Notes

• Assignment 4 due “today” (when I check email tomorrow morning)
• Don’t be afraid to make assumptions, approximate quantities, …
  • In particular, method for computing time step bound (look at max eigenvalue of Jacobian) already made lots of assumptions about linearity etc. so it won’t hurt to make a few more!

Fluid mechanics

• We already figured out the equations of motion for continuum mechanics
  \[ \rho \ddot{x} = \nabla \cdot \sigma + \rho g \]
• Just need a constitutive model
  \[ \sigma = \sigma(x,t,\varepsilon,\dot{\varepsilon}) \]
• We’ll look at the constitutive model for “Newtonian” fluids today
  • Remarkably good model for water, air, and many other simple fluids
  • Only starts to break down in extreme situations, or more complex fluids (e.g. viscoelastic substances)

Inviscid Euler model

• Inviscid=no viscosity
• Great model for most situations
  • Numerical methods end up with viscosity-like error terms anyways…
• Constitutive law is very simple:
  \[ \sigma_{ij} = -p \delta_{ij} \]
  • New scalar unknown: pressure p
• Barotropic flows: p is just a function of density (e.g. perfect gas law \( p = k(p - p_0) + p_0 \) perhaps)
• For more complex flows need heavy-duty thermodynamics: an equation of state for pressure, equation for evolution of internal energy (heat), …

Lagrangian viewpoint

• We’ve been working with Lagrangian methods so far
  • Identify chunks of material, track their motion in time, differentiate world-space position or velocity w.r.t. material coordinates to get forces
  • In particular, use a mesh connecting particles to approximate derivatives (with FVM or FEM)
• Bad idea for most fluids
  • [vortices, turbulence]
  • At least with a fixed mesh…
**Eulerian viewpoint**

- Take a fixed grid in world space, track how velocity changes at a point
- Even for the craziest of flows, our grid is always nice
- (Usually) forget about object space and where a chunk of material originally came from
  - Irrelevant for extreme inelasticity
  - Just keep track of velocity, density, and whatever else is needed

**Conservation laws**

- Identify any fixed volume of space
- Integrate some conserved quantity in it (e.g. mass, momentum, energy, ...)
- Integral changes in time only according to how fast it is being transferred from/to surrounding space
  - Called the flux
  - [divergence form] \[ \frac{\partial}{\partial t} \int_{\Omega} q = -\int_{\partial\Omega} f(q) \cdot n \]
  \[ q_t + \nabla \cdot f = 0 \]

**Conservation of Mass**

- Also called the continuity equation (makes sure matter is continuous)
- Let’s look at the total mass of a volume (integral of density)
- Mass can only be transferred by moving it: flux must be \( \rho u \)
  \[ \frac{\partial}{\partial t} \int_{\Omega} \rho = -\int_{\partial\Omega} \rho u \cdot n \]
  \[ \rho_t + \nabla \cdot (\rho u) = 0 \]

**Material derivative**

- A lot of physics just naturally happens in the Lagrangian viewpoint
  - E.g. the acceleration of a material point results from the sum of forces on it
  - How do we relate that to rate of change of velocity measured at a fixed point in space?
    - Can’t directly: need to get at Lagrangian stuff somehow
- The material derivative of a property \( q \) of the material (i.e. a quantity that gets carried along with the fluid) is
  \[ \frac{Dq}{Dt} \]
Finding the material derivative

- Using object-space coordinates \( p \) and map \( x = X(p) \) to world-space, then material derivative is just

\[
\frac{D}{Dt} q(t,x) = \frac{d}{dt} q(t,X(t,p))
\]

\[
= \frac{\partial q}{\partial t} + \nabla q \cdot \frac{\partial x}{\partial t}
\]

\[
= q_t + u \cdot \nabla q
\]

- Notation: \( u \) is velocity (in fluids, usually use \( u \) but occasionally \( v \) or \( V \), and components of the velocity vector are sometimes \( u,v,w \))

Compressible Flow

- In general, density changes as fluid compresses or expands
- When is this important?
  - Sound waves (and/or high speed flow where motion is getting close to speed of sound - Mach numbers above 0.3?)
  - Shock waves
- Often not important scientifically, almost never visually significant
  - Though the effect of e.g. a blast wave is visible!
  - But the shock dynamics usually can be hugely simplified for graphics

Incompressible flow

- So we'll just look at incompressible flow, where density of a chunk of fluid never changes
  - Note: fluid density may not be constant throughout space - different fluids mixed together...
- That is, \( \frac{D \rho}{Dt} = 0 \)

Simplifying

- Incompressibility:

\[
\frac{D \rho}{Dt} = \rho_t + u \cdot \nabla \rho = 0
\]

- Conservation of mass:

\[
\rho_t + \nabla \cdot (\rho u) = 0
\]

\[
\rho_t + \nabla \rho \cdot u + \rho \nabla \cdot u = 0
\]

- Subtract the two equations, divide by \( \rho \):

\[
\nabla \cdot u = 0
\]

- Incompressible == divergence-free velocity
  - Even if density isn't uniform!
Conservation of momentum

- Short cut - in use material derivative:
  \[ \rho \frac{D}{Dt} \mathbf{\dot{x}} = \nabla \cdot \sigma + \rho g \]

- Or go by conservation law, with the flux due to transport of momentum and due to stress:
  \[ \rho (u_t + u \cdot \nabla u) = \nabla \cdot \sigma + \rho g \]

- Then simplify a bit using conservation of mass
  \[ (\rho u)_t + \nabla \cdot (\rho u u - \sigma) = \rho g \]

Inviscid momentum equation

- Plug in simplest constitutive law \( (\sigma = -p \delta) \) from before to get
  \[ \rho (u_t + u \cdot \nabla u) = -\nabla p + \rho g \]

- Together with conservation of mass: the Euler equations

\[ u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla p = g \]

Incompressible inviscid flow

- So the equations are:
  \[ u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla p = g \]

- 4 equations, 4 unknowns \( (u, p) \)
- Pressure \( p \) is just whatever it takes to make velocity divergence-free
- In fact, incompressibility is a hard constraint; div and grad are transposes of each other and pressure \( p \) is the Lagrange multiplier
  - Just like we figured out constraint forces before…

Pressure solve

- To see what pressure is, take divergence of momentum equation
  \[ \nabla \cdot \left( u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla p \right) = 0 \]

- For constant density, just get Laplacian (and this is Poisson’s equation)
- Important numerical methods use this approach to find pressure
**Projection**

- Note that $\nabla \cdot u_t = 0$ so in fact
  \[ \nabla \cdot \frac{1}{\rho} \nabla p = -\nabla \cdot (u \cdot \nabla u) \]
- After we add $\nabla p/\rho$ to $u \cdot \nabla u$, divergence must be zero
- So if we tried to solve for additional pressure, we get zero
- Pressure solve is linear too
- Thus what we’re really doing is a projection of $u \cdot \nabla u$ onto the subspace of divergence-free functions: $u_t + P(u \cdot \nabla u) = g$

**Viscosity**

- In reality, nearby molecules travelling at different velocities occasionally bump into each other, transferring energy
  - Differences in velocity reduced (damping)
  - Measure this by strain rate (time derivative of strain, or how far velocity field is from rigid motion)
  - Add terms to our constitutive law

**Strain rate**

- At any instant in time, measure how fast chunk of material is deforming from its current state
  - Not from its original state
  - So we’re looking at infinitesimal, incremental strain updates
  - Can use linear Cauchy strain!
  - (In fact, in solids, this leads to a more advanced “true” strain arrived at by integrating infinitesimal strain increments… but not important here)
- So the strain rate tensor is
  \[ \dot{\varepsilon}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

**Viscous stress**

- As with linear elasticity, end up with two parameters if we want isotropy:
  \[ \sigma_{ij}^{viscous} = 2\mu \dot{\varepsilon}_{ij} + \lambda \dot{\varepsilon}_{kk} \delta_{ij} \]
  - $\mu$ and $\lambda$ are coefficients of viscosity (first and second)
  - These are not the Lame coefficients! Just use the same symbols
  - $\lambda$ damps only compression/expansion
  - Usually $\lambda \approx 2/3 \mu$ (exact for monatomic gases)
- So end up with
  \[ \sigma_{ij}^{viscous} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) \]
Navier-Stokes

- Navier-Stokes equations include the viscous stress
- Incompressible version:
  \[ u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla p = g + \frac{1}{\rho} \nabla \cdot \mu(\nabla u + \nabla^T u) \]
  \[ \nabla \cdot u = 0 \]
- Often (but not always) viscosity \( \mu \) is constant, and this reduces to
  \[ u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla p = g + \frac{\mu}{\rho} \nabla^2 u \]

Boundary conditions

- We’ll sidestep this issue until it comes up in numerical methods
- There are some subtle mathematical details (and open problems) relating to what exactly you can or need to specify
- Generally: specify some mix of velocity and traction at the boundary
  - Depends on whether or not you have viscosity

Inviscid boundaries

- Basic choice:
  - At a closed boundary ("wall") use the "no-stick" condition
    \[ u \cdot n = 0 \]
  - Or if the boundary is moving in the normal direction,
    \[ u \cdot n = u_{wall} \cdot n \]
  - At an open boundary ("free surface") specify the traction, i.e. the pressure. For example:
    \[ p = p_{atm} \quad \frac{\partial u}{\partial n} = 0 \]
  - Condition on normal derivative of \( u \) implicit

Viscous (friction) boundaries

- Can use “no-slip” condition on walls:
  \[ u=0 \]
  - Or \( u=u_{wall} \)
- Traction is no longer so simple at a free surface (stress tensor has more than just pressure)
Other quantities

- We may want to carry around auxiliary quantities
  - E.g. temperature, the type of fluid (if we have a mix), concentration of smoke, etc.
- Use material derivative as before
- E.g. if quantity doesn’t change, just is transported (“advected”) around:

\[
\frac{Dq}{Dt} = q_t + u \cdot \nabla q = 0
\]

Pressure stuff

- Just as we solved for pressure before, can do the same here
- But we take the divergence of the viscosity term as well

What now?

- Can solve numerically the full equations
  - Will do this later
  - Not so simple, could be expensive (3D)
- Or make assumptions and simplify them, then solve numerically
  - Simplify flow (e.g. irrotational)
  - Simplify dimensionality (e.g. go to 2D)

Potential flow

- If velocity is irrotational:
  \[
  \nabla \times u = 0
  \]
  - Which it often is in simple laminar flow
- Then there must be a stream function (potential) such that:
  \[
  u = \nabla \phi
  \]
- Solve for incompressibility:
  \[
  \nabla \cdot \nabla \phi = 0
  \]