

# On the Complexity of Reconfiguration Problems

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## Abstract

Reconfiguration problems arise when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are also feasible. We demonstrate that a host of reconfiguration problems derived from NP-complete problems are PSPACE-complete, while some are also NP-hard to approximate. In contrast, several reconfiguration versions of problems in P are solvable in polynomial time.

*Keywords:* approximation, graph algorithm, PSPACE-complete, reachability on solution space

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## 1. Introduction

Consider the bipartite graph with weighted vertices in Figure 1(a) (both solid and dotted edges). It models a situation in which power stations with fixed capacity (the square vertices) provide power to customers with fixed demand (the round vertices). It can be seen as a feasible solution of a particular instance of a search problem which we may call the **POWER SUPPLY** problem [8, 10]: Given a bipartite graph  $G = (U, V, E)$  with weights on the vertices, can  $G$  be partitioned into subtrees, each of which contains exactly one vertex from  $U$ , such that the sum of the demands of the  $V$  vertices (customers) in each subtree is no more than the capacity of the  $U$  vertex (power station) in it?

But suppose now that we are given *two* feasible solutions of this instance (the leftmost and rightmost ones in Figure 1), and we are asked: Can the solution on the left be transformed into the solution on the right *by moving only one customer at a time, and always remaining feasible*? This problem, which we call the **POWER SUPPLY RECONFIGURATION** problem, is an exemplar of the kind of problems we discuss in this paper. (In this particular instance, it turns out that the answer is “yes”; see Figure 1.) As one may have expected, the most basic reconfiguration problem is the **SATISFIABILITY RECONFIGURATION** problem: Given a CNF formula and two satisfying truth assignments  $s_0$  and  $s_1$ , are these connected in the subgraph of the hypercube induced by the satisfying truth assignments? This problem has been shown PSPACE-complete [3].

In more generality, *reconfiguration problems* have the following structure: Fix a search problem  $\mathcal{S}$  (a polynomial-time algorithm which, on instance  $I$  and candidate solution  $y$  of length polynomial in that of  $I$ , determines whether  $y$  is a feasible solution of  $I$ ); and fix a polynomially-testable symmetric *adjacency relation*  $A$  on the set of feasible solutions, that is, a polynomial-time algorithm such that, given an instance  $I$  of  $\mathcal{S}$  and two feasible solutions  $y'$  and  $y''$  of  $I$ , it determines whether  $y'$  and  $y''$  are adjacent. (In almost all problems discussed in this paper, the feasible solutions can be considered as sets of elements, and two solutions are adjacent if their symmetric difference has size

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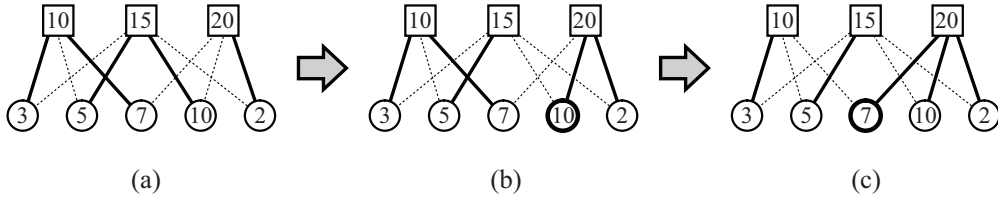


Figure 1: A sequence of feasible solutions for the POWER SUPPLY problem.

1.) The RECONFIGURATION PROBLEM FOR  $\mathcal{S}$  AND  $A$  is the following computational problem: Given instance  $I$  of  $\mathcal{S}$  and two feasible solutions  $y_0$  and  $y_t$  of  $I$ , is there a sequence of feasible solutions  $y_0, y_1, \dots, y_t$  of  $I$  such that  $y_{i-1}$  and  $y_i$  are adjacent for  $i = 1, 2, \dots, t$ ?

Reconfiguration problems can also arise from optimization problems, if one turns the optimization problem into a search problem by giving a threshold. For example, the CLIQUE RECONFIGURATION problem is the following: Given a graph  $G$ , an integer  $k$ , and two cliques  $C_0$  and  $C_t$  of  $G$ , both of size at least  $k$ , is there a way to transform  $C_0$  into  $C_t$  via cliques, each of which results from the previous one by adding or subtracting a single node of  $G$ , without ever going through a clique of size less than  $k - 1$ ?

Reconfiguration problems are useful and entertaining, have been coming up in recent literature [1, 3, 6, 9], and are interesting for a variety of reasons. First, they may reflect, as in the POWER SUPPLY RECONFIGURATION problem above, a situation where we actually seek to implement such a sequence of elementary changes in order to transform the current configuration to a more desirable one, in a context in which intermediate steps must also be fully feasible, and only restricted changes can occur — in our example, no two customers can change providers simultaneously, and we certainly do not wish customers to be without power. In a complex, dynamic environment in which changing circumstances affect the feasible solution of choice, determining whether such adaptation is possible may be crucial. Reconfiguration problems also model questions of *evolvability*: Can genotype  $y_0$  evolve into genotype  $y_t$  via individual mutations which are each of adequate fitness? Here a genotype is considered feasible if its fitness is above a threshold, and two genotypes are considered adjacent if one is a simple mutation of the other. Finally, reconfiguration versions of constraint satisfaction problems (the first kind studied in the literature [3]) yield insights into the structure of the solution space, which may help in understanding heuristics, such as survey propagation, whose performance depends crucially on connectivity and other properties of the solution space.

In this paper, we embark on a systematic investigation of the complexity of reconfiguration problems. Our main focus is showing that a host of reconfiguration problems (including all those mentioned above and many more) are PSPACE-complete. The proof for the POWER SUPPLY RECONFIGURATION problem and those for certain other problems are explained in Section 2. We then point out in Section 3 that certain reconfiguration problems arising from problems in P (such as MINIMUM SPANNING TREE and MATCHING) can be solved in polynomial time. In Section 4 we show certain approximability and inapproximability results for reconfiguration problems. An extended abstract of the paper has been presented in [7].

## 2. PSPACE-completeness

In this section we show that a host of reconfiguration problems are PSPACE-complete. In Section 2.1 we first give a proof for the POWER SUPPLY RECONFIGURATION problem, and in Section 2.2 we then give proof sketches for certain other reconfiguration problems.

### 2.1. POWER SUPPLY RECONFIGURATION

The POWER SUPPLY RECONFIGURATION problem was defined informally in the Introduction. An instance is given in terms of a bipartite graph  $G = (U, V, E)$ , where each vertex in  $U$  is called a *supply vertex* and each vertex in  $V$  is called a *demand vertex*. Each supply vertex  $u \in U$  is assigned a positive integer  $\text{sup}(u)$ , called the *supply of  $u$* , while each demand vertex  $v \in V$  is assigned a positive integer  $\text{dem}(v)$ , called the *demand of  $v$* . We wish to partition  $G$  into subtrees, by deleting edges from  $G$ , such that each subtree  $T$  has exactly one supply vertex whose supply is at least

the sum of demands of all demand vertices in  $T$ . We call an assignment  $f : V \rightarrow U$  a *configuration of  $G$*  if there is an edge  $(v, f(v)) \in E$  for each demand vertex  $v \in V$ . A configuration  $f$  of  $G$  is *feasible* if the following condition holds: for each supply vertex  $u \in U$ ,

$$\text{sup}(u) \geq \sum \{\text{dem}(v) \mid v \in V \text{ such that } f(v) = u\}.$$

The *adjacency relation* on the set of feasible configurations is defined as follows: two feasible configurations  $f$  and  $f'$  are *adjacent* if  $|\{v \in V : f(v) \neq f'(v)\}| = 1$ , that is,  $f'$  can be obtained from  $f$  by changing the assignment of a single demand vertex. Given a bipartite graph  $G = (U, V, E)$  and two feasible configurations  $f_0$  and  $f_t$  of  $G$ , the **POWER SUPPLY RECONFIGURATION** problem is to determine whether there is a sequence of feasible configurations  $f_0, f_1, \dots, f_t$  of  $G$  such that  $f_{i-1}$  and  $f_i$  are adjacent for  $i = 1, 2, \dots, t$ . Note that **POWER SUPPLY RECONFIGURATION**, as well as any reconfiguration problem defined in this paper, does not ask an actual reconfiguration sequence.

Figure 1 illustrates three feasible configurations of a bipartite graph  $G$ , where each supply vertex is drawn as a square, each demand vertex as a circle, and the supply or demand is written inside. Figure 1 also illustrates an example of a transformation from the feasible configuration in Figure 1(a) to one in Figure 1(c), where the demand vertex whose assignment was changed from the previous one is depicted by a thick circle.

We have the following theorem.

**Theorem 1.** **POWER SUPPLY RECONFIGURATION is PSPACE-complete.**

**PROOF.** It is easy to see that this problem, as well as any reconfiguration version of a problem  $S$  in NP, can be solved in polynomial space, as follows. Since  $S$  is in NP, we can enumerate all feasible solutions of  $S$  in nondeterministic polynomial time. Since  $\text{NP} \subseteq \text{PSPACE}$  [11, p. 148], this enumeration can be done in PSPACE. We then nondeterministically traverse the solutions that are adjacent with the current solution. (By the assumption, the adjacency can be checked in polynomial time for each enumerated solution.) Savitch's Theorem [12] says that this NPSPACE algorithm can be converted into a PSPACE algorithm.

We give a polynomial-time reduction from the **SATISFIABILITY RECONFIGURATION** problem to this problem. In that problem we are given a Boolean formula  $\phi$  in conjunctive normal form, say with  $n$  variables  $x_1, x_2, \dots, x_n$  and  $m$  clauses  $C_1, C_2, \dots, C_m$ , and two satisfying truth assignments  $\mathbf{s}_0$  and  $\mathbf{s}_t$  of  $\phi$ . Then, we are asked whether there is a sequence of satisfying truth assignments, starting with  $\mathbf{s}_0$  and ending in  $\mathbf{s}_t$ , and each differing from the previous one in only one variable. This problem is known to be PSPACE-complete [3]. One may assume without loss of generality that the formula  $\phi$  has no clause which contains both positive and negative literals of the same variable. Let  $c$  be the maximum number of occurrences of a literal in the clauses, and hence each literal appears in at most  $c$  clauses in  $\phi$ .

Given such an instance of **SATISFIABILITY RECONFIGURATION**, we construct an instance of **POWER SUPPLY RECONFIGURATION** as follows. We first make a *variable gadget*  $G_{x_i}$  for each variable  $x_i$ ,  $1 \leq i \leq n$ ;  $G_{x_i}$  is a binary tree with three vertices as illustrated in Figure 2(a); the root  $F_i$  is a demand vertex of demand  $c$ , and the two leaves  $x_i$  and  $\bar{x}_i$  are supply vertices of supply  $c$ . Then the bipartite graph  $G_\phi$  corresponding to the formula  $\phi$  is constructed as follows. For

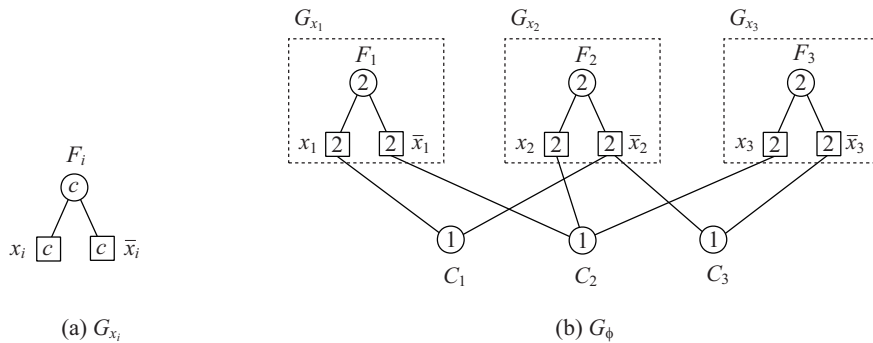


Figure 2: (a) Variable gadget  $G_{x_i}$ , and (b) bipartite graph  $G_\phi$  corresponding to a Boolean formula  $\phi$  with three clauses  $C_1 = (x_1 \vee \bar{x}_2)$ ,  $C_2 = (\bar{x}_1 \vee x_2 \vee x_3)$  and  $C_3 = (\bar{x}_2 \vee \bar{x}_3)$ , and hence  $c = 2$ .

each variable  $x_i$ ,  $1 \leq i \leq n$ , add the variable gadget  $G_{x_i}$  to the graph; and, for each clause  $C_j$ ,  $1 \leq j \leq m$ , add a demand vertex  $C_j$  of demand 1 to the graph. Finally, join a supply vertex  $x_i$  (or  $\bar{x}_i$ ) in  $G_{x_i}$ ,  $1 \leq i \leq n$ , with the clause demand vertex  $C_j$ ,  $1 \leq j \leq m$ , if and only if the literal  $x_i$  (respectively,  $\bar{x}_i$ ) is in the clause  $C_j$ . (See Figure 2(b) as an example.) Clearly,  $G_\phi$  is a bipartite graph.

Consider a feasible configuration of  $G_\phi$ . Then each demand vertex  $F_i$ ,  $1 \leq i \leq n$ , must be assigned to one of  $x_i$  and  $\bar{x}_i$ ; a literal is considered false if  $F_i$  is assigned to its corresponding supply vertex. Notice that, since supply vertices have supply  $c$  and the  $F_i$ 's have demand  $c$ , a false-literal supply vertex cannot provide power to any of the other demand vertices. Hence, all clause demand vertices  $C_j$ ,  $1 \leq j \leq m$ , must be assigned to true-literal supply vertices that occur in them. Since each literal  $x_i$  (or  $\bar{x}_i$ ),  $1 \leq i \leq n$ , appears in at most  $c$  clauses in  $\phi$ , the corresponding supply vertex  $x_i$  (respectively,  $\bar{x}_i$ ) in  $G_{x_i}$  can provide power to all clause demand vertices  $C_j$  whose corresponding clauses have  $x_i$  (respectively,  $\bar{x}_i$ ).

To complete the reduction, we now create two feasible configurations  $f_0$  and  $f_t$  of  $G_\phi$  corresponding to the satisfying truth assignments  $\mathbf{s}_0$  and  $\mathbf{s}_t$  of  $\phi$ , respectively. Each demand vertex  $F_i$ ,  $1 \leq i \leq n$ , is assigned to the supply vertex whose corresponding literal is false, while each clause demand vertex  $C_j$ ,  $1 \leq j \leq m$ , is assigned to an arbitrary true-literal supply vertex adjacent to  $C_j$ . Since  $\mathbf{s}_0$  and  $\mathbf{s}_t$  are satisfying truth assignments of  $\phi$ , both  $f_0$  and  $f_t$  are feasible configurations of  $G_\phi$ . This completes the construction of the corresponding instance of the POWER SUPPLY RECONFIGURATION problem.

We know that a feasible configuration of  $G_\phi$  corresponds to a satisfying truth assignment of  $\phi$  plus an assignment of each clause to a true literal. It is easy to see that this correspondence goes backwards: every satisfying truth assignment of  $\phi$  can be mapped to at least one (in general, to exponentially many) feasible configurations of  $G_\phi$ .

How about adjacent configurations — defined to be configurations differing in the assignment of just one demand vertex? One can easily observe that there are only two types of reassignments to go from a feasible configuration of  $G_\phi$  to an adjacent one, as follows:

- (1) One could change the assignment of a demand vertex  $F_i$  from  $x_i$  to  $\bar{x}_i$ , or vice versa, if no clause demand vertex is currently assigned to supply vertices  $x_i$  or  $\bar{x}_i$ .
- (2) Alternatively, if a clause demand vertex  $C_j$  is adjacent to more than one true-literal supply vertex, then one could change the assignment of  $C_j$  from the current one to another.

Therefore, any sequence of adjacent feasible configurations of  $G_\phi$  can be broken down to subsequences, intermittently with a reassignment of type (1) above; in each subsequence, every two adjacent configurations can go from one to another via a reassignment of type (2) above. Therefore, all feasible configurations in each subsequence correspond to the same satisfying truth assignment of  $\phi$ , while any two consecutive subsequences correspond to adjacent satisfying truth assignments (namely, differing in only one variable).

Conversely, for given any sequence of adjacent satisfying truth assignments of  $\phi$ , there is a corresponding sequence of adjacent feasible configurations of  $G_\phi$ , obtained as follows: Consider a flip of a variable  $x_i$  from true to false. (A flip of  $x_i$  from false to true is similar.) Then we wish to change the assignment of the demand vertex  $F_i$  from the supply vertex  $\bar{x}_i$  to  $x_i$ . (Remember that the literal to which  $F_i$  is assigned is considered false.) We first change the assignments of all clause demand vertices, which are currently assigned to  $x_i$ , to another true-literal supply vertex: since we are about to flip the variable  $x_i$  and we know that the truth assignment of  $\phi$  after the flip will be also satisfiable, there must be a “second” true-literal supply vertex for every clause demand vertex currently assigned to  $x_i$ . After all such reassignments, we finally change the assignment of  $F_i$  from  $\bar{x}_i$  to  $x_i$ .

It is now easy to see that there is a sequence of adjacent satisfying truth assignments of  $\phi$  from  $\mathbf{s}_0$  to  $\mathbf{s}_t$  if and only if there is a sequence of adjacent feasible configurations of  $G_\phi$  from  $f_0$  to  $f_t$ . This completes the proof of Theorem 1.

□

## 2.2. Other intractable reconfiguration problems

There is a wealth of reconfiguration versions of NP-complete problems which can be shown PSPACE-complete via extensions, often quite sophisticated, of the original NP-completeness proofs; in this subsection we only sample the realm of possibilities.

We have already defined the CLIQUE RECONFIGURATION problem in the Introduction as an example of a general scheme whereby any optimization problem can be transformed into a reconfiguration problem by giving a threshold (upper bound for minimization problems, lower bound for maximization problems) for the allowed values of the objective

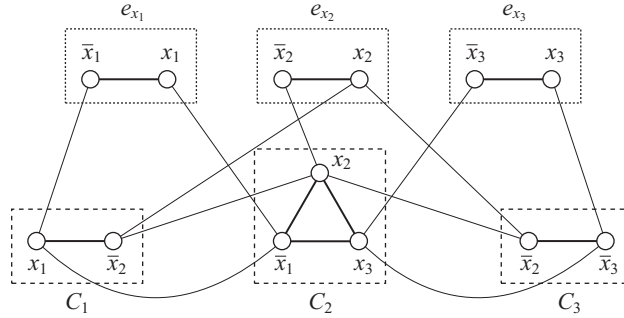


Figure 3: Graph  $\rho(\phi)$  corresponding to a 3SAT formula  $\phi$  with three clauses  $C_1 = (x_1 \vee \bar{x}_2)$ ,  $C_2 = (\bar{x}_1 \vee x_2 \vee x_3)$  and  $C_3 = (\bar{x}_2 \vee \bar{x}_3)$ .

function of the intermediate feasible solutions; the INDEPENDENT SET RECONFIGURATION and VERTEX COVER RECONFIGURATION problems are defined similarly. In the INTEGER PROGRAMMING RECONFIGURATION problem, we are given a 0-1 linear program seeking to maximize  $cx$  subject to  $Ax \leq b$ , and we consider two solutions adjacent if they only differ in one variable.

**Theorem 2.** *The following problems are PSPACE-complete: INDEPENDENT SET RECONFIGURATION, CLIQUE RECONFIGURATION, VERTEX COVER RECONFIGURATION, SET COVER RECONFIGURATION, INTEGER PROGRAMMING RECONFIGURATION.*

*Proof sketch.* We sketch a proof for the INDEPENDENT SET RECONFIGURATION problem. The reduction can be obtained by extending the well-known reduction from the 3SAT problem to the INDEPENDENT SET problem [11]. We construct a graph  $\rho(\phi)$  from a given 3SAT formula  $\phi$  with  $n$  variables and  $m$  clauses, as follows. (As in the proof of Theorem 1, we may assume without loss of generality that the formula  $\phi$  has no clause which contains both positive and negative literals of the same variable.) For each variable  $x$  in  $\phi$ , we add an edge  $e_x$  to the graph; its two endpoints are labeled  $x$  and  $\bar{x}$ . Then, for each clause  $C$  in  $\phi$ , we add a clique of size  $|C|$  to the graph; each node in the clique corresponds to a literal in the clause  $C$ . Finally, we add an edge between two nodes in different components if and only if the nodes correspond to opposite literals. (See Figure 3 as an example.) Then, it is easy to see that  $\rho(\phi)$  has a maximum independent set of size  $k = n + m$  if and only if  $\phi$  is satisfiable;  $n$  nodes are chosen from the endpoints of  $n$  edges corresponding to the variables; a literal is considered true if the corresponding endpoint is chosen. Consider all independent sets of size  $k = n + m$  in  $\rho(\phi)$ ; they can be partitioned into subclasses of the form  $\rho(\mathbf{s})$  corresponding to the satisfying truth assignments  $\mathbf{s}$  of  $\phi$  (the various independent sets in the subclass  $\rho(\mathbf{s})$  correspond to the different possible ways to satisfy each clause by  $\mathbf{s}$ ). It is easy to see that all independent sets in  $\rho(\mathbf{s})$  are connected via intermediate independent sets of size at least  $k - 1$ . Therefore, by similar arguments in the proof of Theorem 1, it is easy to observe that deciding whether two independent sets of size  $k$  in  $\rho(\phi)$  can be transformed into one another via intermediate independent sets of size at least  $k - 1$  is PSPACE-complete.

It is easy to see that a subset  $I \subseteq V$  of vertices in a graph  $G = (V, E)$  is an independent set of  $G$  if and only if  $I$  induces a clique in the complement of  $G$ . Also,  $I$  is an independent set of  $G$  if and only if  $V \setminus I$  is a vertex cover of  $G$  [2, Lemma 3.1]. Thus, the result for INDEPENDENT SET RECONFIGURATION yields those for CLIQUE RECONFIGURATION and VERTEX COVER RECONFIGURATION. Then, the result for SET COVER RECONFIGURATION is immediate since it is a generalization of VERTEX COVER RECONFIGURATION. INTEGER PROGRAMMING RECONFIGURATION generalizes CLIQUE RECONFIGURATION via the well-known integer program for CLIQUE.  $\square$

One might compare our INDEPENDENT SET RECONFIGURATION problem with the SLIDING TOKEN problem, which is also known to be PSPACE-complete [6]. A *Token configuration*  $T$  of a graph  $G$  is an independent set of  $G$  such that a Token is placed on each vertex in  $T$ . In the SLIDING TOKEN problem, we are given a graph  $G$  and two Token configurations (independent sets)  $T_0$  and  $T_t$  of  $G$ , both have the same number of Tokens, and we are asked whether there is a sequence of Token configurations of  $G$ , starting with  $T_0$  and ending in  $T_t$ , and each resulting from the previous one by sliding only one Token from one vertex to an adjacent vertex. Therefore, the two problems have slightly different adjacency relations: in our INDEPENDENT SET RECONFIGURATION problem, a Token can “jump” from one vertex to any other vertex



Figure 4: (a) Initial independent set (Token configuration) and (b) target independent set (Token configuration).

if it results in an independent set of  $G$ ; while, in the SLIDING TOKEN problem, we can just slide a Token *along an edge* of  $G$ . Consider the instance in Figure 4, where the vertices in independent sets (or Token configurations) are colored with black. Then, this is an Yes-instance for INDEPENDENT SET RECONFIGURATION with  $k = 2$ , but a No-instance for SLIDING TOKEN. However, the PSPACE-completeness proof for SLIDING TOKEN by [6] indeed works to prove our result for INDEPENDENT SET RECONFIGURATION. Then, we can prove that INDEPENDENT SET RECONFIGURATION and VERTEX COVER RECONFIGURATION remain PSPACE-complete even for planar graphs of maximum degree 3.

### 3. Reconfiguration Problems in P

Reconfiguration problems arise in relation to problems in P as well. For example, in the MINIMUM SPANNING TREE RECONFIGURATION problem, we are given an edge-weighted graph  $G$ , a threshold  $k$ , and two spanning trees of  $G$ , both of weight at most  $k$ , and wish to transform one tree into another via edge exchanges, without ever getting into a tree with weight  $> k$ . The MATCHING RECONFIGURATION problem is defined similarly (the formal definition will be given later). We show in this section that both problems can be solved in polynomial time.

The result for the MINIMUM SPANNING TREE RECONFIGURATION problem can be obtained from the following more general proposition.

**Proposition 1.** *Let  $\mathbf{M} = (S, \mathcal{B})$  be a matroid, and let  $w : S \rightarrow \mathbb{R}$  be a weight function on  $S$ . Let  $B_0$  and  $B_t$  be two bases in  $\mathcal{B}$  such that  $\max\{w(B_0), w(B_t)\} \leq k$ . Then, there always exists a sequence of  $|B_0 \setminus B_t|$  ( $= |B_t \setminus B_0|$ ) exchanges that transforms one into the other, without ever exceeding weight  $k$ , and maintaining a base at each step.*

PROOF. Since the adjacency relation is symmetric, we may assume without loss of generality that  $w(B_0) \leq w(B_t)$ . Since  $B_0$  and  $B_t$  are bases,  $|B_0| = |B_t|$  and hence let  $m = |B_0 \setminus B_t| = |B_t \setminus B_0|$ . The proposition trivially holds if  $m = 1$ . Therefore, by applying induction, it suffices to prove the following claim: there exists  $y \in B_0 \setminus B_t$  and  $z \in B_t \setminus B_0$  such that  $B_0 - y + z$  is a base in  $\mathcal{B}$  and  $w(B_0 - y + z) \leq w(B_t)$ , where we use the shorthand notation  $B - y + z = (B \setminus \{y\}) \cup \{z\}$ . Observe that  $|(B_0 - y + z) \setminus B_t| = |B_t \setminus (B_0 - y + z)| = m - 1$  and  $w(B_0 - y + z) \leq k$  if the claim holds.

By Brualdi's exchange property [14, Corollary 39.12a], we can always write  $B_0 \setminus B_t = \{y_1, y_2, \dots, y_m\}$  and  $B_t \setminus B_0 = \{z_1, z_2, \dots, z_m\}$  such that  $B_0 - y_i + z_i$  is a base in  $\mathcal{B}$  for every index  $i$ ,  $1 \leq i \leq m$ . Suppose for a contradiction that

$$w(B_0 - y_i + z_i) = w(B_0) - w(y_i) + w(z_i) > w(B_t)$$

for all indices  $i = 1, 2, \dots, m$ . Then,  $w(z_i) - w(y_i) > w(B_t) - w(B_0)$ , and hence

$$\begin{aligned} w(B_t) &= w(B_0) + \sum_{1 \leq i \leq m} (w(z_i) - w(y_i)) \\ &> w(B_0) + \sum_{1 \leq i \leq m} (w(B_t) - w(B_0)) \\ &= w(B_0) + m \cdot (w(B_t) - w(B_0)) \\ &\geq w(B_0) + (w(B_t) - w(B_0)) \\ &= w(B_t), \end{aligned}$$

a contradiction. Therefore, there must exist some index  $i$  such that  $w(B_0 - y_i + z_i) \leq w(B_t)$ , as required.  $\square$

In the MATCHING RECONFIGURATION problem, we are given an unweighted graph  $G$ , a threshold  $k$ , and two matchings  $M_0$  and  $M_t$  of  $G$ , both of size at least  $k$ , and we are asked whether there is a sequence of matchings of  $G$ , starting with  $M_0$  and ending in  $M_t$ , and each resulting from the previous one by either addition or deletion of a single edge in  $G$ , without ever going through a matching of size less than  $k - 1$ .

**Proposition 2.** MATCHING RECONFIGURATION can be solved in polynomial time.

In the remainder of this section, as a proof of Proposition 2, we give a polynomial-time algorithm which solves MATCHING RECONFIGURATION.

We first introduce some terms. Let  $M$  be a matching of a graph  $G$ . A vertex  $v$  is called  $M$ -covered if  $v$  is incident with an edge in  $M$ ; otherwise,  $v$  is called  $M$ -exposed. A path (or a cycle) of  $G$  is called  $M$ -alternating if the edges along the path (respectively, along the cycle) belong alternatively to  $M$  and not to  $M$ . An  $M$ -augmenting path is an  $M$ -alternating path whose endpoints are both  $M$ -exposed. For two matchings  $M$  and  $N$  of  $G$ , we denote by  $M \Delta N$  the symmetric difference of  $M$  and  $N$ , that is,  $M \Delta N = (M \setminus N) \cup (N \setminus M)$ . A path (or a cycle) of  $G$  is called  $(M, N)$ -alternating if the edges along the path (respectively, along the cycle) belong alternatively to  $M$  and to  $N$ . The length of a path  $P$  in a graph is defined as the number of edges in  $P$ .

We may assume without loss of generality that  $|M_0| \leq |M_t|$ . Consider the subgraph  $H$  of  $G$  induced by all edges in  $M_0 \Delta M_t$ . Then, since  $M_0$  and  $M_t$  are both matchings of  $G$ , each vertex in  $H$  has degree at most 2. Therefore,  $H$  consists of single edges,  $(M_0, M_t)$ -alternating paths and  $(M_0, M_t)$ -alternating cycles. The greedy algorithm for transforming  $M_0$  into  $M_t$  is the following. Divide the components of  $H$  into the following four categories:

- (1) single edges of  $M_t \setminus M_0$ ;
- (2)  $(M_0, M_t)$ -alternating paths which start and end with edges of  $M_t \setminus M_0$ ;
- (3)  $(M_0, M_t)$ -alternating cycles; and
- (4) all the rest.

In this category order, transform  $M_0$  into  $M_t$  by repeatedly adding edges of  $M_t \setminus M_0$  and deleting edges of  $M_0 \setminus M_t$  along each component of  $H$ . It is easy to see that intermediate matchings have size at least  $|M_0| - 1 (\geq k - 1)$  for exchanging edges in Category (2). Therefore, we can always exchange the edges in Categories (1) and (2). Moreover, since each component in Categories (1) and (2) is an  $M_0$ -augmenting path, the matching  $M$  obtained by exchanging all edges in Categories (1) and (2) has size at least  $|M_t| (\geq |M_0|)$ . We then exchange the edges in an  $(M_0, M_t)$ -alternating cycle  $C$  in Category (3), as follows: we first delete an arbitrary edge in  $C \cap M_0$ , and then exchange the remaining edges along the obtained  $(M_0, M_t)$ -alternating path. Therefore, intermediate matchings have size at least  $|M| - 2 \geq |M_t| - 2$  for exchanging the edges in Category (3). Similarly, the edges in Category (4) can be exchanged without ever going through a matching of size less than  $|M_t| - 2$ .

We show that the greedy algorithm correctly solves MATCHING RECONFIGURATION in polynomial time.

**Case (a):**  $|M_t| \geq k + 1$ .

In this case, since the greedy algorithm transforms  $M_0$  into  $M_t$  without ever going through a matching of size less than  $|M_t| - 2$ , all the intermediate matchings have size at least  $|M_t| - 2 \geq k - 1$ , as required.

**Case (b):**  $|M_t| = k$ , and  $M_t$  is *not* a maximum matching of  $G$ .

In this case, we first transform  $M_t$  into a matching  $M'_t$  of size  $k + 1$  along an arbitrary  $M_t$ -augmenting path  $P$ ; clearly, the intermediate matchings for exchanging the edges in  $P$  have size at least  $|M_t| - 1 = k - 1$ . Then, the greedy algorithm can transform  $M_0$  into  $M'_t$  so that all intermediate matchings are of size  $\geq k - 1$ . Finally, we transform  $M'_t$  into  $M_t$  along the path  $P$ . In this way, a desired sequence always exists for this case.

**Case (c):**  $|M_t| = k$ , and  $M_t$  is a maximum matching of  $G$ .

Since  $k \leq |M_0| \leq |M_t|$ ,  $M_0$  is also a maximum matching of  $G$ . Then,  $H$  consists only of  $(M_0, M_t)$ -alternating paths with even-length and  $(M_0, M_t)$ -alternating cycles; otherwise, this contradicts that  $M_0$  and  $M_t$  are both maximum matchings of  $G$ . Therefore,  $H$  contains components only of Categories (3) and (4).

Since every component in Category (4) is an even-length  $(M_0, M_t)$ -alternating path, each path starts with an edge of  $M_t \setminus M_0$  and ends at an edge of  $M_0 \setminus M_t$ . It is easy to see that all intermediate matchings have size at least  $|M_t| - 1 \geq k - 1$  for exchanging edges in the path. Therefore, if  $H$  contains no component of Category (3), then the greedy algorithm can transform  $M_0$  into  $M_t$  without ever going through a matching of size less than  $k - 1$ .

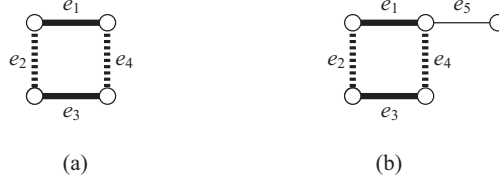


Figure 5: (a) No-instance and (b) Yes-instance for MATCHING RECONFIGURATION, where  $M_0 = \{e_1, e_3\}$ ,  $M_t = \{e_2, e_4\}$  and  $k = 2$  in both instances.

Suppose now that  $H$  contains components of Category (3). In this case, there does not always exist a desired sequence of matchings. (See Figure 5 as an example.) Nonetheless, existence can be determined in polynomial time by the following lemma.

**Lemma 1.** *Suppose that both  $M_0$  and  $M_t$  are maximum matchings of  $G$ , and let  $k = |M_0| = |M_t|$ . Then, there exists a sequence of matchings which transforms  $M_0$  into  $M_t$  so that all intermediate matchings have size at least  $k - 1$  if and only if, for every  $(M_0, M_t)$ -alternating cycle  $C$ , there exists an  $M_0$ -alternating path in  $G$  starting with an  $M_0$ -exposed vertex and ending at a vertex in  $C$ .*

For the example in Figure 5(b), the  $(M_0, M_t)$ -alternating cycle  $\{e_1, e_2, e_3, e_4\}$  has such an  $M_0$ -alternating path  $\{e_5\}$ . By Lemma 1 one can easily determine whether there exists a desired sequence for Case (c) in polynomial time; we simply check if there exists such an  $M_0$ -alternating path  $P$  in  $G$ , assuming that each vertex in an  $(M_0, M_t)$ -alternating cycle is the endpoint of  $P$ .

From now on, we prove Lemma 1 to complete the proof of Proposition 2. We first show a useful fact, which is a part of the Edmonds-Gallai decomposition [14].

For a graph  $G = (V, E)$ , let

$$D(G) = \{v \in V \mid \text{there exists a maximum matching } N \text{ of } G \text{ in which } v \text{ is } N\text{-exposed}\}.$$

For a maximum matching  $M$  of  $G$ , let

$$\text{EVEN}(M) = \{v \in V \mid \text{there exists an even-length } M\text{-alternating path from an } M\text{-exposed vertex to } v\}.$$

Note that we regard an  $M$ -alternating path of length 0 as even-length path, and hence  $\text{EVEN}(M)$  contains all  $M$ -exposed vertices. We have the following lemma.

**Lemma 2.** *For every maximum matching  $M$  of a graph  $G$ ,  $\text{EVEN}(M) = D(G)$ .*

**PROOF.** We first show that  $\text{EVEN}(M) \subseteq D(G)$ . Let  $v$  be an arbitrary vertex in  $\text{EVEN}(M)$ . Then, there exists an even-length  $M$ -alternating path  $P$  from an  $M$ -exposed vertex to  $v$ . Consider the matching  $M' = M \Delta P$ . (Note that  $M' = M$  if  $P$  is an  $M$ -alternating path of length 0.) Since the length of  $P$  is even,  $M'$  is also a maximum matching of  $G$  and  $v$  is  $M'$ -exposed. We thus have  $v \in D(G)$ .

We then show that  $\text{EVEN}(M) \supseteq D(G)$ . Let  $v$  be an arbitrary vertex in  $D(G)$ . If  $v$  is  $M$ -exposed, then  $v \in \text{EVEN}(M)$ , of course. Suppose now that  $v$  is  $M$ -covered. Since  $v \in D(G)$ , there exists a maximum matching  $N$  of  $G$  in which  $v$  is  $N$ -exposed. Consider the subgraph  $H_{M,N}$  of  $G$  induced by all edges in  $M \Delta N$ . Then, since  $M$  and  $N$  are both maximum matchings of  $G$ ,  $H_{M,N}$  consists only of  $(M, N)$ -alternating paths with even-length and  $(M, N)$ -alternating cycles. Since  $v$  is  $M$ -covered and  $N$ -exposed,  $v$  must be an endpoint of an even-length  $(M, N)$ -alternating path  $P$ . Clearly, the other endpoint of the path  $P$  is  $M$ -exposed (and  $N$ -covered), and hence  $v \in \text{EVEN}(M)$ .  $\square$

Lemma 2 immediately implies the following corollary.

**Corollary 1.** *For every two maximum matchings  $M$  and  $N$  of  $G$ ,  $\text{EVEN}(M) = \text{EVEN}(N)$ .*



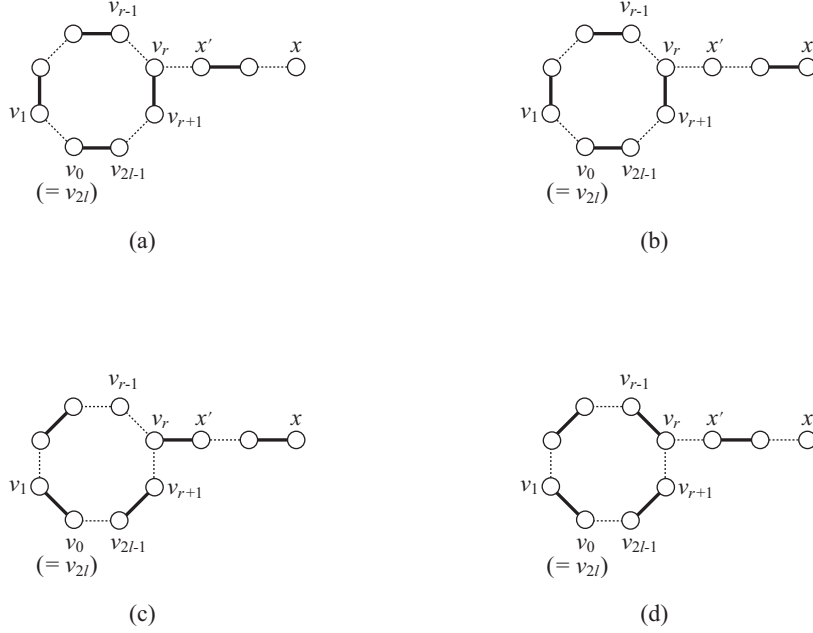


Figure 6: Exchanging the edges in an  $(M_0, M_t)$ -alternating cycle  $C = \{v_0, v_1, \dots, v_{2l}\}$  using an  $M_0$ -alternating path  $P$  starting with an  $M_0$ -exposed vertex  $x$  and ending at  $v_r \in C$ , where each edge in a matching is drawn as a thick line.

We are now ready to prove Lemma 1.

**[Proof of Lemma 1]**

*Necessity:* Suppose that, for every  $(M_0, M_t)$ -alternating cycle, there exists an  $M_0$ -alternating path in  $G$  starting with an  $M_0$ -exposed vertex and ending at a vertex in the cycle. It suffices to show that we can exchange the edges in Category (3) such that all intermediate matchings are of size  $\geq k - 1$ .

Let  $C = \{v_0, v_1, \dots, v_{2l}\}$  be an  $(M_0, M_t)$ -alternating cycle where  $v_{2l} = v_0$ , and suppose that there exists an  $M_0$ -alternating path  $P$  starting with an  $M_0$ -exposed vertex  $x$  and ending at  $v_r$  in  $C$ . (See Figure 6(a).) Let  $x'$  be the vertex in  $P$  adjacent with  $v_r$ , as illustrated in Figure 6(a). Note that, since  $v_r$  is in  $C$ , the edge  $(x', v_r)$  is not in  $M_0$ . Then, we exchange the edges in  $C$  as follows: first, exchange the edges of the path  $\{x, \dots, x'\}$  along  $P$ , and obtain a matching  $M$  in which  $x'$  is  $M$ -exposed (see Figs. 6(a) and (b)); then, exchange the edges of the path  $\{x', v_r, v_{r+1}, \dots, v_{r-1}\}$  in this order (see Figs. 6(b) and (c)); finally, exchange the edges of the path  $\{v_{r-1}, v_r, x', \dots, x\}$  in this order (see Figs. 6(c) and (d)). Clearly, all intermediate matchings have size  $\geq k - 1$ .

Let  $M'$  be the matching of  $G$  obtained by the edge exchanges above. Let  $E(C)$  be the set of edges in  $C$ . Since  $M_0 \cap M' = M_0 \setminus E(C)$ , we can exchange the edges of each  $(M_0, M_t)$ -alternating cycle independently. In this way, we can exchange the edges of all components of Category (3) such that all intermediate matchings are of size  $\geq k - 1$ , and hence there exists a way to transform  $M_0$  into  $M_t$  without ever going through a matching of size less than  $k - 1$ .

*Sufficiency:* Suppose that Category (3) contains an  $(M_0, M_t)$ -alternating cycle  $C$  such that there is no  $M_0$ -alternating path in  $G$  starting with an  $M_0$ -exposed vertex and ending at a vertex in  $C$ . Then, no vertex in  $C$  is contained in  $\text{EVEN}(M_0)$ . Suppose for a contradiction that there is a sequence of matchings which transforms  $M_0$  into  $M_t$  such that all intermediate matchings are of size  $\geq k - 1$ . Let  $M_0, M_1, \dots, M_t$  be such a sequence of matchings whose length (i.e. the number of intermediate matchings) is minimum. Let  $M_q$  be the first matching in the sequence for which we remove an edge  $(u, v)$  of  $M_0$  that belongs to  $C$ . Then, since  $k$  is equal to the maximum size of a matching in  $G$ , we clearly have  $|M_{q-1}| = k, |M_q| = k - 1$  and

$$\text{Exposed}(M_q) = \text{Exposed}(M_{q-1}) \cup \{u, v\},$$

where  $\text{Exposed}(M)$  is the set of all  $M$ -exposed vertices in  $G$  for a matching  $M$  of  $G$ . Since all intermediate matchings are of size  $\geq k-1$ , the matching  $M_{q+1}$  must be obtained from  $M_q$  by adding some edge  $(y, z)$ . Note that  $y$  and  $z$  must be both in  $\text{Exposed}(M_q)$ . If both  $y$  and  $z$  are also in  $\text{Exposed}(M_{q-1})$ , then this contradicts the fact that  $M_{q-1}$  is a maximum matching of  $G$ . We thus assume that  $y = u$ . If  $z = v$ , then  $M_{q-1} = M_{q+1}$ ; this contradicts that  $M_0, M_1, \dots, M_t$  is a minimum-length sequence. Therefore,  $z$  is some vertex in  $\text{Exposed}(M_{q-1})$ . But then, the path  $\{z, u, v\}$  is an even-length  $M_{q-1}$ -alternating path. Since  $z$  is  $M_{q-1}$ -exposed and  $M_{q-1}$  is a maximum matching of  $G$ ,  $v$  is in  $\text{EVEN}(M_{q-1})$ . By Corollary 1,  $\text{EVEN}(M_0) = \text{EVEN}(M_{q-1})$  and hence  $v \in \text{EVEN}(M_0)$ . This contradicts that no vertex in  $C$  is contained in  $\text{EVEN}(M_0)$ .  $\square$

Besides `MATROID RECONFIGURATION` and `MATCHING RECONFIGURATION`, it turns out that all polynomial-time solvable special cases of `SATISFIABILITY`, as characterized by Schaefer [13], give rise to polynomially solvable reconfiguration problems:

**Theorem 3 ([3]).** `SATISFIABILITY RECONFIGURATION for linear, Horn, dual Horn and 2-literal clauses are all in P.`

## 4. Approximation

We have seen that an optimization problem gives rise to a reconfiguration problem by bounding the objective of intermediate configurations. In turn, we can get a natural optimization problem if we try to *optimize the worst objective among all configurations* in the reconfiguration sequence. For example, in the problem that we call the `MAXMIN CLIQUE RECONFIGURATION` problem, we are given a graph and two cliques  $C_0$  and  $C_t$ , and we are asked to maximize the minimum size of any clique in a sequence which transforms  $C_0$  into  $C_t$  by additions and removals of single nodes. In this section, we give some inapproximability and approximability results for such optimization problems.

### 4.1. Inapproximability

In this subsection, we show inapproximability results for two max-min type reconfiguration problems.

We first give the following theorem for the `MAXMIN CLIQUE RECONFIGURATION` problem.

**Theorem 4.** `MAXMIN CLIQUE RECONFIGURATION cannot be approximated within any constant factor unless P = NP.`

**PROOF.** We give a polynomial-time reduction in an approximation-preserving manner from the (ordinary) `CLIQUE` problem to this problem. For a given graph  $G$  with  $n$  nodes, we construct a new graph  $G'$  with  $3n$  nodes as the corresponding instance of `MAXMIN CLIQUE RECONFIGURATION`: a set of  $n$  nodes is connected as  $G$ , while two new sets of  $n$  nodes are connected each as a clique (these two cliques of  $G'$  are called  $C_0$  and  $C_t$ ); finally, there are edges in  $G'$  between each new node and each node in  $G$ .

Consider any sequence of cliques of  $G'$ , each resulting from the previous one by insertion or deletion of a single node, starting from  $C_0$  and ending in  $C_t$ . We claim that one of them will be a clique of  $G$  — this follows directly from the absence of any edges between  $C_0$  and  $C_t$ . Conversely, for every clique  $C$  of  $G$ , there exists a sequence from  $C_0$  to  $C_t$  via  $C$ : add the nodes of  $C$  to the clique  $C_0$  and obtain the clique  $C_0 \cup C$ , then remove those of  $C_0$  and obtain  $C$ , then add those of  $C_t$  and obtain  $C \cup C_t$ , and finally remove those of  $C$  and obtain  $C_t$ . Since  $|C_0| = |C_t| = n$  and  $|C| \leq n$ , the minimum clique size in the sequence is the size of  $C$ , and hence solving (or approximating) this instance of `MAXMIN CLIQUE RECONFIGURATION` is the same as solving (respectively, approximating) the `CLIQUE` problem for  $G$ . Since it is known that `CLIQUE` cannot be approximated within any constant factor unless  $P = NP$  [4], the result follows.  $\square$

In the `MAXMIN MAXSAT RECONFIGURATION` problem, we are given a SAT formula and two truth assignments  $s_0$  and  $s_t$  (which are not necessarily satisfying), and we are asked to maximize the minimum number of clauses satisfied by any truth assignment in a path in the hypercube between  $s_0$  and  $s_t$ . Then, a similar argument establishes the following theorem.

**Theorem 5.** `MAXMIN MAXSAT RECONFIGURATION cannot be approximated within a factor better than  $\frac{15}{16}$  unless P = NP.`

PROOF. We give a polynomial-time reduction in an approximation-preserving manner from the (ordinary) MAXSAT problem to this problem. Suppose that we are given an instance  $\phi$  of MAXSAT with  $n$  variables  $x_1, x_2, \dots, x_n$  and  $m$  clauses  $C_1, C_2, \dots, C_m$ . We construct a new formula  $\phi'$  in which each clause  $C_j$ ,  $1 \leq j \leq m$ , is replaced by  $(C_j \vee y \vee z)$  where  $y$  and  $z$  are new variables, and the additional clause  $(\bar{y} \vee \bar{z})$  with weight  $m$ . Notice that every truth assignment of  $\phi'$  with  $z \neq y$  satisfies all  $2m$  clauses, and hence the truth assignments  $\mathbf{s}_0 : z = 1, y = 0, x_1 = x_2 = \dots = x_n = 1$  and  $\mathbf{s}_t : z = 0, y = 1, x_1 = x_2 = \dots = x_n = 0$  are both satisfying all  $2m$  clauses.

For each truth assignment  $\mathbf{s}$  of the original formula  $\phi$ , let  $\mathbf{s}'$  be a truth assignment of the corresponding formula  $\phi'$  such that  $z = y$  (namely, either  $z = y = 0$  or  $z = y = 1$ ) and each  $x_i$ ,  $1 \leq i \leq n$ , is as in  $\mathbf{s}$ . Then, it is easy to see that there is a path in the  $(n + 2)$ -dimensional hypercube from  $\mathbf{s}_0$  to  $\mathbf{s}_t$  via  $\mathbf{s}'$  such that  $y \neq z$  in all intermediate truth assignments except for  $\mathbf{s}'$ . Clearly, every truth assignment, except for  $\mathbf{s}'$ , in the path satisfies all  $2m$  clauses, and hence the objective value for the path is the number of clauses satisfied by  $\mathbf{s}'$ .

Consider now an optimal path in the  $(n + 2)$ -dimensional hypercube between  $\mathbf{s}_0$  and  $\mathbf{s}_t$ . Since at  $\mathbf{s}_0 : z = 1, y = 0$  and at  $\mathbf{s}_t : z = 0, y = 1$ , there must exist a truth assignment  $\mathbf{s}^*$  on this path such that  $z = y$ . Since the clause  $(\bar{y} \vee \bar{z})$  has weight  $m$  and the path is assumed optimal, it must be that  $z = y = 0$ . Thus, the remaining variables  $x_i$ ,  $1 \leq i \leq n$ , must spell an optimal satisfying truth assignment of the original formula  $\phi$ . Hence, from the optimal value  $\text{OPT}'$  for the corresponding instance of MAXMIN MAXSAT RECONFIGURATION, we can compute the optimal value  $\text{OPT}$  for the original instance  $\phi$  of MAXSAT: since at  $\mathbf{s}^* : z = y = 0$ , we have

$$\text{OPT} = \text{OPT}' - m. \quad (1)$$

Suppose now that we have an  $\alpha$ -approximation for MAXMIN MAXSAT RECONFIGURATION, and hence we can compute an approximate value  $A'$  for the corresponding instance such that

$$A' \geq \alpha \cdot \text{OPT}'. \quad (2)$$

One may assume without loss of generality that  $A' \geq m$ ; otherwise there must exist at least one truth assignment such that  $z = y = 1$  in the path; but, by replacing all such truth assignments with  $z = y = 0$ , we can easily obtain a better objective  $\geq m$ . Thus, there exists a truth assignment for the original formula  $\phi$  which satisfies a number  $(A' - m)$  of clauses. Let  $A = A' - m$ . By Eqs. (1) and (2) we have

$$A = A' - m \geq \alpha \cdot \text{OPT}' - m = \alpha \cdot \text{OPT} + (\alpha - 1)m. \quad (3)$$

Since  $m \geq \text{OPT}$ , by Eq. (3) we have  $A \geq (2\alpha - 1) \cdot \text{OPT}$ . Therefore, we can obtain a  $(2\alpha - 1)$ -approximation for MAXSAT, from an  $\alpha$ -approximation for MAXMIN MAXSAT RECONFIGURATION. Since it is known that MAXSAT cannot be approximated within a factor better than  $\frac{7}{8}$  unless  $P = NP$  [5], the result follows.  $\square$

#### 4.2. Approximability

In this subsection, we show approximability results for two min-max type reconfiguration problems.

In the MINMAX SET COVER RECONFIGURATION problem, we are given an universal set  $U$ , a family  $\mathcal{S}$  of subsets of  $U$ , each of subsets has a nonnegative cost, and two covers  $C_0$  and  $C_t$  of  $U$ , where a cover  $C$  of  $U$  is a subfamily of  $\mathcal{S}$  whose union is  $U$ . Then, we are asked to minimize the maximum total cost of any cover in a sequence which transforms  $C_0$  into  $C_t$  via covers of  $U$ , each of which results from the previous one by adding or deleting a single set in  $\mathcal{S}$ .

**Theorem 6.** *There is a linear-time 2-approximation algorithm for MINMAX SET COVER RECONFIGURATION.*

PROOF. For a cover  $C$  of  $U$ , we denote by  $\omega(C)$  the sum of costs of all subsets in  $C$ . Consider an optimal sequence  $C_0, C_1, \dots, C_t$  for an instance of MINMAX SET COVER RECONFIGURATION. Let  $\text{OPT}$  be the objective value for the sequence, and hence  $\text{OPT} = \max\{\omega(C_i) \mid 0 \leq i \leq t\}$ . Therefore, we clearly have

$$\max\{\omega(C_0), \omega(C_t)\} \leq \text{OPT}. \quad (4)$$

As our approximation solution, we consider the following sequence of covers: (i) add the subsets in  $C_t \setminus C_0$  one by one to  $C_0$ , and obtain the cover  $C_0 \cup C_t$  of  $U$ ; (ii) delete the subsets in  $C_0 \setminus C_t$  one by one from  $C_0 \cup C_t$ , and obtain  $C_t$ . Clearly, our approximate value  $A$  is  $A = \omega(C_0 \cup C_t)$ , and hence by Eq. (4) we have

$$A = \omega(C_0 \cup C_t) \leq \omega(C_0) + \omega(C_t) \leq 2 \cdot \max\{\omega(C_0), \omega(C_t)\} \leq 2 \cdot \text{OPT}.$$

This completes the proof of Theorem 6.  $\square$

Returning to the POWER SUPPLY problem, there is a natural optimization version of the problem, in which the constraint that the total demand of all demand vertices in each subtree  $T$  be within the supply of the supply vertex in  $T$  is replaced by a “soft” criterion: we allow that the total demand in  $T$  exceeds the supply in  $T$ , but wish to minimize the sum of the “deficient power” of all supply vertices in the graph.

We now define the MINMAX POWER SUPPLY RECONFIGURATION problem. For a configuration  $f$  of a bipartite graph  $G = (U, V, E)$  and a supply vertex  $u \in U$ , the *deficient power*  $d(f, u)$  of  $u$  on  $f$  is defined as follows:

$$d(f, u) = \sum \{ \text{dem}(v) \mid v \in V \text{ such that } f(v) = u \} - \text{sup}(u).$$

If  $f$  is infeasible, then there is at least one supply vertex  $u$  such that  $d(f, u) > 0$ . On the other hand, if  $f$  is feasible, then  $d(f, u) \leq 0$  for all supply vertices  $u \in U$ ; in fact, a nonpositive deficient power  $d(f, u)$  represents the *marginal power* of  $u$  on  $f$ . The *cost*  $c(f)$  of a configuration  $f$  is defined as follows:

$$c(f) = \sum_{u \in U} |d(f, u)|.$$

Note that  $c(f)$  contains the marginal power of supply vertices, because it is difficult to change the supplies quickly and hence we waste the marginal power. Clearly,  $c(f) = \sum_{u \in U} \text{sup}(u) - \sum_{v \in V} \text{dem}(v)$  for every feasible configuration  $f$  of  $G$ . In the problem that we call the MINMAX POWER SUPPLY RECONFIGURATION problem, we are given a bipartite graph  $G = (U, V, E)$  and two feasible configurations  $f_0$  and  $f_i$  of  $G$ , and we are asked to minimize the maximum cost of any configuration in a sequence which transforms  $f_0$  into  $f_i$  by reassignments of single demand vertices. Then, we have the following observation.

**Observation 1.** *The objective value for a sequence which transforms  $f_0$  into  $f_i$  is  $\sum_{u \in U} \text{sup}(u) - \sum_{v \in V} \text{dem}(v)$  if and only if all configurations in the sequence are feasible. Moreover, such a sequence is optimal if it exists.*

In the remainder of this subsection, we give a linear-time 2-approximation algorithm for the MINMAX POWER SUPPLY RECONFIGURATION problem if a given bipartite graph  $G$  has exactly two supply vertices. We first show that the problem is strongly NP-hard even for more restricted instances.

**Lemma 3.** *MINMAX POWER SUPPLY RECONFIGURATION is strongly NP-hard, even for the restricted problem consisting of instances on a complete bipartite graph with exactly two supply vertices.*

**PROOF.** We give a polynomial-time reduction from the 3-PARTITION problem [2] to this problem for a complete bipartite graph with exactly two supply vertices. In 3-PARTITION, we are given a positive integer bound  $b$ , and a set  $A$  of  $3m$  elements  $a_1, a_2, \dots, a_{3m}$ ; each element  $a_i \in A$  has a positive integer size  $s(a_i)$  such that  $b/4 < s(a_i) < b/2$  and such that  $\sum_{a \in A} s(a) = mb$ . Then, the 3-PARTITION problem is to determine whether  $A$  can be partitioned into  $m$  disjoint subsets  $A_1, A_2, \dots, A_m$  such that  $\sum_{a \in A_j} s(a) = b$  for each  $j$ ,  $1 \leq j \leq m$ . 3-PARTITION is known to be strongly NP-complete [2].

For a given instance of 3-PARTITION, we first construct a complete bipartite graph  $G = (U, V, E)$  with  $|U| = 2$ , as follows:  $U$  consists of two supply vertices  $u_1$  and  $u_2$  such that  $\text{sup}(u_1) = mb$  and  $\text{sup}(u_2) = (m+1)b$ ; and  $V$  consists of  $4m$  demand vertices  $v_1, v_2, \dots, v_{3m}$  and  $b_1, b_2, \dots, b_m$  such that  $\text{dem}(v_i) = s(a_i)$  for each  $i$ ,  $1 \leq i \leq 3m$ , and  $\text{dem}(b_j) = b$  for each  $j$ ,  $1 \leq j \leq m$ . We then give two feasible configurations  $f_0$  and  $f_i$  of  $G$ , as follows:

$$f_0(x) = \begin{cases} u_1 & \text{if } x = v_i, 1 \leq i \leq 3m; \\ u_2 & \text{if } x = b_j, 1 \leq j \leq m, \end{cases}$$

and

$$f_i(x) = \begin{cases} u_2 & \text{if } x = v_i, 1 \leq i \leq 3m; \\ u_1 & \text{if } x = b_j, 1 \leq j \leq m. \end{cases}$$

Clearly,  $d(f_0, u_1) = d(f_i, u_1) = 0$  and  $d(f_0, u_2) = d(f_i, u_2) = -b$  (that is, only the supply vertex  $u_2$  has an amount  $b$  of marginal power), and hence  $c(f_0) = c(f_i) = b$ .

It is easy to see that there exists a desired partition  $\{A_1, A_2, \dots, A_m\}$  for a given instance of 3-PARTITION if and only if there exists a sequence which consists of only feasible configurations of  $G$  for the corresponding instance of MINMAX POWER SUPPLY RECONFIGURATION. Therefore, by Observation 1 we can answer whether the set  $A$  has a desired partition by determining whether the optimal value is  $b$  or not for the corresponding instance of MINMAX POWER SUPPLY RECONFIGURATION.  $\square$

By Lemma 3 it is very unlikely that the MINMAX POWER SUPPLY RECONFIGURATION problem can be solved even in pseudo-polynomial time. However, the problem can be solved in linear time for the following special case.

Suppose in the remainder of this subsection that we are given a bipartite graph  $G = (U, V, E)$  having exactly two supply vertices  $u_1$  and  $u_2$ . (Note that  $G$  is not necessarily complete.) For two given feasible configurations  $f_0$  and  $f_t$  of  $G$ , let  $W = \{v \in V \mid f_0(v) \neq f_t(v)\}$ , that is,  $W$  is the set of demand vertices which must be reassigned to the other supply vertex. Notice that all (demand) vertices in  $W$  are adjacent to both the two supply vertices. Let  $v^*$  be a demand vertex in  $W$  having the maximum demand, that is,  $\text{dem}(v^*) = \max\{\text{dem}(v) \mid v \in W\}$ . Then, we have the following lemma.

**Lemma 4.** *If  $c(f_0) \geq 2 \cdot \text{dem}(v^*)$ , then an optimal sequence for the instance consists of only feasible configurations of  $G$ , and it can be found in linear time.*

PROOF. Suppose without loss of generality that  $W \neq \emptyset$ . If all demand vertices in  $W$  are assigned to the same supply vertex  $u$  on  $f_0$ , then we just reassign the demand vertices in  $W$  from  $u$  to the other one by one. Notice that all intermediate configurations are feasible since both  $f_0$  and  $f_t$  are feasible. Therefore, we may assume in the following that each of the two supply vertices has at least one demand vertex in  $W$ .

Since  $f_0$  is feasible,  $c(f_0) = \text{sup}(u_1) + \text{sup}(u_2) - \sum_{v \in V} \text{dem}(v)$  and the cost  $c(f_0)$  denotes the sum of marginal power of the two supply vertices. Moreover, since the sum is at least  $2 \cdot \text{dem}(v^*)$ , one of the two supply vertices has marginal power of at least  $\text{dem}(v^*)$ . Therefore, we can change the assignment of at least one demand vertex  $v \in W$  from the initial supply vertex to the target one, since  $\text{dem}(v) \leq \text{dem}(v^*)$ . Clearly, the resulting configuration  $f_1$  is also feasible, and hence it satisfies  $c(f_1) = c(f_0) \geq 2 \cdot \text{dem}(v^*)$ . In this way, by reassigning the demand vertices in  $W$  one by one, we can obtain a desired sequence  $f_0, f_1, \dots, f_t$  which consists of only feasible configurations. By Observation 1 the sequence is an optimal solution. The length of the sequence is  $|W| (\leq |V|)$  since each demand vertex in  $W$  moves exactly once and any of the other demand vertices does not move in the sequence. We can thus find an optimal solution in linear time.  $\square$

Using Lemma 4, we have the following theorem.

**Theorem 7.** *There is a linear-time 2-approximation algorithm for MINMAX POWER SUPPLY RECONFIGURATION if a given bipartite graph has exactly two supply vertices.*

PROOF. Let OPT be the optimal value for a given instance of MINMAX POWER SUPPLY RECONFIGURATION. Since the demand vertex  $v^*$  having the maximum demand in  $W$  must be reassigned at least once in any sequence from  $f_0$  to  $f_t$ , it is easy to observe that

$$\text{OPT} \geq \text{dem}(v^*). \quad (5)$$

By Lemma 4 it suffices to consider the case  $c(f_0) < 2 \cdot \text{dem}(v^*)$ . Note that, since  $f_0$  is feasible,  $\text{sup}(u_1) + \text{sup}(u_2) < 2 \cdot \text{dem}(v^*) + \sum_{v \in V} \text{dem}(v)$  in this case. Consider a slightly modified instance in which the supplies of the two supply vertices are increased by the same amount  $\varepsilon$  so that the total supply is equal to  $2 \cdot \text{dem}(v^*) + \sum_{v \in V} \text{dem}(v)$ , that is, the supply  $\overline{\text{sup}}(u)$  of a supply vertex  $u$  in the modified instance is  $\overline{\text{sup}}(u) = \text{sup}(u) + \varepsilon$  where

$$\varepsilon = \frac{1}{2} \left( 2 \cdot \text{dem}(v^*) + \sum_{v \in V} \text{dem}(v) - \text{sup}(u_1) - \text{sup}(u_2) \right).$$

In the modified instance, both the configurations  $f_0$  and  $f_t$  remain feasible and  $\bar{c}(f_0) = \bar{c}(f_t) = 2 \cdot \text{dem}(v^*)$ , where  $\bar{c}(f)$  denotes the cost of a configuration  $f$  in the modified instance. Therefore, by Lemma 4 we can find in linear time a sequence which consists of only feasible configurations for the modified instance; by Observation 1, the objective value is  $2 \cdot \text{dem}(v^*)$ . Note that some configurations in the sequence may be infeasible for the original instance. Consider an arbitrary configuration  $f$  in the sequence which is infeasible for the original instance; let  $V_1 \subseteq V$  be the set of demand vertices such that  $f(v) = u_1$ , and let  $V_2 = V \setminus V_1$ . Since  $f$  is feasible for the modified instance, we have

$$\bar{c}(f) = \left( \overline{\text{sup}}(u_1) - \sum_{v \in V_1} \text{dem}(v) \right) + \left( \overline{\text{sup}}(u_2) - \sum_{v \in V_2} \text{dem}(v) \right) = 2 \cdot \text{dem}(v^*). \quad (6)$$

On the other hand, since  $f$  is infeasible for the original instance, exactly one of  $d(f, u_1)$  and  $d(f, u_2)$  is positive, say  $u_1$ ; otherwise, either  $f$  is feasible or  $f_0$  would be infeasible in the original instance. Then, we have

$$\begin{aligned} c(f) &= \left( \sum_{v \in V_1} \text{dem}(v) - \text{sup}(u_1) \right) + \left( \text{sup}(u_2) - \sum_{v \in V_2} \text{dem}(v) \right) \\ &= \left( \sum_{v \in V_1} \text{dem}(v) - \overline{\text{sup}}(u_1) + \varepsilon \right) + \left( \overline{\text{sup}}(u_2) - \varepsilon - \sum_{v \in V_2} \text{dem}(v) \right) \\ &\leq \overline{\text{sup}}(u_2) - \sum_{v \in V_2} \text{dem}(v) \end{aligned}$$

since  $\sum_{v \in V_1} \text{dem}(v) - \overline{\text{sup}}(u_1) \leq 0$ . Then, by Eq. (6) we have  $c(f) \leq \bar{c}(f) = 2 \cdot \text{dem}(v^*)$ . By Eq. (5) we thus have  $c(f) \leq 2 \cdot \text{OPT}$ . Since the cost of a feasible configuration is smaller than the cost of an infeasible configuration, the objective value of this sequence in the original instance is at most  $2 \cdot \text{OPT}$ , as required.  $\square$

## 5. Open Problems

There are many open problems raised by this work, and we mention some of these below:

- Can the MATCHING RECONFIGURATION problem for edge-weighted graphs be solved also in polynomial time? We conjecture that the answer is positive.
- Is the TRAVELING SALESMAN RECONFIGURATION problem (where two tours are adjacent if they differ in two edges) PSPACE-complete?
- Are there better approximation algorithms for the MINMAX POWER SUPPLY RECONFIGURATION problem? Lower bounds?
- Are the problems in Section 4 PSPACE-hard to approximate (not just NP-hard)?

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