Uncertainty

I think that when we know that we actually do live in uncertainty, then we ought to admit it; it is of great value to realize that we do not know the answers to different questions. This attitude of mind – this attitude of uncertainty – is vital to the scientist, and it is this attitude of mind which the student must first acquire.

Richard P. Feynman

Review

Decision network:

- DAG with three sorts of nodes: decision (rectangle), random (ellipse), utility (diamond)
- Domain for each decision and random node (no domain for utility)
- Factor for each random node and the utility (no initial factor for decision nodes)
- A decision function maps assignments to the parents of a decision node to a value in the domain of the decision node.
- A policy assigns a decision function to each decision node.
- In VE: sum out random variables until a decision variable D is in a factor F where all of the other variables in F are parents of D; then maximize D.

Overview of Course

	dynamics	observable	repr	stage	preference	rationality
search	det	fully	states	indef	goals	perfect
forward plan	det	fully	feats	indef	goals	perfect
regression plan	det	fully	feats	indef	goals	perfect
CSP planning	det	fully	feats	static	goals	perfect
SLS planning	det	fully	feats	static	goals	bounded
decision nets	stoch	partial	feats	finite	utility	perfect
MDPs	stoch	fully	states	infinite	utility	perfect
Dynamic DNs	stoch	fully	feats	infinite	utility	perfect

Decision-theoretic Planning

What should an agent do when

- it gets rewards (including punishments) and tries to maximize its rewards received

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- actions can be stochastic; the outcome of an action can't be fully predicted
- there is a model that specifies the (probabilistic) outcome of actions and the rewards
- the world is fully observable

Initial Assumptions

- flat or modular or hierarchical
- explicit states or features or individuals and relations
- static or finite stage or indefinite stage or infinite stage
- fully observable or partially observable
- deterministic or stochastic dynamics
- goals or complex preferences
- single agent or multiple agents
- knowledge is given or knowledge is learned
- perfect rationality or bounded rationality

Utility and time

- Would you prefer \$1000 today or \$1000 next year?
- What price would you pay now to have an eternity of happiness?
- How can you trade off pleasures today with pleasures in the future?

Utility and time

- How would you compare the following sequences of rewards (per week):

  A: \$1000000, \$0, \$0, \$0, \$0, \$0,...
  B: \$1000, \$1000, \$1000, \$1000, \$1000,...
  C: \$1000, \$0, \$0, \$0, \$0,...
  D: \$1, \$1, \$1, \$1, \$1,...
  E: \$1, \$2, \$3, \$4, \$5,...

Rewards and Values

Suppose the agent receives a sequence of rewards $r_1, r_2, r_3, r_4, \ldots$ in time. What utility should be assigned? "Return" or "value"

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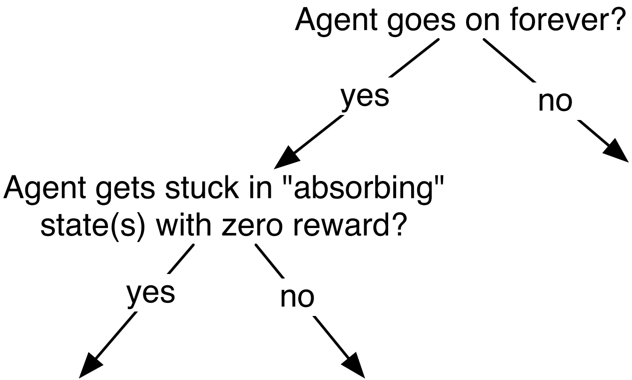
- total reward $V = \sum_{i=1}^{\infty} r_i$
- average reward $V = \lim_{n\to\infty} (r_1 + \cdots + r_n)/n$

Average vs Accumulated Rewards



Rewards and Values

Suppose the agent receives a sequence of rewards $r_1, r_2, r_3, r_4, \dots$ in time.

- discounted return $V = r_1 + \gamma r_2 + \gamma^2 r_3 + \gamma^3 r_4 + \cdots$
  γ is the discount factor $0 \leq \gamma \leq 1$.

Properties of the Discounted Rewards

- The discounted return for rewards $r_1, r_2, r_3, r_4, \ldots$ is

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World State

- The world state is the information such that if the agent knew the world state, no information about the past is relevant to the future. Markovian assumption.
- S_i is state at time i, and A_i is the action at time i:

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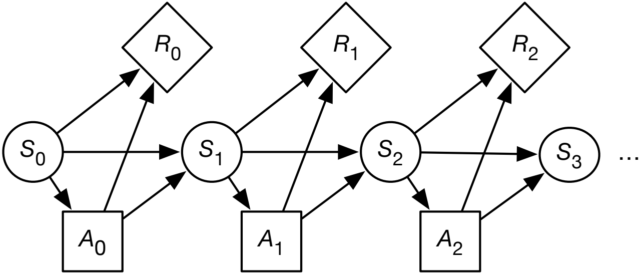
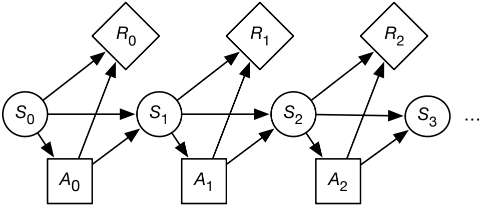
  $P(s' \mid s, a)$ is the probability that the agent will be in state s' immediately after doing action a in state s.
- The dynamics is stationary if the distribution is the same for each time point.

Decision Processes

- A Markov decision process augments a Markov chain with actions and rewards:



Markov Decision Processes



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- set S of states.
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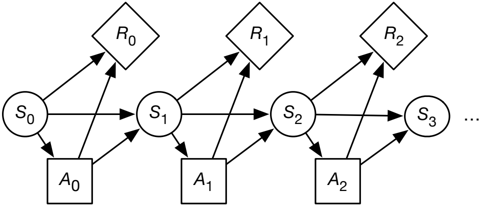

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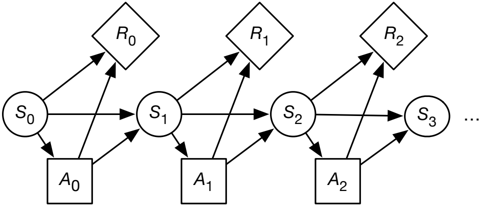

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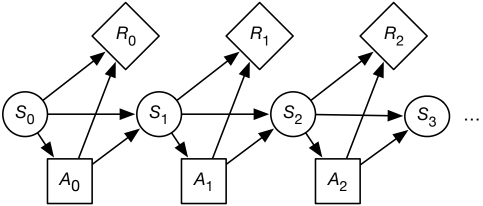

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- γ is discount factor.

Example: to party or relax?

Each week *Sam* has to decide whether to party or relax:

- States: $\{healthy, sick\}$
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healthy	relax	0.95
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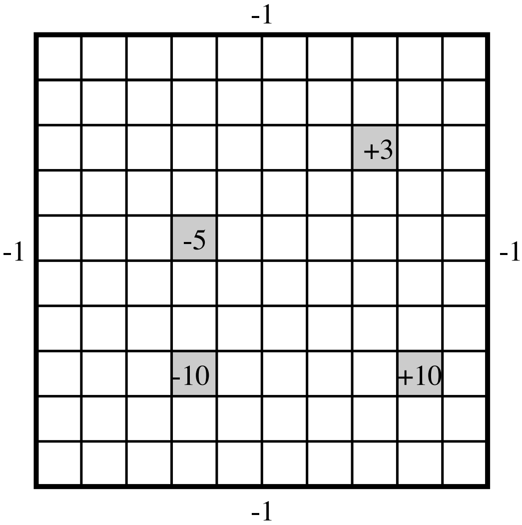
- Reward:

State	Action	Reward
healthy	relax	7
healthy	party	10
sick	relax	0
sick	party	2

Example: Simple Grid World



Grid World Model

- Actions: up, down, left, right.
- 100 states corresponding to the positions of the robot.
- Robot goes in the commanded direction with probability 0.7, and one of the other directions with probability 0.1.
- If it crashes into an outside wall, it remains in its current position and has a reward of -1.
- Four special rewarding states; the agent gets the reward when leaving.

Planning Horizons

The planning horizon is how far ahead the planner looks to make a decision.

- The robot gets flung to one of the corners at random after leaving a positive ($+10$ or $+3$) reward state.
  - the process never halts
  - infinite horizon

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The planning horizon is how far ahead the planner looks to make a decision.

- The robot gets flung to one of the corners at random after leaving a positive ($+10$ or $+3$) reward state.
  - the process never halts
  - infinite horizon
- The robot gets $+10$ or $+3$ in the state, then it stays there getting no reward. These are absorbing states.
  - The robot will eventually reach an absorbing state.
  - indefinite horizon

Information Availability

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[This lecture only considers FOMDPs.
POMDPS are much harder to solve.]

Policies

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  $$\pi : S \rightarrow A$$

  Given a state s, $\pi(s)$ specifies what action the agent who is following π will do.

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- An optimal policy is one with maximum expected discounted reward.
- For a fully-observable MDP with stationary dynamics and rewards with infinite or indefinite horizon, there is always an optimal stationary policy.

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How many stationary policies are there?

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How many stationary policies are there?
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For the grid world with 100 states and 4 actions,
how many stationary policies are there?

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$$\begin{aligned} V^\pi(s) &= Q^\pi(s,\pi(s)) \\ Q^\pi(s,a) &= R(s,a) + \gamma \sum_{s'} P(s' \mid a,s) V^\pi(s') \end{aligned}$$

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  The error reduces proportionally to $\dfrac{\gamma^k}{1-\gamma}$

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- This converges to the optimal value functions, if each state and action is visited infinitely often in the limit.
- It can either store $V[s]$ or $Q[s, a]$.

Asynchronous VI: storing $V[s]$

- Repeat forever:
  - Select state s
  - $V[s] \leftarrow$

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Policy Iteration

- Set π_0 arbitrarily, let $i = 0$
- Repeat:
  - evaluate $Q^{\pi_i}(s, a)$
  - let $\pi_{i+1}(s) = argmax_a Q^{\pi_i}(s, a)$
  - set $i = i + 1$
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Evaluating $Q^{\pi_i}(s, a)$ means finding a solution to a set of $|S| \times |A|$ linear equations with $|S| \times |A|$ unknowns.

It can also be approximated iteratively.

Modified Policy Iteration

Set $\pi[s]$ arbitrarily
Set $Q[s,a]$ arbitrarily
Repeat forever:

- Repeat for a while:
  - Select state s, action a
  - $Q[s,a] \leftarrow \sum_{s'} P(s' \mid s,a)\,(R(s,a,s') + \gamma Q[s', \pi[s']])$
- $\pi[s] \leftarrow argmax_a Q[s,a]$

Q, V, π, R

$$Q^*(s,a) = \sum_{s'} P(s' \mid a,s)\left(R(s,a,s') + \gamma V^*(s')\right)$$
$$= R(s,a) + \gamma \sum_{s'} P(s' \mid a,s) V^*(s')$$
$$V^*(s) = \max_a Q^*(s,a)$$
$$\pi^*(s) = argmax_a Q^*(s,a)$$

where

$$R(s,a) = \sum_{s'} P(s' \mid a,s) R(s,a,s')$$

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