1D linear elasticity

- Taking the limit as the number of springs and masses goes to infinity (and the forces and masses go to zero):

$$\ddot{x}(p) = \frac{1}{\rho}\frac{\partial}{\partial p}\left(E(p)\left(\frac{\partial}{\partial p}x(p) - 1\right)\right)$$

  - If density and Young's modulus constant,

$$\frac{\partial^2 x}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 x}{\partial p^2}$$

Sound waves

- Try solution $x(p,t)=x_0(p-ct)$
- And $x(p,t)=x_0(p+ct)$
- So speed of "sound" in rod is $\sqrt{\frac{E}{\rho}}$
- Courant-Friedrichs-Levy (CFL) condition:
  - Numerical methods only will work if information transmitted numerically at least as fast as in reality (here: the speed of sound)
  - Usually the same as stability limit for good explicit methods [what are the eigenvalues here]
  - Implicit methods transmit information infinitely fast

Why?

- Are sound waves important?
  - Visually? Usually not
- However, since speed of sound is a material property, it can help us get to higher dimensions
- Speed of sound in terms of one spring is

$$c = \sqrt{\frac{kL}{m}}$$

- So in higher dimensions, just pick k so that c is constant
  - m is mass around spring [triangles, tets]
  - Optional reading: van Gelder

Damping

- Figuring out how to scale damping is more tricky
- Go to differential equation (no mesh)

$$\frac{\partial^2 x}{\partial t^2} = \frac{1}{\rho}\frac{\partial}{\partial p}\left(E\left(\frac{\partial x}{\partial p} - 1\right) + D\frac{\partial v}{\partial p}\right)$$

- So spring damping should be

$$f_{i+\frac{1}{2}} = k_{i+\frac{1}{2}}\frac{x_{i+1} - x_i - L_{i+\frac{1}{2}}}{L_{i+\frac{1}{2}}} + d_{i+\frac{1}{2}}\frac{v_{i+1} - v_i}{L_{i+\frac{1}{2}}}$$

Extra effects with springs

- (Brittle) fracture
  - Whenever a spring is stretched too far, break it
  - Issue with loose ends…
- Plasticity
  - Whenever a spring is stretched too far, change the rest length part of the way
- More on this later

Mass-spring problems

- [anisotropy]
- [stretching, Poisson's ratio]
- So we will instead look for a generalization of "percent deformation" to multiple dimensions: elasticity theory

Studying Deformation

- Let's look at a deformable object
  - World space: points x in the object as we see it
  - Object space (or rest pose): points p in some reference configuration of the object
  - (Technically we might not have a rest pose, but usually we do, and it is the simplest parameterization)
- So we identify each point x of the continuum with the label p, where x=X(p)
- The function X(p) encodes the deformation

Going back to 1D

- Worked out that dX/dp-1 was the key quantity for measuring stretching and compression
- Nice thing about differentiating: constants (translating whole object) don't matter
- Call A= $\partial X/\partial p$ the deformation gradient

Strain

- A isn't so handy, though it somehow encodes exactly how stretched/compressed we are
  - Also encodes how rotated we are: who cares?
- We want to process A somehow to remove the rotation part
- [difference in lengths]
- A^TA-I is exactly zero when A is a rigid body rotation
- Define Green strain

$$G = \tfrac{1}{2}\left(A^T A - I\right)$$

Why the half??

- [Look at 1D, small deformation]
- $A=1+\varepsilon$
- $A^TA-I = A^2-1 = 2\varepsilon+\varepsilon^2 \approx 2\varepsilon$
- Therefore $G \approx \varepsilon$, which is what we expect
- Note that for large deformation, Green strain grows quadratically
  - maybe not what you expect!
- Whole cottage industry: defining strain differently

Cauchy strain tensor

- Get back to linear, not quadratic
- Look at "small displacement"
  - Not only is the shape only slightly deformed, but it only slightly rotates (e.g. if one end is fixed in place)
- Then displacement x-p has gradient D=A-I
- Then $G = \tfrac{1}{2}\left(D^T D + D + D^T\right)$
- And for small displacement, first term negligible
- Cauchy strain $\varepsilon = \tfrac{1}{2}\left(D + D^T\right)$
- Symmetric part of deformation gradient
  - Rotation is skew-symmetric part

Analyzing Strain

- Strain is a 3x3 "tensor" (fancy name for a matrix)
- Always symmetric
- What does it mean?
- Diagonalize: rotate into a basis of eigenvectors
  - Entries (eigenvalues) tells us the scaling on the different axes
  - Sum of eigenvalues (always equal to the trace=sum of diagonal, even if not diagonal): approximate volume change
- Or directly analyze: off-diagonals show skew (also known as shear)

Force

- In 1D, we got the force of a spring by simply multiplying the strain by some material constant (Young's modulus)
- In multiple dimensions, strain is a tensor, but force is a vector…
- And in the continuum limit, force goes to zero anyhow---so we have to be a little more careful

Conservation of Momentum

- In other words F=ma
- Decompose body into "control volumes"
- Split F into
  - f_{body} (e.g. gravity, magnetic forces, …) force per unit volume
  - and traction t (on boundary between two chunks of continuum: contact force) dimensions are force per unit area (like pressure)

$$\int_{\Omega_W} f_{body}\, dx + \int_{\partial\Omega_W} t\, ds = \int_{\Omega_W} \rho \ddot{X}\, dx$$

Cauchy's Fundamental Postulate

- Traction t is a function of position x and normal n
  - Ignores rest of boundary (e.g. information like curvature, etc.)
- **Theorem**
  - If t is smooth (be careful at boundaries of object, e.g. cracks) then t is linear in n: $t=\sigma(x)n$
- σ is the Cauchy stress tensor (a matrix)
- It also is force per unit area
- Diagonal: normal stress components
- Off-diagonal: shear stress components

Cauchy Stress

- From conservation of angular momentum can derive that Cauchy stress tensor σ is symmetric: $\sigma = \sigma^T$
- Thus there are only 6 degrees of freedom (in 3D)
  - In 2D, only 3 degrees of freedom
- What is σ?
  - That's the job of **constitutive modeling**
  - Depends on the material (e.g. water vs. steel vs. silly putty)

Divergence Theorem

- Try to get rid of integrals
- First make them all volume integrals with divergence theorem:

$$\int_{\partial\Omega_W} \sigma n\, ds = \int_{\Omega_W} \nabla \cdot \sigma\, dx$$

- Next let control volume shrink to zero:

$$f_{body} + \nabla \cdot \sigma = \rho \ddot{X}$$

  - Note that integrals and normals were in world space, so is the divergence (it's w.r.t. x not p)

Constitutive Modeling

- This can get very complicated for complicated materials
- Let's start with simple elastic materials
- We'll even leave damping out
- Then stress σ only depends on strain, however we measure it (say G or ε)

Linear elasticity

- Very nice thing about Cauchy strain: it's linear in deformation
  - No quadratic dependence
  - Easy and fast to deal with
- Natural thing is to make a linear relationship with Cauchy stress σ
- Then the full equation is linear!

Young's modulus

- Obvious first thing to do: if you pull on material, resists like a spring:
  σ=Eε
- E is the Young's modulus
- Let's check that in 1D (where we know what should happen with springs)

$$\nabla \cdot \sigma = \rho \ddot{x}$$

$$\frac{\partial}{\partial x}\left(E\left(\frac{\partial X}{\partial p} - 1\right)\right) = \rho \ddot{x}$$

Poisson Ratio

- Real materials are essentially incompressible (for large deformation - neglecting foams and other weird composites…)
- For small deformation, materials are usually somewhat incompressible
- Imagine stretching block in one direction
  - Measure the contraction in the perpendicular directions
  - Ratio is ν, Poisson's ratio
- [draw experiment; $\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}}$]

What is Poisson's ratio?

- Has to be between -1 and 0.5
- 0.5 is exactly incompressible
  - [derive]
- Negative is weird, but possible [origami]
- Rubber: close to 0.5
- Steel: more like 0.33
- Metals: usually 0.25-0.35
- [cute: cork is almost 0]