Notes

- I am back, but still catching up
- Assignment 2 is due today (or next time I'm in the dept following today)
- Final project proposals:
  - I haven't sorted through my email, but make sure you send me something now (even quite vague)
  - Let's make sure everyone has their project started this weekend or early next week

Multi-Dimensional Plasticity

- Simplest model: total strain is sum of elastic and plastic parts: $\varepsilon=\varepsilon_e+\varepsilon_p$
- Stress only depends on elastic part (so rest state includes plastic strain): $\sigma=\sigma(\varepsilon_e)$
- If σ is too big, we yield, and transfer some of ε_e into ε_p so that σ is acceptably small

Multi-Dimensional Yield criteria

- Lots of complicated stuff happens when materials yield
  - Metals: dislocations moving around
  - Polymers: molecules sliding against each other
  - Etc.
- Difficult to characterize exactly when plasticity (yielding) starts
  - Work hardening etc. mean it changes all the time too
- Approximations needed
  - Big two: Tresca and Von Mises

Yielding

- First note that shear stress is the important quantity
  - Materials (almost) never can permanently change their volume
  - Plasticity should ignore volume-changing stress
- So make sure that if we add kI to σ it doesn't change yield condition

Tresca yield criterion

- This is the simplest description:
  - Change basis to diagonalize σ
  - Look at normal stresses (i.e. the eigenvalues of σ)
  - No yield if $\sigma_{max}-\sigma_{min} \leq \sigma_Y$
- Tends to be conservative (rarely predicts yielding when it shouldn't happen)
- But, not so accurate for some stress states
  - Doesn't depend on middle normal stress at all
- Big problem (mathematically): not smooth

Von Mises yield criterion

- If the stress has been diagonalized:

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2} \leq \sigma_Y$$

- More generally: $\sqrt{\tfrac{3}{2}}\sqrt{\|\sigma\|_F^2 - \tfrac{1}{3}Tr(\sigma)^2} \leq \sigma_Y$
- This is the same thing as the Frobenius norm of the deviatoric part of stress
  - i.e. after subtracting off volume-changing part:

$$\sqrt{\tfrac{3}{2}}\left\|\sigma - \tfrac{1}{3}Tr(\sigma)I\right\|_F \leq \sigma_Y$$

Linear elasticity shortcut

- For linear (and isotropic) elasticity, apart from the volume-changing part which we cancel off, stress is just a scalar multiple of strain
  - (ignoring damping)
- So can evaluate von Mises with elastic strain tensor too (and an appropriately scaled yield strain)

Perfect plastic flow

- Once yield condition says so, need to start changing plastic strain
- The magnitude of the change of plastic strain should be such that we stay on the yield surface
  - I.e. maintain f(σ)=0
    (where f(σ)≤0 is, say, the von Mises condition)
- The direction that plastic strain changes isn't as straightforward
- "Associative" plasticity: $\dot{\varepsilon}_p = \gamma \frac{\partial f}{\partial \sigma}$

Algorithm

- After a time step, check von Mises criterion: is $f(\sigma) = \sqrt{\tfrac{3}{2}}\|dev(\sigma)\|_F - \sigma_Y > 0$?
- If so, need to update plastic strain:

$$\begin{aligned}\varepsilon_p^{new} &= \varepsilon_p + \gamma \frac{\partial f}{\partial \sigma} \\ &= \varepsilon_p + \gamma \sqrt{\tfrac{3}{2}} \frac{dev(\sigma)}{\|dev(\sigma)\|_F}\end{aligned}$$

  - with γ chosen so that $f(\sigma^{new})=0$ (easy for linear elasticity)

Sand (Granular Materials)

- Things get a little more complicated for sand, soil, powders, etc.
- Yielding actually involves friction, and thus is pressure (the trace of stress) dependent
- Flow rule can’t be associated
- See Zhu and Bridson, SIGGRAPH’05 for quick-and-dirty hacks… :-)

Multi-Dimensional Fracture

- Smooth stress to avoid artifacts (average with neighbouring elements)
- Look at largest eigenvalue of stress in each element
- If larger than threshhold, introduce crack perpendicular to eigenvector
- Big question: what to do with the mesh?
  - Simplest: just separate along closest mesh face
  - Or split elements up: O’Brien and Hodgins SIGGRAPH’99
  - Or model crack path with embedded geometry: Molino et al. SIGGRAPH’04

Fluids

Fluid mechanics

- We already figured out the equations of motion for continuum mechanics $\rho\ddot{x} = \nabla \cdot \sigma + \rho g$
- Just need a constitutive model

$$\sigma = \sigma(x, t, \varepsilon, \dot{\varepsilon})$$

- We'll look at the constitutive model for "Newtonian" fluids next
  - Remarkably good model for water, air, and many other simple fluids
  - Only starts to break down in extreme situations, or more complex fluids (e.g. viscoelastic substances)

Inviscid Euler model

- Inviscid=no viscosity
- Great model for most situations
  - Numerical methods usually end up with viscosity-like error terms anyways…
- Constitutive law is very simple: $\sigma_{ij} = -p\delta_{ij}$
  - New scalar unknown: pressure p
  - Barotropic flows: p is just a function of density (e.g. perfect gas law $p=k(\rho-\rho_0)+p_0$ perhaps)
  - For more complex flows need heavy-duty thermodynamics: an equation of state for pressure, equation for evolution of internal energy (heat), …

Lagrangian viewpoint

- We've been working with Lagrangian methods so far
  - Identify chunks of material, track their motion in time, differentiate world-space position or velocity w.r.t. material coordinates to get forces
  - In particular, use a mesh connecting particles to approximate derivatives (with FVM or FEM)
- Bad idea for most fluids
  - [vortices, turbulence]
  - At least with a fixed mesh…

Eulerian viewpoint

- Take a fixed grid in world space, track how velocity changes at a point
- Even for the craziest of flows, our grid is always nice
- (Usually) forget about object space and where a chunk of material originally came from
  - Irrelevant for extreme inelasticity
  - Just keep track of velocity, density, and whatever else is needed

Conservation laws

- Identify any fixed volume of space
- Integrate some conserved quantity in it (e.g. mass, momentum, energy, …)
- Integral changes in time only according to how fast it is being transferred from/to surrounding space
  - Called the flux
  - [divergence form]

$$\frac{\partial}{\partial t}\int_{\Omega} q = -\int_{\partial\Omega} f(q)\cdot n$$

$$q_t + \nabla \cdot f = 0$$

Conservation of Mass

- Also called the continuity equation (makes sure matter is continuous)
- Let's look at the total mass of a volume (integral of density)
- Mass can only be transferred by moving it: flux must be ρu

$$\frac{\partial}{\partial t}\int_{\Omega} \rho = -\int_{\partial\Omega} \rho u\cdot n$$

$$\rho_t + \nabla \cdot (\rho u) = 0$$

Material derivative

- A lot of physics just naturally happens in the Lagrangian viewpoint
  - E.g. the acceleration of a material point results from the sum of forces on it
  - How do we relate that to rate of change of velocity measured at a fixed point in space?
  - Can't directly: need to get at Lagrangian stuff somehow
- The material derivative of a property q of the material (i.e. a quantity that gets carried along with the fluid) is $\dfrac{Dq}{Dt}$

Finding the material derivative

- Using object-space coordinates p and map x=X(p) to world-space, then material derivative is just

$$\begin{aligned}\frac{D}{Dt}q(t,x) &= \frac{d}{dt}q\big(t,X(t,p)\big)\\ &= \frac{\partial q}{\partial t} + \nabla q\cdot\frac{\partial x}{\partial t}\\ &= q_t + u\cdot\nabla q\end{aligned}$$

- Notation: u is velocity (in fluids, usually use u but occasionally v or V, and components of the velocity vector are sometimes u,v,w)

Compressible Flow

- In general, density changes as fluid compresses or expands
- When is this important?
  - Sound waves (and/or high speed flow where motion is getting close to speed of sound - Mach numbers above 0.3?)
  - Shock waves
- Often not important scientifically, almost never visually significant
  - Though the **effect** of e.g. a blast wave is visible! But the shock dynamics usually can be hugely simplified for graphics

Incompressible flow

- So we'll just look at incompressible flow, where density of a chunk of fluid never changes
  - Note: fluid density may not be constant throughout space - different fluids mixed together…
- That is, $D\rho/Dt=0$

Simplifying

- Incompressibility: $$\frac{D\rho}{Dt} = \rho_t + u \cdot \nabla \rho = 0$$
- Conservation of mass: $$\rho_t + \nabla \cdot (\rho u) = 0$$ $$\rho_t + \nabla \rho \cdot u + \rho \nabla \cdot u = 0$$
- Subtract the two equations, divide by ρ:

$$\nabla \cdot u = 0$$

- Incompressible == divergence-free velocity
  - Even if density isn't uniform!

Conservation of momentum

- Short cut: in $\rho \ddot{x} = \nabla \cdot \sigma + \rho g$

  use material derivative:

$$\rho \frac{Du}{Dt} = \nabla \cdot \sigma + \rho g$$
$$\rho(u_t + u \cdot \nabla u) = \nabla \cdot \sigma + \rho g$$

- Or go by conservation law, with the flux due to transport of momentum and due to stress:
  - Equivalent, using conservation of mass

$$(\rho u)_t + \nabla \cdot (u \rho u - \sigma) = \rho g$$

Inviscid momentum equation

- Plug in simplest consitutive law (σ=-pδ) from before to get

$$\rho(u_t + u \cdot \nabla u) = -\nabla p + \rho g$$

$$u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla p = g$$

  - Together with conservation of mass: the Euler equations

Incompressible inviscid flow

- So the equations are:
$$u_t + u \cdot \nabla u + \tfrac{1}{\rho} \nabla p = g$$
$$\nabla \cdot u = 0$$
- 4 equations, 4 unknowns (u, p)
- Pressure p is just whatever it takes to make velocity divergence-free
  - Actually a "Lagrange multiplier" for enforcing the incompressibility constraint

Pressure solve

- To see what pressure is, take divergence of momentum equation

$$\nabla \cdot \left(u_t + u \cdot \nabla u + \tfrac{1}{\rho} \nabla p - g\right) = 0$$
$$\nabla \cdot \left(\tfrac{1}{\rho} \nabla p\right) = -\nabla \cdot \left(u_t + u \cdot \nabla u - g\right)$$

- For constant density, just get Laplacian (and this is Poisson's equation)
- Important numerical methods use this approach to find pressure

Projection

- Note that $\nabla \cdot u_t = 0$ so in fact
$$\nabla \cdot \tfrac{1}{\rho} \nabla p = -\nabla \cdot \left(u \cdot \nabla u - g\right)$$
- After we add ∇p/ρ to u•∇u, divergence must be zero
- So if we tried to solve for additional pressure, we get zero
- Pressure solve is linear too
- Thus what we're really doing is a **projection** of u•∇u-g onto the subspace of divergence-free functions: $u_t + P(u \cdot \nabla u - g) = 0$