla p = -\nabla G$$

Bernoulli's equation

- Every term in the simplified momentum equation is a gradient: integrate to get

$$\phi_t + \tfrac{1}{2}u^2 + \frac{p}{\rho} = -G$$

  - (Remember Bernoulli's law for pressure?)
- This tells us how the potential should evolve in time

Water waves

- For small waves (no breaking), can reduce geometry of water to 2D heightfield
- Can reduce the physics to 2D also
  - How do surface waves propagate?
- Plan of attack
  - Start with potential flow, Bernoulli's equation
  - Write down boundary conditions at water surface
  - Simplify 3D structure to 2D

Set up

- We'll take y=0 as the height of the water at rest
- H is the depth (y=-H is the sea bottom)
- h is the current height of the water at (x,z)
- Simplification: velocities very small (small amplitude waves)

Boundaries

- At sea floor (y=-H), v=0 $\phi_y = 0$
- At sea surface (y=h≈0), v=h_t
  - Note again - assuming very small horizontal motion

$$\phi_y = h_t$$

- At sea surface (y=h≈0), p=0
  - Or atmospheric pressure, but we only care about pressure differences
  - Use Bernoulli's equation, throw out small velocity squared term, use p=0,

$$\phi_t = -gh$$

Finding a wave solution

- Plug in ϕ=f(y)sin(K•(x,z)-ωt)
  - In other words, do a Fourier analysis in horizontal component, assume nothing much happens in vertical
  - Solving $\nabla^2\phi$=0 with boundary conditions on ϕ_y gives

$$\phi = A\frac{\omega}{|K|}\frac{\cosh(|K|(y+H))}{\sinh(|K|H)}\sin(K\cdot(x,z)-\omega t)$$

  - Pressure boundary condition then gives (with k=|K|, the magnitude of K)

$$\omega = \sqrt{gk\tanh kH}$$

Dispersion relation

- So the wave speed c is

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh kH}$$

- Notice that waves of different wave-numbers k have different speeds
  - Separate or disperse in time
- For deep water (H big, k reasonable -- not tsunamis) tanh(kH)≈1

$$c = \sqrt{\frac{g}{k}}$$

Simulating the ocean

- Far from land, a reasonable thing to do is
  - Do Fourier decomposition of initial surface height
  - Evolve each wave according to given wave speed (dispersion relation)
    - Update phase, use FFT to evaluate
- How do we get the initial spectrum?
  - Measure it! (oceanography)

Energy spectrum

- Fourier decomposition of height field:

$$h(x,z,t) = \sum_{i,j}\hat{h}(i,j,t)e^{\sqrt{-1}(i,j)\cdot(x,z)}$$

- "Energy" in K=(i,j) is $S(K) = \left|\hat{h}(K)\right|^2$
- Oceanographic measurements have found models for expected value of S(K) (statistical description)

Phillips Spectrum

- For a "fully developed" sea
  - wind has been blowing a long time over a large area, statistical distribution of spectrum has stabilized
- The Phillips spectrum is: [Tessendorf…]

$$S(K) = A\frac{1}{k^4}\exp\left(\frac{-1}{(kL)^2} - (kl)^2\right)\left(\frac{|K \cdot W|}{|K\|W|}\right)^2$$

  - A is an arbitrary amplitude
  - $L=|W|^2/g$ is largest size of waves due to wind velocity W and gravity g
  - Little l is the smallest length scale you want to model

Fourier synthesis

- From the prescribed S(K), generate actual Fourier coefficients

$$\hat{h}(K,0) = \tfrac{1}{\sqrt{2}}\left(X_1 + X_2\sqrt{-1}\right)\sqrt{S(K)}$$

  - X_i is a random number with mean 0, standard deviation 1 (Gaussian)
  - Uniform numbers from unit circles aren't terrible either
- Want real-valued h, so must have

$$\hat{h}(K) = \hat{h}(-K)^*$$

  - Or give only half the coefficients to FFT routine and specify you want real output

Time evolution

- Dispersion relation gives us ω(K)
- At time t, want $h(x,t) = \sum_{K=(i,j)} \hat{h}(K,0)e^{\sqrt{-1}(K\cdot x - \omega t)}$

$$= \sum_{K=(i,j)} \hat{h}(K,0)e^{-\sqrt{-1}\omega t}e^{\sqrt{-1}K\cdot x}$$

- So then coefficients at time t are
  - For j≥0:
  - Others: figure out from conjugacy condition (or leave it up to real-valued FFT to fill them in)

$$\hat{h}(i,j,t) = \hat{h}(i,j,0)e^{-\sqrt{-1}\omega t}$$

Picking parameters

- Need to fix grid for Fourier synthesis (e.g. 1024x1024 height field grid)
- Grid spacing shouldn't be less than e.g. 2cm (smaller than that - surface tension, nonlinear wave terms, etc. take over)
  - Take little l (cut-off) a few times larger
- Total grid size should be greater than but still comparable to L in Phillips spectrum (depends on wind speed and gravity)
- Amplitude A shouldn't be too large
  - Assumed waves weren't very steep

Note on FFT output

- FFT takes grid of coefficients, outputs grid of heights
- It's up to you to map that grid (0…n-1, 0…n-1) to world-space coordinates
- In practice: scale by something like L/n
  - Adjust scale factor, amplitude, etc. until it looks nice
- Alternatively: look up exactly what your FFT routines computes, figure out the "true" scale factor to get world-space coordinates

Tiling issues

- Resulting grid of waves can be tiled in x and z
- Handy, except people will notice if they can see more than a couple of tiles
- Simple trick: add a second grid with a non-rational multiple of the size
  - Golden mean (1+sqrt(5))/2=1.61803… works well
  - The sum is no longer periodic, but still can be evaluated anywhere in space and time easily enough

Choppy waves

- See Tessendorf for more explanation
- Nonlinearities cause real waves to have sharper peaks and flatter troughs than linear Fourier synthesis gives
- Can manipulate height field to give this effect
  - Distort grid with (x,z) -> (x,z)+λD(x,z,t)

$$D(x,t) = \sum_{K} -\sqrt{-1}\frac{K}{|K|}\hat{h}(K,t)e^{\sqrt{-1}K\cdot x}$$

Choppiness problems

- The distorted grid can actually tangle up (Jacobian has negative determinant - not 1-1 anymore)
  - Can detect this, do stuff (add particles for foam, spray?)
- Can't easily use superposition of two grids to defeat periodicity… (but with a big enough grid and camera position chosen well, not an issue)