Notes

- Assignment 4 due "today" (when I check email tomorrow morning)
- Don't be afraid to make assumptions, approximate quantities, …
  - In particular, method for computing time step bound (look at max eigenvalue of Jacobian) already made lots of assumptions about linearity etc. so it won't hurt to make a few more!

Fluid mechanics

- We already figured out the equations of motion for continuum mechanics

$$\rho\ddot{x} = \nabla \cdot \sigma + \rho g$$

- Just need a constitutive model

$$\sigma = \sigma\left(x, t, \varepsilon, \dot{\varepsilon}\right)$$

- We'll look at the constitutive model for "Newtonian" fluids today
  - Remarkably good model for water, air, and many other simple fluids
  - Only starts to break down in extreme situations, or more complex fluids (e.g. viscoelastic substances)

Inviscid Euler model

- Inviscid=no viscosity
- Great model for most situations
  - Numerical methods end up with viscosity-like error terms anyways…
- Constitutive law is very simple:

$$\sigma_{ij} = -p\delta_{ij}$$

  - New scalar unknown: pressure p
  - Barotropic flows: p is just a function of density (e.g. perfect gas law $p=k(\rho-\rho_0)+p_0$ perhaps)
  - For more complex flows need heavy-duty thermodynamics: an equation of state for pressure, equation for evolution of internal energy (heat), …

Lagrangian viewpoint

- We've been working with Lagrangian methods so far
  - Identify chunks of material,
    track their motion in time,
    differentiate world-space position or velocity w.r.t. material coordinates to get forces
  - In particular, use a mesh connecting particles to approximate derivatives (with FVM or FEM)
- Bad idea for most fluids
  - [vortices, turbulence]
  - At least with a fixed mesh…

Eulerian viewpoint

- Take a fixed grid in world space, track how velocity changes at a point
- Even for the craziest of flows, our grid is always nice
- (Usually) forget about object space and where a chunk of material originally came from
  - Irrelevant for extreme inelasticity
  - Just keep track of velocity, density, and whatever else is needed

Conservation laws

- Identify any fixed volume of space
- Integrate some conserved quantity in it (e.g. mass, momentum, energy, …)
- Integral changes in time only according to how fast it is being transferred from/to surrounding space
  - Called the flux
  - [divergence form]

$$\frac{\partial}{\partial t}\int_{\Omega} q = -\int_{\partial\Omega} f(q)\cdot n$$

$$q_t + \nabla \cdot f = 0$$

Conservation of Mass

- Also called the continuity equation (makes sure matter is continuous)
- Let's look at the total mass of a volume (integral of density)
- Mass can only be transferred by moving it: flux must be ρu

$$\frac{\partial}{\partial t}\int_{\Omega} \rho = -\int_{\partial\Omega} \rho u \cdot n$$

$$\rho_t + \nabla \cdot (\rho u) = 0$$

Material derivative

- A lot of physics just naturally happens in the Lagrangian viewpoint
  - E.g. the acceleration of a material point results from the sum of forces on it
  - How do we relate that to rate of change of velocity measured at a fixed point in space?
  - Can't directly: need to get at Lagrangian stuff somehow
- The material derivative of a property q of the material (i.e. a quantity that gets carried along with the fluid) is

$$\frac{Dq}{Dt}$$

Finding the material derivative

- Using object-space coordinates p and map x=X(p) to world-space, then material derivative is just

$$\frac{D}{Dt}q(t,x) = \frac{d}{dt}q\big(t, X(t,p)\big)$$
$$= \frac{\partial q}{\partial t} + \nabla q \cdot \frac{\partial x}{\partial t}$$
$$= q_t + u \cdot \nabla q$$

- Notation: u is velocity (in fluids, usually use u but occasionally v or V, and components of the velocity vector are sometimes u,v,w)

Compressible Flow

- In general, density changes as fluid compresses or expands
- When is this important?
  - Sound waves (and/or high speed flow where motion is getting close to speed of sound - Mach numbers above 0.3?)
  - Shock waves
- Often not important scientifically, almost never visually significant
  - Though the **effect** of e.g. a blast wave is visible! But the shock dynamics usually can be hugely simplified for graphics

Incompressible flow

- So we'll just look at incompressible flow, where density of a chunk of fluid never changes
  - Note: fluid density may not be constant throughout space - different fluids mixed together…
- That is, Dρ/Dt=0

Simplifying

- Incompressibility: $\dfrac{D\rho}{Dt} = \rho_t + u \cdot \nabla \rho = 0$
- Conservation of mass: $\rho_t + \nabla \cdot (\rho u) = 0$

$$\rho_t + \nabla \rho \cdot u + \rho \nabla \cdot u = 0$$

- Subtract the two equations, divide by ρ:

$$\nabla \cdot u = 0$$

- Incompressible == divergence-free velocity
  - Even if density isn't uniform!

Conservation of momentum

- Short cut - in $\rho\ddot{x} = \nabla \cdot \sigma + \rho g$ use material derivative:

$$\rho \frac{Du}{Dt} = \nabla \cdot \sigma + \rho g$$
$$\rho(u_t + u \cdot \nabla u) = \nabla \cdot \sigma + \rho g$$

- Or go by conservation law, with the flux due to transport of momentum and due to stress:
  - Then simplify a bit using conservation of mass

$$(\rho u)_t + \nabla \cdot (u \rho u - \sigma) = \rho g$$

Inviscid momentum equation

- Plug in simplest consitutive law (σ=-pδ) from before to get

$$\rho(u_t + u \cdot \nabla u) = -\nabla p + \rho g$$
$$u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla p = g$$

  - Together with conservation of mass: the Euler equations

Incompressible inviscid flow

- So the equations are: $u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla p = g$

$$\nabla \cdot u = 0$$

- 4 equations, 4 unknowns (u, p)
- Pressure p is just whatever it takes to make velocity divergence-free
- In fact, incompressibility is a hard constraint; div and grad are transposes of each other and pressure p is the Lagrange multiplier
  - Just like we figured out constraint forces before…

Pressure solve

- To see what pressure is, take divergence of momentum equation

$$\nabla \cdot \left(u_t + u \cdot \nabla u + \tfrac{1}{\rho} \nabla p\right) = 0$$
$$\nabla \cdot \left(\tfrac{1}{\rho} \nabla p\right) = -\nabla \cdot (u_t + u \cdot \nabla u)$$

- For constant density, just get Laplacian (and this is Poisson’s equation)
- Important numerical methods use this approach to find pressure

Projection

- Note that $\nabla \cdot u_t = 0$ so in fact

$$\nabla \cdot \tfrac{1}{\rho} \nabla p = -\nabla \cdot (u \cdot \nabla u)$$

- After we add $\nabla p/\rho$ to $u \cdot \nabla u$, divergence must be zero
- So if we tried to solve for additional pressure, we get zero
- Pressure solve is linear too
- Thus what we're really doing is a **projection** of $u \cdot \nabla u$ onto the subspace of divergence-free functions: $u_t + P(u \cdot \nabla u) = g$

Viscosity

- In reality, nearby molecules travelling at different velocities occasionally bump into each other, transferring energy
  - Differences in velocity reduced (damping)
  - Measure this by strain rate (time derivative of strain, or how far velocity field is from rigid motion)
  - Add terms to our constitutive law

Strain rate

- At any instant in time, measure how fast chunk of material is deforming from its current state
  - **Not** from its original state
  - So we're looking at infinitesimal, incremental strain updates
  - Can use linear Cauchy strain!
  - (In fact, in solids, this leads to a more advanced "true" strain arrived at by integrating infinitesimal strain increments… but not important here)
- So the strain rate tensor is $\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

Viscous stress

- As with linear elasticity, end up with two parameters if we want isotropy:

$$\sigma_{ij}^{viscous} = 2\mu\dot{\varepsilon}_{ij} + \lambda\dot{\varepsilon}_{kk}\delta_{ij}$$

  - μ and λ are coefficients of viscosity (first and second)
  - These are not the Lame coefficients! Just use the same symbols
  - λ damps only compression/expansion
- Usually $\lambda \approx -2/3\mu$ (exact for monatomic gases)
- So end up with $\sigma_{ij}^{viscous} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)$

Navier-Stokes

- Navier-Stokes equations include the viscous stress
- Incompressible version:

$$u_t + u \cdot \nabla u + \tfrac{1}{\rho} \nabla p = g + \tfrac{1}{\rho} \nabla \cdot \mu \left(\nabla u + \nabla u^T \right)$$

$$\nabla \cdot u = 0$$

- Often (but **not** always) viscosity μ is constant, and this reduces to

$$u_t + u \cdot \nabla u + \tfrac{1}{\rho} \nabla p = g + \tfrac{\mu}{\rho} \nabla^2 u$$

Boundary conditions

- We'll sidestep this issue until it comes up in numerical methods
- There are some subtle mathematical details (and open problems) relating to what exactly you can or need to specify
- Generally: specify some mix of velocity and traction at the boundary
  - Depends on whether or not you have viscosity

Inviscid boundaries

- Basic choice:
  - At a closed boundary ("wall") use the "no-stick" condition $u \cdot n = 0$
  - Or if the boundary is moving in the normal direction, $u \cdot n = u_{wall} \cdot n$
  - At an open boundary ("free surface") specify the traction, i.e. the pressure. For example:

$$p = p_{atm} \qquad \frac{\partial u}{\partial n} = 0$$

  - Condition on normal derivative of u implicit

Viscous (friction) boundaries

- Can use "no-slip" condition on walls:
  u=0
  - Or $u = u_{wall}$
- Traction is no longer so simple at a free surface (stress tensor has more than just pressure)

Other quantities

- We may want to carry around auxiliary quantities
  - E.g. temperature, the type of fluid (if we have a mix), concentration of smoke, etc.
- Use material derivative as before
- E.g. if quantity doesn't change, just is transported ("advected") around:

$$\frac{Dq}{Dt} = q_t + \underbrace{u \cdot \nabla q}_{advection} = 0$$

Pressure stuff

- Just as we solved for pressure before, can do the same here
- But we take the divergence of the viscosity term as well

What now?

- Can solve numerically the full equations
  - Will do this later
  - Not so simple, could be expensive (3D)
- Or make assumptions and simplify them, then solve numerically
  - Simplify flow (e.g. irrotational)
  - Simplify dimensionality (e.g. go to 2D)

Potential flow

- If velocity is irrotational:

$$\nabla \times u = 0$$

  - Which it often is in simple laminar flow
- Then there must be a stream function (potential) such that: $u = \nabla \phi$
- Solve for incompressibility: $\nabla \cdot \nabla \phi = 0$