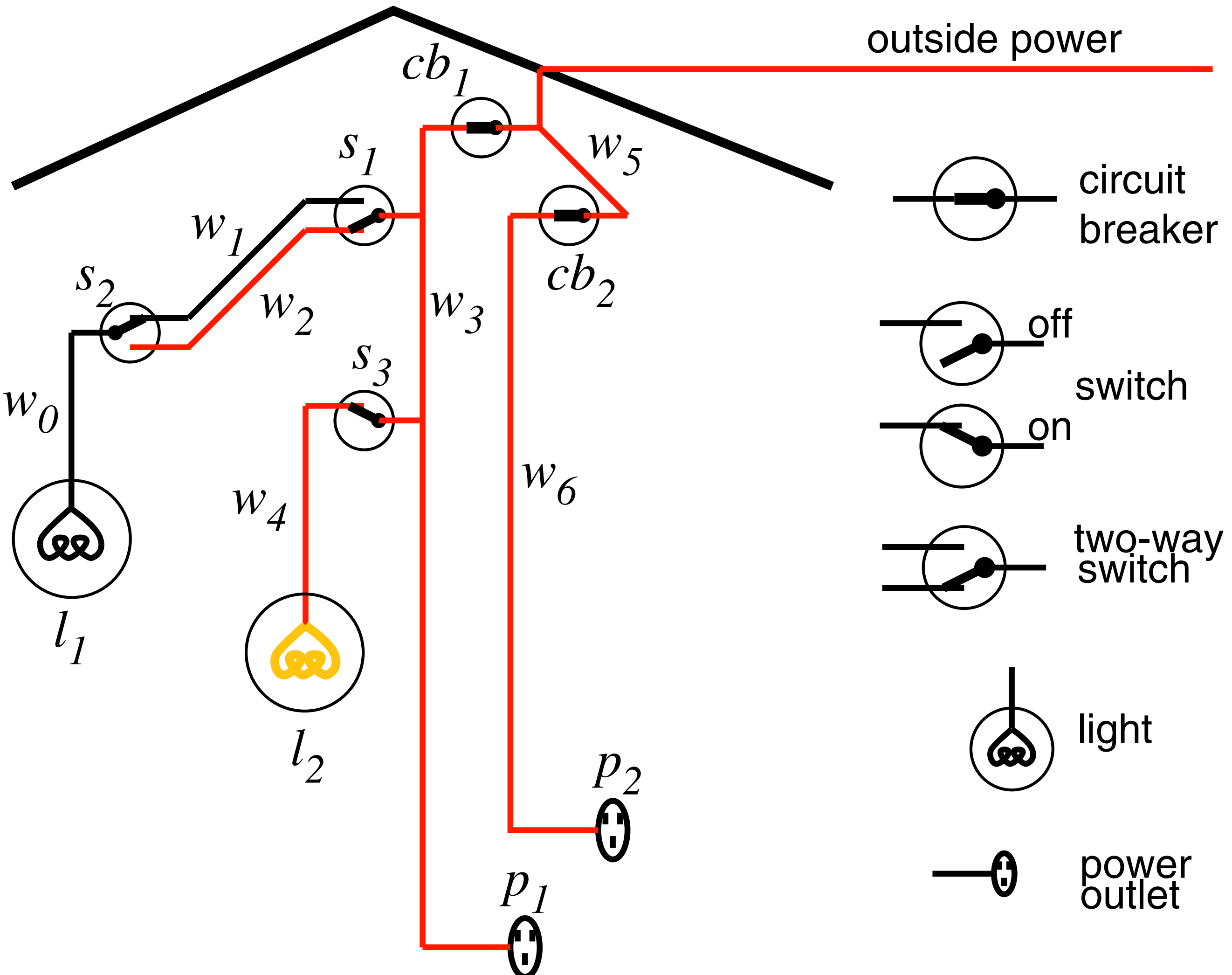
Conditional independence

Random variable X is independent of random variable Y given random variable Z if, for all $x_i \in dom(X)$, $y_j \in dom(Y)$, $y_k \in dom(Y)$ and $z_m \in dom(Z)$,

$$\begin{aligned} & P(X = x_i | Y = y_j \wedge Z = z_m) \\ = \quad & P(X = x_i | Y = y_k \wedge Z = z_m) \\ = \quad & P(X = x_i | Z = z_m). \end{aligned}$$

That is, knowledge of Y's value doesn't affect your belief in the value of X, given a value of Z.

Example domain (diagnostic assistant)

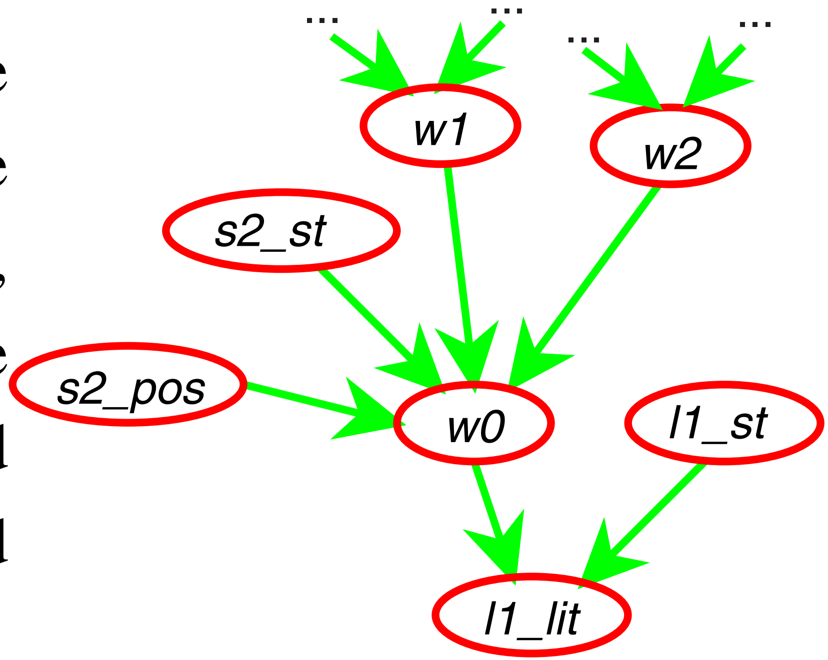


Examples of conditional independence

- The identity of the queen of Canada is independent of whether light $l1$ is lit given whether there is outside power.
- Whether there is someone in a room is independent of whether a light $l2$ is lit given the position of switch $s3$.
- Whether light $l1$ is lit is independent of the position of light switch $s2$ given whether there is power in wire w_0.
- Every other variable may be independent of whether light $l1$ is lit given whether there is power in wire w_0 and the status of light $l1$ (if it's *ok*, or if not, how it's broken).

Idea of belief networks

Whether $l1$ is lit ($l1\_lit$) depends only on the status of the light ($l1\_st$) and whether there is power in wire $w0$. Thus, $l1\_lit$ is independent of the other variables given $l1\_st$ and $w0$. In a belief network, $w0$ and $l1\_st$ are parents of $l1\_lit$.



Similarly, $w0$ depends only on whether there is power in $w1$, whether there is power in $w2$, the position of switch $s2$ ($s2\_pos$), and the status of switch $s2$ ($s2\_st$).

Belief networks

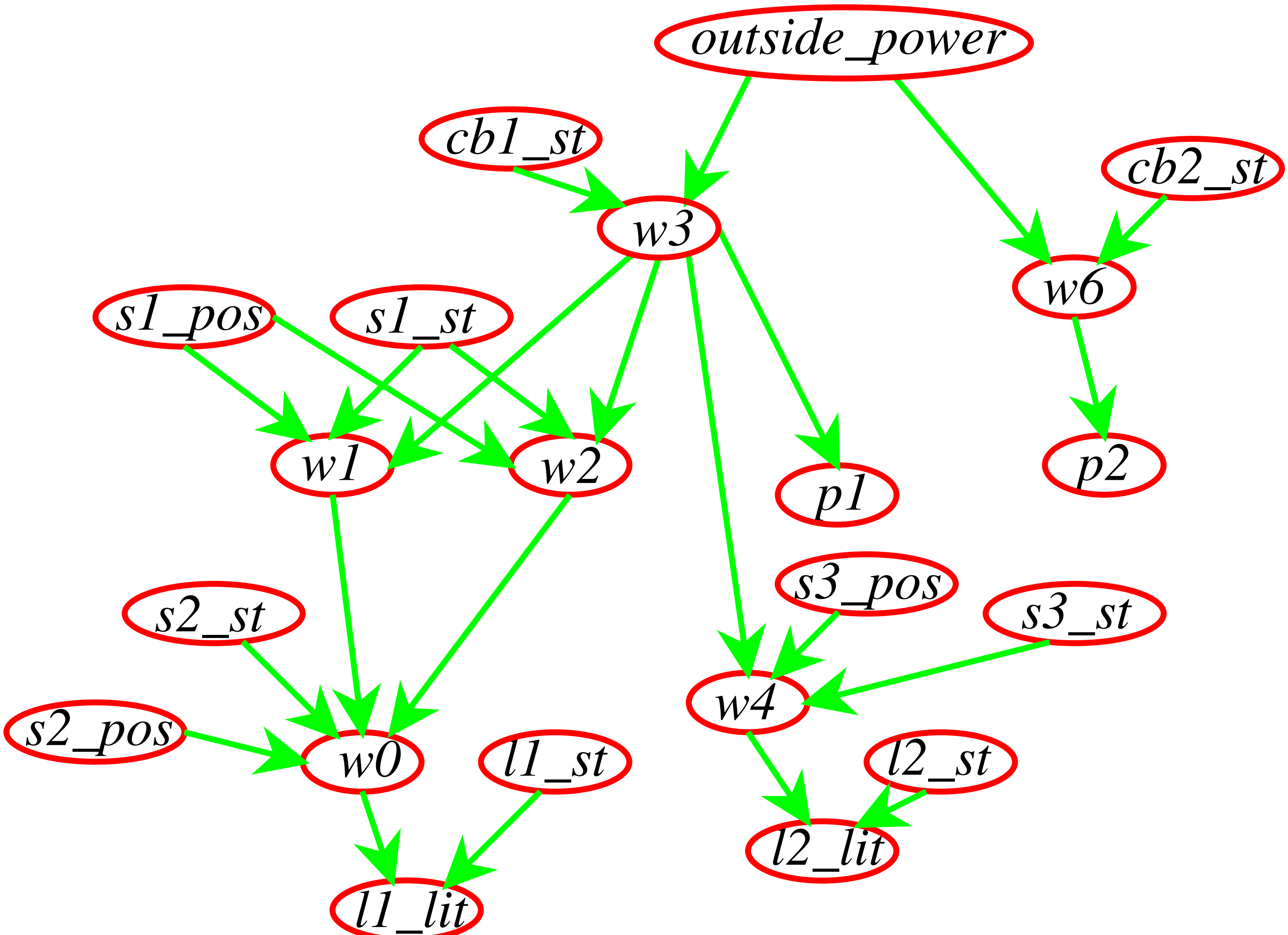
- Totally order the variables of interest: $X_1, \ldots, X_n$
- Theorem of probability theory (chain rule): $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i | X_1, \ldots, X_{i-1})$
- The parents π_{X_i} of X_i are those predecessors of X_i that render X_i independent of the other predecessors. That is, $\pi_{X_i} \subseteq X_1, \ldots, X_{i-1}$ and $P(X_i | \pi_{X_i}) = P(X_i | X_1, \ldots, X_{i-1})$
- So $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i | \pi_{X_i})$
- A belief network is a graph: the nodes are random variables; there is an arc from the parents of each node into that node.

Components of a belief network

A belief network consists of:

- a directed acyclic graph with nodes labeled with random variables
- a domain for each random variable
- a set of conditional probability tables for each variable given its parents (including prior probabilities for nodes with no parents).

Example belief network



Example belief network (continued)

The belief network also specifies:

- The domain of the variables:

  $w_0, \ldots, w_6$ have domain $\{live, dead\}$

  s_1_pos, s_2_pos, and s_3_pos have domain $\{up, down\}$

  s_1_st has $\{ok, upside\_down, short, intermittent, broken\}$.

- Conditional probabilities, including:

  $P(w_1 = live | s_1\_pos = up \wedge s_1\_st = ok \wedge w_3 = live)$

  $P(w_1 = live | s_1\_pos = up \wedge s_1\_st = ok \wedge w_3 = dead)$

  $P(s_1\_pos = up)$

  $P(s_1\_st = upside\_down)$

Belief network summary

- A belief network is automatically acyclic by construction.
- A belief network is a directed acyclic graph (DAG) where nodes are random variables.
- The parents of a node n are those variables on which n directly depends.
- A belief network is a graphical representation of dependence and independence:
  - A variable is independent of its non-descendants given its parents.

Constructing belief networks

To represent a domain in a belief network, you need to consider:

- What are the relevant variables?
- What values should these variables take?
- What is the relationship between them? This should be expressed in terms of local influence.
- How does the value of one variable depend on the variables that locally influence it (its parents)? This is expressed in terms of the conditional probability tables.

Using belief networks

The power network can be used in a number of ways:

- Conditioning on the status of the switches and circuit breakers, whether there is outside power and the position of the switches, you can simulate the lighting.
- Given values for the switches, the outside power, and whether the lights are lit, you can determine the posterior probability that each switch or circuit breaker is *ok* or not.
- Given some switch positions and some outputs and some intermediate values, you can determine the probability of any other variable in the network.

Inferring conditional probabilities (I)

If observation e doesn't involve X or a descendent of X:

$$
\begin{aligned}
& P(X|e) \\
&= \sum_{v \in dom(\pi_X)} P(X \wedge \pi_X = v|e) \\
&= \sum_{v \in dom(\pi_X)} P(X|\pi_X = v \wedge e)P(\pi_X = v|e) \\
&= \sum_{v \in dom(\pi_X)} P(X|\pi_X = v)P(\pi_X = v|e)
\end{aligned}
$$

Inferring conditional probabilities (II)

If you observe $Y = y \wedge e$ where Y is a descendent of X:

$$P(X|Y = y \wedge e) \quad = \quad \frac{P(Y = y|X \wedge e)P(X|e)}{P(Y = y|e)}$$