Appendix: Lemmas and Proofs

Proposition 1. *Let $\mathcal{M}$ be an OGBN and $\mathcal{S}$ be a maximal well-defined conjunction of variable assignments. For any variable $X \notin vars(\mathcal{S})$, $\mathcal{S} \models \neg dom(X)$.*

Proof. Assume the statement were not true. Let $X_k \notin vars(\mathcal{S})$ be the first variable such that $\mathcal{S}_{<k} \not\models \neg dom(X_k)$. Let $\boldsymbol{D} = vars(dom(X_k)) \setminus vars(\mathcal{S})$. We could find an assignment D to the variables in $\boldsymbol{D}$ such that $\mathcal{S} \wedge D \wedge X_k = x_k$ is well-defined, and so $\mathcal{S}$ were not maximal. □

Lemma 1. *Let $\mathcal{M}$ be an OGBN with the total ordering $X_1, \ldots, X_n$ of variables, and $\mathcal{M}^+$ be the corresponding EBN. Let $\mathcal{S}^+ \equiv X^+_{\pi(1)} = x_{\pi(1)} \wedge \cdots \wedge X^+_{\pi(k)} = x_{\pi(k)}$ be a conjunction of variable assignments. If there exists some $X_i^+ \in \mathcal{S}^+$ such that $\mathcal{S}^+ \models dom(X_i)^+$ and $\mathcal{S}^+ \models X_i^+ = \bot$, then $P_{\mathcal{M}^+}(\mathcal{S}^+) = 0$.*

Proof. It suffices to show that for any full conjunction $\mathcal{S}'_{full}$ such that $\mathcal{S}'_{full} \models \mathcal{S}^+$, $P_{\mathcal{M}^+}(\mathcal{S}'_{full}) = 0$.

By the construction of the EBN, $P_{\mathcal{M}^+}(X_i^+ = \bot \mid \mathcal{S}'_{\{\boldsymbol{Pa}(X_i^+)\}}) = 0$, where $\mathcal{S}'_{\{\boldsymbol{Pa}(X_i^+)\}}$ denotes the part of $\mathcal{S}'_{full}$ involving the variables in $\boldsymbol{Pa}(X_i^+)$. The chain rule for belief networks thus gives

$$P_{\mathcal{M}^+}(\mathcal{S}'_{full}) = \prod_{j=1}^{n} P_{\mathcal{M}^+}(\mathcal{S}'_{\{X_j\}} \mid \mathcal{S}'_{\{\boldsymbol{Pa}(X_j^+)\}}) = 0.$$

□

Definition 7. A conjunction $\mathcal{S}$ of variable assignments defined in an OGBN $\mathcal{M}$ is **realistic** if for any variable $X_i \in vars(\mathcal{S})$, $\mathcal{M}.Ont \not\models \mathcal{S}_{<i} \to \neg dom(X_i)$.

Realistic conjunctions are exactly those that may have positive probabilities in an OGBN. A (maximal) well-defined conjunction can be constructed from a realistic conjunction by adding variable assignments to it. It is also clear that any well-defined conjunction is also realistic.

Lemma 2. *Let $\mathcal{M}$ be an OGBN with the total ordering $X_1, \ldots, X_n$ of variables, and $\mathcal{M}^+$ be the corresponding EBN. Let $\mathcal{S}^+ \equiv X^+_{\pi(1)} = x_{\pi(1)} \wedge \cdots \wedge X^+_{\pi(k)} = x_{\pi(k)}$ be a conjunction of variable assignments and $\mathcal{S}$ be the conjunction such that $\mathcal{S} \models X_i = x_i$ iff $\mathcal{S}^+ \models X_i^+ = x_i$ and $x_i \neq \bot$. If $\mathcal{S}$ is not realistic, then $P_{\mathcal{M}^+}(\mathcal{S}^+) = 0$.*

Proof. It suffices to show that for any full conjunction $\mathcal{S}'_{full}$ such that $\mathcal{S}'_{full} \models \mathcal{S}^+$, $P_{\mathcal{M}^+}(\mathcal{S}'_{full}) = 0$.

Let $X_u \in vars(\mathcal{S})$ be the first variable such that $\mathcal{M}.Ont \models \mathcal{S}_{<u} \to \neg dom(X_u))$. By the construction of the EBN, $P_{\mathcal{M}^+}(X_u^+ = x_u \mid \mathcal{S}'_{\{\boldsymbol{Pa}(X_u^+)\}}) = 0$. The chain rule for belief networks then gives $P_{\mathcal{M}^+}(\mathcal{S}'_{full}) = 0$. □

Corollary 1. *Let $\mathcal{M}$ be an OGBN with the total ordering $X_1, \ldots, X_n$ of variables, and $\mathcal{M}^+$ be the corresponding EBN. Let $\mathcal{S}$ be a maximal well-defined conjunction of variable assignments, and $\mathcal{S}^+_{full}$ be the full conjunction such that X_i^+ is assigned the value as X_i if $X_i \in vars(\mathcal{S})$ and $\bot$ otherwise. $P_{\mathcal{M}^+}(\mathcal{S}^+) = P_{\mathcal{M}^+}(\mathcal{S}^+_{full})$.*

Proof. The probability of $\mathcal{S}^+$ in $\mathcal{M}^+$ can be computed as

$$P_{\mathcal{M}^+}(\mathcal{S}^+) = \sum_{\mathcal{S}'_{full} \models \mathcal{S}^+} P_{\mathcal{M}^+}(\mathcal{S}'_{full}),$$

where $\mathcal{S}'_{full}$ is any full conjunction of variables assignments that entails $\mathcal{S}^+$. By Proposition 1 and Lemma 2, since $\mathcal{S}$ is maximal, $P_{\mathcal{M}^+}(\mathcal{S}'_{full}) = 0$ for any $\mathcal{S}'_{full} \not\equiv \mathcal{S}^+_{full}$. Thus, $P_{\mathcal{M}^+}(\mathcal{S}^+) = P_{\mathcal{M}^+}(\mathcal{S}^+_{full})$. □

Lemma 3. *Let $\mathcal{M}$ be an OGBN and $\mathcal{M}^+$ be the corresponding EBN. Let $\mathcal{S} \equiv X_{\pi(1)} = x_{\pi(1)} \wedge \cdots \wedge X_{\pi(k)} = x_{\pi(k)}$ be a maximal well-defined conjunction of variable assignments. $P_{\mathcal{M}}(\mathcal{S}) = P_{\mathcal{M}^+}(\mathcal{S}^+)$.*

Proof. $P_{\mathcal{M}}(\mathcal{S})$ is computed as

$$P_{\mathcal{M}}(\mathcal{S}) = \prod_{i=1}^{k} P_{\mathcal{M}}(X_{\pi(i)} = x_{\pi(i)} \mid c_{\pi(i)}),$$

where $c_{\pi(i)}$ is the parent context for $X_{\pi(i)}$ such that $\mathcal{S} \models c_{\pi(i)}$.

Similarly for the corresponding extended belief network,

$$P_{\mathcal{M}^+}(\mathcal{S}^+) = \sum_{\mathcal{S}'} P_{\mathcal{M}^+}(\mathcal{S}^+ \wedge \mathcal{S}') \tag{1}$$

$$= \sum_{\mathcal{S}'} \prod_{j=1}^{n} P_{\mathcal{M}^+}(X_1^+ = x_1 \mid (\mathcal{S}^+ \wedge \mathcal{S}')_{\{\boldsymbol{Pa}(X_j^+)\}}) \tag{2}$$

$$= \prod_{i=1}^{k} P_{\mathcal{M}^+}(X^+_{\pi(i)} = x_{\pi(i)} \mid c^+_{\pi(i)}), \tag{3}$$

where $\mathcal{S}'$ is any conjunction that assigns a value to every variable not assigned in $\mathcal{S}^+$. Equation 3 follows because, by construction, $P_{\mathcal{M}^+}(X^+_{\pi(i)} = x_{\pi(i)} \mid (\mathcal{S}^+ \wedge \mathcal{S}')_{\{\boldsymbol{Pa}(X^+_{\pi(i)})\}}) = P_{\mathcal{M}^+}(X^+_{\pi(i)} = x_{\pi(i)} \mid c^+_{\pi(i)})$, and the variables in $\mathcal{S}'$ are all summed out.

We have the correspondence $P(X_{\pi(i)} = x_{\pi(i)} \mid c_{\pi(i)}) = P_{\mathcal{M}^+}(X^+_{\pi(i)} = x_{\pi(i)} \mid c^+_{\pi(i)})$ between $\mathcal{M}$ and $\mathcal{M}^+$, and so $P_{\mathcal{M}}(\mathcal{S}) = P_{\mathcal{M}^+}(\mathcal{S}^+)$. □

Proof of Theorem 1. Let $\mathcal{M}^+$ be the corresponding EBN of $\mathcal{M}$. Consider a maximal well-defined conjunction $\mathcal{S}$ of variable assignments. By Corollary 1 and Lemma 3, $P_{\mathcal{M}}(\mathcal{S}) = P_{\mathcal{M}^+}(\mathcal{S}^+_{full})$, where $\mathcal{S}^+_{full}$ is as defined in Corollary 1. By Lemma 1 and Lemma 2, any other full conjunction in $\mathcal{M}^+$ has probability 0. Since $\mathcal{M}^+$ is known to represent a coherent probability distribution, it follows that $\mathcal{M}$ also encodes a coherent probability distribution. □

Validity of 3Q-INFERENCE

We first show that an OGBN $\mathcal{M}$ and its corresponding EBN $\mathcal{M}^+$ encode the same probabilities over all realistic (thus, including well-defined) conjunctions of variable assignments. The conjunctions that are not realistic are irrelevant and not specified in the OGBN.

Proof of Theorem 2. The probability of $\mathcal{S}$ can be calculated as

$$P_{\mathcal{M}}(\mathcal{S}) = \sum_{\mathcal{S}_{max} \models \mathcal{S}} P_{\mathcal{M}}(\mathcal{S}_{max}),$$

where $\mathcal{S}_{max}$ is any maximal well-defined conjunction that entails $\mathcal{S}$. Similarly, the probability of $\mathcal{S}^+$ is

$$P_{\mathcal{M}^+}(\mathcal{S}^+) = \sum_{\mathcal{S}'_{full} \models \mathcal{S}^+} P_{\mathcal{M}^+}(\mathcal{S}'_{full}),$$

where $\mathcal{S}'_{full}$ is any full conjunction that entails $\mathcal{S}^+$.

By Corollary 1 and Lemma 3, for any $\mathcal{S}_{max}$, $P_{\mathcal{M}}(\mathcal{S}_{max}) = P_{\mathcal{M}^+}(\mathcal{S}^+_{full})$, where $\mathcal{S}^+_{full}$ is the full conjunction in $\mathcal{M}^+$ such that X_i^+ is assigned the value as X_i if $X_i \in vars(\mathcal{S}_{max})$ and $\perp$ otherwise. By Lemma 1 and Lemma 2, all other full conjunctions in $\mathcal{M}^+$ have probability 0. Hence, the desired equality follows. □

Corollary 2. *Let $\mathcal{M}$ be an OGBN and $\mathcal{M}^+$ be the corresponding EBN. Consider any two conjunctions of variable assignments, $\mathcal{S}_1$ and $\mathcal{S}_2$, such that $\mathcal{S}_1$ is realistic, $\mathcal{S}_1 \wedge \mathcal{S}_2$ is realistic, and $P_{\mathcal{M}}(\mathcal{S}_1) > 0$. $P_{\mathcal{M}}(\mathcal{S}_2 \mid \mathcal{S}_1) = P_{\mathcal{M}^+}(\mathcal{S}_2^+ \mid \mathcal{S}_1^+)$.*

Proof. Elementary probability theory gives

$$P_{\mathcal{M}^+}(\mathcal{S}_2^+ \mid \mathcal{S}_1^+) = \frac{P_{\mathcal{M}^+}(\mathcal{S}_1^+ \wedge \mathcal{S}_2^+)}{P_{\mathcal{M}^+}(\mathcal{S}_1^+)}.$$

By Theorem 2, $P_{\mathcal{M}}(\mathcal{S}_1) = P_{\mathcal{M}^+}(\mathcal{S}_1^+)$ and $P_{\mathcal{M}}(\mathcal{S}_1 \wedge \mathcal{S}_2) = P_{\mathcal{M}^+}(\mathcal{S}_1^+ \wedge \mathcal{S}_2^+)$. The desired result follows. □

Corollary 3. *Let $\mathcal{M}$ be an OGBN and $\mathcal{M}^+$ be the corresponding EBN. Consider any variable Q and well-defined evidence $\mathcal{E}$ such that $\mathcal{E} \models dom(Q)$ and $P_{\mathcal{M}}(\mathcal{E}) > 0$. $P_{\mathcal{M}}(Q = q \mid \mathcal{E}) = P_{\mathcal{M}^+}(Q^+ = q \mid \mathcal{E}^+)$ for any $q \neq \perp$, and $P_{\mathcal{M}^+}(Q^+ = \perp \mid \mathcal{E}^+) = 0$.*

Proof. Since $\mathcal{E} \models dom(Q)$, the conjunction $\mathcal{E} \wedge Q = q$ is realistic. By Corollary 2, $P_{\mathcal{M}}(Q = q \mid \mathcal{E}) = P_{\mathcal{M}^+}(Q^+ = q \mid \mathcal{E}^+)$. Because $P_{\mathcal{M}}(Q \mid \mathcal{E})$ is a probability distribution over $range(Q)$, it follows that $P_{\mathcal{M}^+}(Q^+ = \perp \mid \mathcal{E}^+) = 0$. □

We proceed to show other simple identities that hold in the corresponding EBN $\mathcal{M}^+$.

Proposition 2. *Let $\mathcal{M}$ be an OGBN and $\mathcal{M}^+$ be the corresponding extended belief network. For any variable Q and well-defined evidence $\mathcal{E}$ such that $\mathcal{E} \models \neg dom(Q)$ and $P_{\mathcal{M}}(\mathcal{E}) > 0$, $P_{\mathcal{M}^+}(Q^+ = \perp \mid \mathcal{E}^+) = 1$.*

Proof. Consider any $q \neq \perp$. By Lemma 2, $P_{\mathcal{M}^+}(\mathcal{E}^+ \wedge Q^+ = q) = 0$ since $\mathcal{E} \wedge Q = q$ is not realistic. It follows that

$$\begin{aligned} P_{\mathcal{M}^+}(\mathcal{E}^+) &= \sum_v P_{\mathcal{M}^+}(\mathcal{E}^+ \wedge Q^+ = v) \\ &= P_{\mathcal{M}^+}(\mathcal{E}^+ \wedge Q^+ = \perp). \end{aligned}$$

This result gives

$$\begin{aligned} P_{\mathcal{M}^+}(Q^+ = \perp \mid \mathcal{E}^+) &= \frac{P_{\mathcal{M}^+}(\mathcal{E}^+ \wedge Q^+ = \perp)}{P_{\mathcal{M}^+}(\mathcal{E}^+)} \\ &= 1. \end{aligned}$$

□

An immediate consequence of Proposition 2 is that, when $\mathcal{E} \models \neg dom(Q)$, $P_{\mathcal{M}^+}(Q^+ = q \mid \mathcal{E}^+) = 0$ for any $q \neq \perp$.

Lemma 4. *Let $\mathcal{M}$ be an OGBN and $\mathcal{M}^+$ be the corresponding EBN. For any variable Q and well-defined evidence $\mathcal{E}$ such that $\mathcal{E} \not\models \neg dom(Q)$ and $P_{\mathcal{M}}(\mathcal{E}) > 0$, $P_{\mathcal{M}^+}(Q^+ = q \mid \mathcal{E}^+) = P_{\mathcal{M}^+}(dom(Q)^+ \wedge Q^+ = q \mid \mathcal{E}^+)$ for any $q \neq \perp$.*

Proof. It suffices to show that $P_{\mathcal{M}^+}(\neg dom(Q)^+ \wedge Q^+ = q \mid \mathcal{E}^+) = 0$ for any $q \neq \perp$.

If $\mathcal{E} \models dom(Q)$, it is an immediate result that $P_{\mathcal{M}^+}(\neg dom(Q)^+ \wedge Q^+ = q \mid \mathcal{E}^+) = 0$. Otherwise,

$$\begin{aligned} &P_{\mathcal{M}^+}(\neg dom(Q)^+ \wedge Q^+ = q \mid \mathcal{E}^+) \\ &= \frac{P_{\mathcal{M}^+}(\mathcal{E}^+ \wedge \neg dom(Q)^+ \wedge Q^+ = q)}{P_{\mathcal{M}^+}(\mathcal{E}^+)}. \end{aligned}$$

By Theorem 2, $P_{\mathcal{M}^+}(\mathcal{E}^+ \wedge \neg dom(Q)^+ \wedge Q^+ = q) = 0$, and the result follows. □

Lemma 5. *Let $\mathcal{M}$ be an OGBN and $\mathcal{M}^+$ be the corresponding EBN. For any variable Q and well-defined evidence $\mathcal{E}$ such that $P_{\mathcal{M}}(\mathcal{E}) > 0$, $P_{\mathcal{M}^+}(Q^+ = \perp \mid \mathcal{E}^+) = 1 - P_{\mathcal{M}^+}(dom(Q)^+ \mid \mathcal{E}^+)$.*

Proof. If $\mathcal{E} \models dom(Q)$, then $P_{\mathcal{M}^+}(dom(Q)^+ \mid \mathcal{E}^+) = 1$, and by Corollary 3, $P_{\mathcal{M}^+}(Q^+ = \perp \mid \mathcal{E}^+) = 0$.

If $\mathcal{E} \models \neg dom(Q)$, then $P_{\mathcal{M}^+}(dom(Q)^+ \mid \mathcal{E}^+) = 0$, and by Proposition 2, $P_{\mathcal{M}^+}(Q^+ = \perp \mid \mathcal{E}^+) = 1$.

Otherwise, $\mathcal{E} \not\models dom(Q)$ and $\mathcal{E} \not\models \neg dom(Q)$.

$$\begin{aligned} &P_{\mathcal{M}^+}(Q^+ = \perp \mid \mathcal{E}^+) && (4) \\ &= \sum_{\mathcal{S}^+} P_{\mathcal{M}^+}(Q^+ = \perp \mid \mathcal{E}^+ \wedge \mathcal{S}^+) \times P_{\mathcal{M}^+}(\mathcal{S}^+ \mid \mathcal{E}^+) && (5) \\ &= \sum_{\mathcal{S}'} P_{\mathcal{M}^+}(Q^+ = \perp \mid \mathcal{E}^+ \wedge \mathcal{S}') \times P_{\mathcal{M}^+}(\mathcal{S}' \mid \mathcal{E}^+) && (6) \\ &= \sum_{\mathcal{S}'} P_{\mathcal{M}^+}(\mathcal{S}' \mid \mathcal{E}) && (7) \\ &= P_{\mathcal{M}^+}(\neg dom(Q)^+ \mid \mathcal{E}^+) && (8) \\ &= 1 - P_{\mathcal{M}^+}(dom(Q)^+ \mid \mathcal{E}^+), && (9) \end{aligned}$$

where $\mathcal{S}^+$ is any conjunction of variable assignments such that $vars(\mathcal{S}^+) = vars(dom(Q)^+) \setminus vars(\mathcal{E}^+)$, and $\mathcal{S}'$ is any $\mathcal{S}^+$ such that $\mathcal{E}^+ \wedge \mathcal{S}^+ \models \neg dom(Q)^+$. Equation 6 follows because, by Corollary 3, $P_{\mathcal{M}^+}(Q^+ = \perp \mid \mathcal{E}^+ \wedge \mathcal{S}^+) = 0$ for any $\mathcal{S}^+$ such that $\mathcal{E}^+ \wedge \mathcal{S}^+ \models dom(Q)^+$; Equation 7 follows since, by Proposition 2, $P_{\mathcal{M}^+}(Q^+ = \perp \mid \mathcal{E}^+ \wedge \mathcal{S}') = 1$. (Note that $\mathcal{E}^+ \wedge \mathcal{S}^+$ includes all variables in $vars(dom(Q)^+)$, and so $\mathcal{E}^+ \wedge \mathcal{S}^+ \not\models dom(Q)^+ \Leftrightarrow \mathcal{E}^+ \wedge \mathcal{S}^+ \models \neg dom(Q)^+$.) □

We now apply the previous results to prove the validity of the inference scheme 3Q-INFERENCE by showing that any query results in identical posterior probabilities with an OGBN and with its corresponding EBN.

Proof of Theorem 3. We prove by considering the three distinct cases.

Case 1: $\mathcal{M}.Ont \models \mathcal{E} \rightarrow \neg dom(Q)$. 3Q-INFERENCE specifies $P(Q^+ = \perp \mid \mathcal{E}) = 1$. By Proposition 2, $P_{\mathcal{M}^+}(Q^+ = \perp \mid \mathcal{E}^+) = 1$.

Case 2: $\mathcal{M}.Ont \models \mathcal{E} \rightarrow dom(Q)$. 3Q-INFERENCE specifies, for any $q \neq \perp$, $P(Q^+ = q \mid \mathcal{E}) = P_{\mathcal{M}}(Q = q \mid \mathcal{E})$ and

$P(Q^+ = \bot \mid \mathcal{E}) = 0$. By Corollary 3, for any $q \neq \bot$, $P_{\mathcal{M}}(Q^+ = q \mid \mathcal{E}^+) = P_{\mathcal{M}^+}(Q^+ = q \mid \mathcal{E}^+)$ and thus $P_{\mathcal{M}^+}(Q^+ = \bot \mid \mathcal{E}^+) = 0$.

Case 3: $\mathcal{M}.Ont \not\models \mathcal{E} \to \neg dom(Q)$ and $\mathcal{M}.Ont \not\models \mathcal{E} \to dom(Q)$. 3Q-INFERENCE specifies $P(Q^+ = \bot \mid \mathcal{E}) = 1 - P_{\mathcal{M}}(dom(Q) \mid \mathcal{E})$ and, for any $q \neq \bot$, $P(Q^+ = q \mid \mathcal{E}) = P_{\mathcal{M}}(dom(Q) \mid \mathcal{E}) \times P_{\mathcal{M}}(Q = q \mid dom(Q) \wedge \mathcal{E})$.

By Corollary 2 and Lemma 5,

$$\begin{aligned} 1 - P_{\mathcal{M}}(dom(Q) \mid \mathcal{E}) &= 1 - P_{\mathcal{M}^+}(dom(Q)^+ \mid \mathcal{E}^+) \\ &= P_{\mathcal{M}^+}(Q^+ = \bot \mid \mathcal{E}^+). \end{aligned}$$

Since $dom(Q) \wedge \mathcal{E}$ is well-defined, Corollary 3 gives $P_{\mathcal{M}}(Q = q \mid dom(Q) \wedge \mathcal{E}) = P_{\mathcal{M}^+}(Q^+ = q \mid dom(Q)^+ \wedge \mathcal{E}^+)$ for any $q \neq \bot$. Together with Corollary 2, it follows that

$$\begin{aligned} &P_{\mathcal{M}}(dom(Q) \mid \mathcal{E}) \times P_{\mathcal{M}}(Q = q \mid dom(Q) \wedge \mathcal{E}) \\ &= P_{\mathcal{M}^+}(dom(Q)^+ \mid \mathcal{E}^+) \times P_{\mathcal{M}^+}(Q^+ = q \mid dom(Q)^+ \wedge \mathcal{E}^+) \\ &= P_{\mathcal{M}^+}(dom(Q)^+ \wedge Q^+ = q \mid \mathcal{E}^+) \\ &= P_{\mathcal{M}^+}(Q^+ = q \mid \mathcal{E}^+), \end{aligned}$$

where the final equality follows from Lemma 4. □

We show that inference algorithms for belief networks can be used for OGBNs to compute the correct probabilities.

Theorem 4. *Let $\mathcal{M}^+$ be an OGBN. Suppose we treated the graph structure and CPDs of $\mathcal{M}^+$ as for a regular belief network (i.e., ignoring whether the variables are well-defined and the missing possible value "undefined"), called $\mathcal{M}'$. Let $\mathcal{S}$ be a well-defined conjunction for $\mathcal{M}^+$, $P_{\mathcal{M}'}(\mathcal{S}) = P_{\mathcal{M}^+}(\mathcal{S}_{max})$.*

Proof. Since the probability of $\mathcal{S}$ is the sum of the probabilities of all maximal well-defined conjunctions that entail $\mathcal{S}$, it suffices to show that $P_{\mathcal{M}'}(\mathcal{S}_{max}) = P_{\mathcal{M}^+}(\mathcal{S}_{max})$ for any maximal well-defined conjunction $\mathcal{S}_{max}$.

Let $\mathcal{S}_{max} \equiv X_{\pi(1)} = x_{\pi(1)} \wedge \cdots \wedge X_{\pi(k)} = x_{\pi(k)}$. $P_{\mathcal{M}'}(\mathcal{S}_{max})$ can be computed by summing the probabilities of all full conjunctions that entail $\mathcal{S}_{max}$, or equivalently, by "summing out" variables not in $vars(\mathcal{S}_{max})$:

$$P_{\mathcal{M}'}(\mathcal{S}_{max}) \tag{10}$$

$$= \sum_{\substack{\{x_j \in range(X_j): \\ X_j \notin vars(\mathcal{S}_{max})\}}} P_{\mathcal{M}'}(\mathcal{S}_{full}) \tag{11}$$

$$= \sum_{\substack{\{x_j \in range(X_j): \\ X_j \notin vars(\mathcal{S}_{max})\}}} \prod_{i=1}^{n} P_{\mathcal{M}'}(X_i = x_i \mid c_i) \tag{12}$$

$$= \prod_{i=1}^{k} P_{\mathcal{M}'}(X_{\pi(i)} = x_{\pi(i)} \mid c_{\pi(i)}) \sum_{\{x_j\}} \prod_{j} P_{\mathcal{M}'}(X_j = x_j \mid c_j) \tag{13}$$

$$= P_{\mathcal{M}^+}(\mathcal{S}_{max}). \tag{14}$$

where c_i is the parent context for X_i such that $\mathcal{S}_{full} \models c_i$. Equation 13 follows since, by construction, $c_{\pi(i)}$ does not involve any $X_j \notin vars(\mathcal{S}_{max})$ (even if X_j is a parent variable of $X_{\pi(i)}$). Equation 14 follows since the exact same conditional probabilities are used. □

This essentially means that, although $\mathcal{M}'$ can encode arbitrary probabilities for conjunctions that are not well-defined, we can still apply any belief network algorithm to an OGBN in 3Q-INFERENCE, as we only make probabilistic queries that are well defined by the ontology. Note that this is not true even for realistic conjunctions because for $\mathcal{M}'$, unlike how $\mathcal{M}$ is defined, the probability of a realistic conjunction is not just the sum of the probabilities of all maximal well-defined conjunctions that entail it.