Continuum Mechanics

- We'll stick with the Lagrangian viewpoint for now
- Let's look at a deformable object
  - World space: points x in the object as we see it
  - Object space (or rest pose): points p in some reference configuration of the object
  - (Technically we might not have a rest pose, but usually we do, and it is the simplest parametrization)
- So we identify each point x of the continuum with the label p, where x=X(p)

Conservation of mass

- Look at a control volume
- Mass of control volume is

$$\int_{\Omega_W} \rho(x)dx = \int_{\Omega_O} \rho\big(X(p)\big)\det\left(\frac{\partial X}{\partial p}\right)dp = \int_{\Omega_O} \rho(p)J\,dp = \int_{\Omega_O} \rho_0(p)dp$$

- Usually we discretize into particles (with fixed mass) and don't need to worry about this
  - But if you adaptively change the discretization, may need to use this

Conservation of Momentum

- In other words F=ma
- Control volume again
- Split F into
  - f_{body} (e.g. gravity, magnetic forces, …) force per unit volume
  - and traction t (on boundary between two chunks of continuum: contact) force per unit area (like pressure)

$$\int_{\Omega_W} f_{body}\,dx + \int_{\partial\Omega_W} t\,ds = \int_{\Omega_W} \rho\ddot{X}\,dx$$

Cauchy's Fundamental Postulate

- Traction t is a function of position x and normal n
  - Ignores rest of boundary (curvature, etc.)
- **Theorem**
  - If t is smooth (be careful at boundaries of object, e.g. cracks) then t is linear in n: $t=\sigma(x)n$
- σ is the Cauchy stress tensor (a matrix)
- It also is force per unit area
- Diagonal: normal stress components
- Off-diagonal: shear stress components

Cauchy Stress

- From conservation of angular momentum can derive that Cauchy stress tensor σ is symmetric: $\sigma = \sigma^T$
- Thus there are only 6 degrees of freedom (in 3D)
  - In 2D, only 3 degrees of freedom
- What is σ?
  - That's the job of **constitutive modeling**
  - Depends on the material (e.g. water vs. steel vs. silly putty)

Divergence Theorem

- Try to get rid of integrals
- First make them all volume integrals with divergence theorem:

$$\int_{\partial\Omega_W} \sigma n\, ds = \int_{\partial\Omega_W} \sigma^T \cdot n\, ds = \int_{\Omega_W} \nabla \cdot \sigma^T\, dx = \int_{\Omega_W} \nabla \cdot \sigma\, dx$$

- Next let control volume shrink to zero:

$$f_{body} + \nabla \cdot \sigma = \rho \ddot{x}$$

Constitutive Modeling

- This can get very complicated for complicated materials
- Let's start with simple elastic materials
- We'll even leave damping out
- Then stress σ only depends on deformation gradient $A = \partial X / \partial p$
  - No memory of past deformations
  - Always will want to return to original shape (when A is just a rotation matrix)
  - Note we don't care about translation at all (X)

Strain

- A isn't so handy to deal with, though it somehow encodes exactly how stretched/compressed we are
  - It also encodes how rotated we are, which we don't care about
- We want to process A somehow to remove the rotation part
- [difference in lengths]
- $A^T A - I$ is exactly zero when A is a rigid body rotation
- Define Green strain $G = \frac{1}{2}\left(A^T A - I\right)$

Why the half??

- [Look at 1D, small deformation]
- $A=1+\varepsilon$
- $A^TA-I = A^2-1 = 2\varepsilon+\varepsilon^2 \approx 2\varepsilon$
- Therefore $G \approx \varepsilon$, which is what we expect
- Note that for large deformation, Green strain grows quadratically
  - maybe not what you expect!
- Whole cottage industry: defining strain differently

Cauchy strain tensor

- Look at "small displacement"
  - Not only is the shape only slightly deformed, but it only slightly rotates
    (e.g. if one end is fixed in place)
- Then displacement x-p has gradient $D=A-I$
- Then $G = \frac{1}{2}\left(D^T D + D + D^T\right)$
- And for small displacement, first term negligible
- Cauchy strain $\varepsilon = \frac{1}{2}\left(D + D^T\right)$
- Symmetric part of deformation gradient
  - Rotation is skew-symmetric part

Linear elasticity

- Very nice thing about Cauchy strain: it's linear in deformation
  - No quadratic dependence
  - Easy and fast to deal with
- Natural thing is to make a linear relationship with Cauchy stress σ
- Then the full equation is linear!

Young's modulus

- Obvious first thing to do: if you pull on material, resists like a spring:
  $\sigma=E\varepsilon$
- E is the Young's modulus
- Let's check that in 1D (where we know what should happen with springs)

$$\nabla \cdot \sigma = \rho \ddot{x}$$

- Almost:
  (close enough)

$$\frac{\partial}{\partial x}\left(E\left(\frac{\partial X}{\partial p} - 1\right)\right) = \rho \ddot{x}$$

Poisson Ratio

- Real materials are essentially incompressible (for large deformation - neglecting foams and other weird composites…)
- For small deformation, materials are usually somewhat incompressible
- Imagine stretching block in one direction
  - Measure the contraction in the perpendicular directions
  - Ratio is ν, Poisson's ratio
- [draw experiment; $\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}}$]

What is Poisson's ratio?

- Has to be between -1 and 0.5
- 0.5 is exactly incompressible
  - [derive]
- Negative is weird, but possible [origami]
- Rubber: close to 0.5
- Steel: more like 0.33
- Metals: usually 0.25-0.35
- [cute: cork is almost 0]

Putting it together

$$E\varepsilon_{11} = \sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}$$
$$E\varepsilon_{22} = -\nu\sigma_{11} + \sigma_{22} - \nu\sigma_{33}$$
$$E\varepsilon_{33} = -\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33}$$

- Can invert this to get normal stress, but what about shear stress?
  - [draw sheared block]
- When the dust settles,

$$E\varepsilon_{ij} = (1+\nu)\sigma_{ij} \quad i \neq j$$

Inverting…

$$\sigma = E\left(\frac{1}{1+\nu}I + \frac{\nu}{(1+\nu)(1-2\nu)}1 \otimes 1\right)\varepsilon$$

- For convenience, relabel these expressions
  - λ and μ are called the Lamé coefficients
  - [incompressibility]

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$
$$\mu = \frac{E}{2(1+\nu)}$$
$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

Linear elasticity

- Putting it together and assuming constant coefficients, simplifies to

$$\begin{aligned}\rho\dot{v} &= f_{body} + \lambda\nabla\varepsilon_{kk} + 2\mu\nabla\cdot\varepsilon \\ &= f_{body} + \lambda\nabla\cdot\nabla x + \mu\left(\nabla\cdot\nabla x + \nabla\nabla\cdot x\right) \\ &= f_{body} + \left(\lambda+\mu\right)\Delta x + \mu\nabla\nabla\cdot x\end{aligned}$$

- A PDE!
  - We'll talk about solving it later

Rayleigh damping

- We'll need to look at strain rate
  - How fast object is deforming
  - We want a damping force that resists change in deformation
- Just the time derivative of strain
- For Rayleigh damping of linear elasticity

$$\sigma_{ij}^{damp} = \phi\dot{\varepsilon}_{kk}\delta_{ij} + 2\psi\dot{\varepsilon}_{ij}$$

Problems

- Linear elasticity is very nice for small deformation
  - Linear form means lots of tricks allowed for speed-up, simpler to code, …
- But it's useless for large deformation, or even zero deformation but large rotation
  - (without hacks)
  - [draw tangent to circle]
- Thus we need to go back to Green strain

(Almost) Linear Elasticity

- Use the same constitutive model as before, but with Green strain tensor
- This is the simplest general-purpose elasticity model
- Note this is different from "Green elasticity"
  - Slightly better foundation: write down a potential energy (a scalar function of strain) and take the gradient to get stress

2D Elasticity

- Let's simplify life before starting numerical methods
- The world isn't 2D of course, but want to track only deformation in the plane
- Have to model why
  - Plane stress: very thin material, $\sigma_{3\cdot}=0$
    [explain, derive $\varepsilon_{3\cdot}$ and new law, note change in incompressibility singularity]
  - Plane strain: very thick material, $\varepsilon_{3\cdot}=0$
    [explain, derive $\sigma_{3\cdot}$]

Finite Volume Method

- Simplest approach: finite volumes
  - We picked arbitrary control volumes before
  - Now pick fractions of triangles from a triangle mesh
    - Split each triangle into 3 parts, one for each corner
    - E.g. Voronoi regions
    - Be consistent with mass!
  - Assume A is constant in each triangle (piecewise linear deformation)
  - [work out]
  - Note that exact choice of control volumes not critical - constant times normal integrates to zero