Rainbow Hamilton cycles and lopsidependency

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Abstract

The Lovász Local Lemma is a powerful probabilistic tool used to prove the existence of combinatorial structures which avoid a set of constraints. A standard way to apply the local lemma is to prove that the set of constraints satsify a lopsidependency condition and obtain a lopsidependency graph. For instance, Erdős and Spencer used this framework to posit the existence of Latin transversals in matrices provided no symbol appears too often in the matrix.

The local lemma has been used in various ways to infer the existence of rainbow Hamilton cycles in complete graphs when each colour is used at most $O(n)$ times. However, the existence of a lopsidependency graph for Hamilton cycles has neither been proved nor refuted. All previous approaches have had to prove a variant of the local lemma or reduce the problem of finding Hamilton cycles to finding another combinatorial structure, such as Latin transversals. In this paper, we revisit the question of whether or not Hamilton cycles have a lopsidependency graph and give a positive answer for this question. We also use the resampling oracle framework of Harvey and Vondrák to give a polynomial time algorithm for finding rainbow Hamilton cycles in complete graphs.

Keywords: Hamilton cyles, Lovász Local Lemma

1. Introduction

In combinatorics, the Lovász Local Lemma (LLL) is a very powerful probabilistic tool which has seen many applications (for some classic examples, see [1, 2]). The original LLL was only applicable to probability spaces where the events formed a "dependency graph". This was later extended to the setting of "lopsidependency graphs" by Erdős and Spencer [3]. A similar extension of the LLL was independently obtained by Albert, Frieze, and Reed in their work on "rainbow Hamilton cycles" [4]. Lu, Mohr, and Szekely have undertaken a study of probability spaces and events that have a lopsidependency graph [5]. Some examples from their work include random matchings in complete uniform hypergraphs, random spanning trees in complete graphs, and random permutations. Although Albert, Frieze, and Reed did apply the LLL to random Hamilton cycles in complete graphs, they did not show that this scenario leads to a lopsidependency graph. To our knowledge, that statement is neither proven nor refuted by any result appearing

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in the literature. We prove indeed random Hamilton cycles do lead to a lopsidependency graph, thereby extending the list of examples collected in the survey of Lu, Mohr, and Szekely.

Over the past few years, there has also been much work on algorithmic forms of the LLL, even for settings involving lopsidependency graphs [6, 7, 8, 9, 10, 11, 12, 13]. Harvey and Vondrák [11] define an abstract notion of resampling oracles, and show that the LLL has an algorithmic proof in any scenario with resampling oracles. They also show that the existence of resampling oracles implies that the scenario involves a lopsidependency graph. We design efficient resampling oracles for the scenario of random Hamilton cycles in complete graphs, implying that this scenario involves a lopsidependency graph.

Finally, we discuss a recent enhancement of the LLL known as the cluster expansion criterion. This gives sharper results in several applications of the LLL. We use this criterion in the scenario of random Hamilton cycles to give new results on rainbow Hamilton cycles that slightly strengthen those of Albert, Frieze, and Reed. Furthermore our results are algorithmic due to the framework of Harvey and Vondrák and our efficient resampling oracles.

1.1. Background

A cycle in a graph is called a *Hamilton* cycle if every vertex appears exactly once. If the graph is edge-coloured then the Hamilton cycle is called *rainbow* if distinct edges are assigned distinct colours. Define the function $k(n)$ to be the minimum value that satisfies the following condition. No matter how we edge-colour the complete graph K_n, if every colour appears at most $k(n)$ times then there exists a rainbow Hamilton cycle.

If we pick a vertex v and assign the same colour to all edges incident to v then this graph does not contain a rainbow Hamilton cycle. So an easy upper bound is $k(n) < n-1$. Hahn and Thomassen [14] conjectured that this is essentially tight. More precisely, they conjectured that for some constant $\gamma > 0$ and any n sufficiently large, we have $k(n) \geq \gamma n$.

The earliest result in this direction is due to Hahn and Thomassen [14] who proved that $k(n) = \Omega\left(n^{1/3}\right)$. Frieze and Reed [15] improved this to $k(n) = \Omega\left(\frac{n}{\log n}\right)$. Finally, Albert, Frieze, and Reed [4] closed the gap.

Theorem 1. *(Albert, Frieze, Reed [4]) Let $\gamma < 1/64$[1]. There exists $n_0 = n_0(\gamma)$ such that if $n \geq n_0$ then the following holds. For any edge-colouring of K_n, if any colour is appears on at most γn edges then K_n contains a rainbow Hamilton cycle.*

Other related works. The present work considers the existence of a rainbow Hamilton cycle under an adversarial colouring in the complete graph. It is also interesting to ask when rainbow Hamilton cycles exist under different settings. The existence of rainbow Hamilton cycles in the Erdős-Renyi random graph model and a uniform random colouring was studied by [16, 17, 18]. Let $G_{n,p,c}$ be the random graph on n vertices where each edge is included with probability p and each edge receives one of c colours uniformly at random. Ferber and Krivelevich [18] show that, w.h.p.[2], the random graph $G_{n,p,c}$

[1] The original paper claimed that $\gamma < 1/32$. This was later corrected to $\gamma < 1/64$. The comment can be found at `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v2i1r10/comment`.

[2] A sequence of events E_n is said to occur with high probability (w.h.p.) if $\lim_{n\to\infty} \Pr[E_n] = 1$.

contains a rainbow Hamilton cycle as long as $c \geq (1+\varepsilon)n$ and $p \geq \frac{\log n + \log\log n + \omega(1)}{n}$. This result is tight as $c \geq n$ colours are required and it is well-known that the threshold of p is required just to have Hamiltonicity [19]. Generalizations of this result to hypergraphs have also appeared in the literature (see [18, 20]).

There are also some results for other graph models. Janson and Wormald [21] showed that a random d-regular graph with a random colouring where each colour appears $d/2$ times ($d \geq 8$ is even) has a rainbow Hamilton cycle w.h.p. Bal et al. [22] study the existence of rainbow Hamilton cycles in a random geometric graphs where each edge is given a uniform colour from a set of $\Theta(n)$ colours. They show that, w.h.p., a rainbow Hamilton cycle "emerges" as soon as the minimum degree of the graph is at least 2. This is best possible as any graph with minimum degree less than 2 cannot even by Hamiltonian.

1.2. The Lovász Local Lemma

We first review some results related to the Lovász Local Lemma.

Definition 2. Let Ω be a probability space and $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a collection of "bad" events from Ω. Let G be a graph with vertex set $[n] = \{1, \ldots, n\}$ and edge set $E \subseteq \binom{[n]}{2}$. Denote with $\Gamma(i)$ the neighbourhood of i and $\Gamma^+(i) = \Gamma(i) \cup \{i\}$. We say that G is a *dependency* graph for $\mathcal{F}$ if for all $i \in [n]$ and $J \subseteq [n] \setminus \Gamma^+(i)$

$$\Pr\left[F_i \mid \cap_{j \in J} \overline{F_j}\right] = \Pr\left[F_i\right]. \tag{1}$$

If, instead, (1) is replaced by

$$\Pr\left[F_i \mid \cap_{j \in J} \overline{F_j}\right] \leq \Pr\left[F_i\right] \tag{2}$$

then G is a *lopsidependency* graph for $\mathcal{F}$.

Remark 3. *Observe that if G is a dependency (resp. lopsidependency) graph for the events $\{F_i\}_{i\in[n]}$ and $I \subseteq [n]$ then the vertex-induced subgraph $G[I]$ is a dependency (resp. lopsidependency) graph for the events $\{F_i\}_{i\in I}$. Indeed, since $I \subseteq [n]$, if (1) (resp. (2)) holds for $\{F_i\}_{i\in[n]}$ then (1) (resp. (2)) holds for $\{F_i\}_{i\in I}$.*

Theorem 4. *(Lovász Local Lemma [2, 23, 3]) Let $F_1, \ldots, F_n$ be a set of events with associated lopsidependency graph G. Suppose there exists $x_1, \ldots, x_n \in [0, 1)$ such that for all $i \in [n]$*

$$\Pr\left[F_i\right] \leq x_i \prod_{j \in \Gamma(i)} (1 - x_j).$$

Then $\Pr\left[\bigcap_i \overline{F_i}\right] > 0$.

To obtain sharper constants, we will use a stronger from of the LLL, due to Bissacot et al. [24], known as cluster expansion. An algorithmic version of Theorem 5 is given as Theorem 8.

Theorem 5. *([24]) Let $F_1, \ldots, F_n$ be a set of events with associated lopsidependency graph G. Let $\mathrm{Ind}(i)$ be the set of all independent subsets of $\Gamma^+(i)$. Suppose there exists $y_1, \ldots, y_n > 0$ such that for all $i \in [n]$*

$$\Pr[F_i] \leq \frac{y_i}{\sum_{J \in \mathrm{Ind}(i)} \prod_{j \in J} y_j}.$$

Then $\Pr\left[\bigcap_{i=1}^n \overline{F_i}\right] > 0$.

1.3. Algorithmic aspects of the Lovász Local Lemma

The original proof of the Lovász Local Lemma was not constructive and provided no efficient algorithm for finding an element in $\cap_i \overline{F_i}$. This led to a lot of work on making the local lemma algorithmic [25, 26, 27, 28]. In 2009, Moser and Tardos [13] made a breakthrough by making Theorem 4 algorithmic under the "independent variable model". It was later shown that the Moser-Tardos algorithm can be extended to the cluster expansion criterion [29] and also Shearer's criterion [12].

There have been a number of extensions of the Moser-Tardos algorithm, each of which relax the independent variable model in a different way. In [6], Achlioptas and Iliopoulos devised a random walk algorithm for finding "flawless objects". This approach generalizes the Moser-Tardos algorithm and was applicable to some scenarios not covered by the Moser-Tardos algorithm. One such application is searching for rainbow Hamilton cycles in complete hypergraphs. In a follow-up paper [7], they showed that the random walk framework was able to make the cluster expansion criterion algorithmic in a setting that was beyond the Moser-Tardos model. In [9], Harris and Srinivasan made the LLL algorithmic for certain events involving random permutations. This is another application where the Moser-Tardos algorithm is not applicable. The Moser-Tardos algorithm works only when the underlying probability measure is a product measure. Moreover, Moser and Tardos gave no discussion on how to resample from a probability space that was not a product space. In particular, since the space of random permutations is not a product space, the Moser-Tardos algorithm is not applicable.

In [11], Harvey and Vondrák gave another relaxation of the independent random variable assumption by introducing the notion of resampling oracles. This made the LLL algorithmic in more general probability spaces. Their work also gave an algorithmic viewpoint on lopsidependency graphs.

Definition 6. (Resampling oracles [11]) Let Ω be a probability space with probability measure μ and $F_1, \ldots, F_n$ be a set of events from Ω. Let G be a graph with vertex set $[n]$. Let $r_i \colon \Omega \to \Omega$ be a randomized function. We call r_i a *resampling oracle* for F_i with respect to the graph G if the following two conditions hold.

1. If $\omega \sim \mu|_{F_i}$ then $r_i(\omega) \sim \mu$. Here, $\mu|_{F_i}$ denotes the probability measure on Ω conditioned on F_i.
2. Suppose $j \notin \Gamma^+(i)$. If $\omega \notin F_j$ then $r_i(\omega) \notin F_j$.

The first condition says that if ω is randomly distributed conditioned on F_i then applying the resampling oracle r_i removes the conditioning on F_i. The second condition says that applying the resampling oracle can only cause an event F_j if $F_j \in \Gamma^+(F_i)$.

Lemma 7. *([11]) Suppose that there exists resampling oracles $r_1, \ldots, r_n$ for the events $F_1, \ldots, F_n$ with respect to a graph G. Then G is a lopsidependency graph for $F_1, \ldots, F_n$.*

The main theorem that we will need from [11] is the following.

Theorem 8. *([11]) Let $F_1, \ldots, F_n$ be a collection of events and let G be its associated lopsidependency graph. Let $\mathrm{Ind}(i)$ be the set of independent subsets of $\Gamma^+(i)$. Suppose there exists $y_1, \ldots, y_n > 0$ such that for all i*

$$\Pr[F_i] \le \frac{y_i}{\sum_{J \in \mathrm{Ind}(i)} \prod_{j \in J} y_j}.$$

Then $\Pr\left[\cap_i \overline{F_i}\right] > 0$*. Moreover, there exists a randomized algorithm that finds a point* $\omega \in \cap_i \overline{F}_i$ *using* $O\left(\sum_{i=1}^{n} y_i \sum_{j=1}^{n} \log(1+y_j)\right)$ *resampling oracle calls in expectation.*

1.4. Our contribution

Let $\mathcal{A}$ be the collection of all subsets of edges in K_n. For all $A \in \mathcal{A}$ let E_A be the event that a Hamilton cycle chosen at random contains all edges in A. Define a graph G with vertex set $\mathcal{A}$. If $A, B \in \mathcal{A}$ then add an edge betwen A and B if A and B are not vertex disjoint.

Is G a lopsidependency graph for the events $\{E_A\}_{A \in \mathcal{A}}$? Albert, Frieze, and Reed [4] do not answer this question, and instead formulate a variant of the LLL which was applicable to their scenario. If it were lopsidependent, we would be able to prove the result of [4] without having to prove a variant of the local lemma. We answer this question positively.

Lemma 9. *Let* Ω *be the set of all* $(n-1)!/2$ *Hamilton cycles in* K_n *endowed with the uniform measure. For each* $A \subseteq E(K_n)$*, define* E_A *to be the event that a randomly chosen Hamilton cycle contains all edges in* A*. Define the graph* G *with vertex set* $2^{E(K_n)}$ *and an edge between* $A, B \subseteq E(K_n)$ *if* $A \neq B$ *and* A, B *are not vertex disjoint. Then* G *is a lopsidependency graph for the events* $\{E_A\}_{A \subseteq E(K_n)}$.

As we noted in Remark 3, Lemma 9 implies that any vertex-induced subgraph of G is a lopsidependency graph for the associated events.

We will present a proof of Lemma 9 by using Lemma 7. In particular, we will show that there exist resampling oracles for Hamilton cycles. Going via this route allows us to construct a polynomial time algorithm to find a rainbow Hamilton cycle in K_n, provided each colour is used at most $O(n)$ times. This yields a constructive proof of Theorem 1, with an improved value of γ. We also show that K_n contains many disjoint rainbow Hamilton cycles, provided each colour is used at most $O(n)$ times. Moreover, a set of disjoint rainbow Hamilton cycles can be found in polynomial time. Our proofs below will also find explicit lower bounds on the constants hidden in the $O(\cdot)$.

A self-contained proof of Lemma 9 using basic counting arguments is given in Appendix A. However, going this direction does not provide the means to give a polynomial time algorithm to find a rainbow Hamilton cycle in K_n. Moreover, our proof via Lemma 9 using resampling oracles implies that the graph satisfies "lopsided association", which is a stronger condition then lopsidependency.[3]

2. Proof of main lemma

Proof of Lemma 9. A resampling oracle for the event E_A is described in Algorithm 1. For the algorithm, we may assume that $A = \{x_0y_0, x_1y_1, \ldots, x_my_m\}$ where $x_0 < y_0 \leq x_1 < y_1 \leq \ldots \leq x_m < y_m$ and $m < n$. See Figure 1 for a diagram of the resampling oracle.

The following two claims together with Lemma 7 proves the lemma.

[3] A graph G is a *lopsided association* graph for the events $\{F_i\}$ if $\Pr[F_i \cap A] \geq \Pr[F_i] \cdot \Pr[A]$ for all events A such that the indicator variable of A is a non-decreasing function of the indicator variables of $\{F_j\}_{j \notin \Gamma^+(i)}$ (see also [11]). To recover the lopsidependency condition, set $A = \cap_{j \notin \Gamma^+(i)} F_j$.

Algorithm 1 Resampling oracle for Hamilton cycles

function $r_A(H)$
    if $A = \emptyset$ **then**
        return H
    end if
    Pick $x_0y_0 \in A$ arbitrarily
    $A' \leftarrow A \setminus \{x_0y_0\}$
    $K_n^c \leftarrow K_n/A'$, $H_c \leftarrow H/A'$ (discarding multiple edges)
    Pick a vertex u uniformly from $V(K_n^c) \setminus \{x_0\}$
    if $u = y_0$ **then**
        $u' \leftarrow x_0$
    else
        Set u' as the unique neighbour of u along the path from x_0 to u avoiding y_0.
            ▷ Note that if u is a neighbour of x_0 but $u \neq y_0$ then $u' = x_0$.
    end if
    $H_c' \leftarrow (H_c \setminus \{x_0y_0, uu'\}) \cup \{x_0u, y_0u'\}$
    Partition A' into vertex-disjoint paths
    Uncontract the paths in A', reversing each with probability $\frac{1}{2}$ to get H'
    return $r_{A'}(H')$
end function

Claim 10. *Let $A \subseteq E(K_n)$. If H is a uniformly random Hamilton cycle conditioned on $A \subseteq H$ then $r_A(H)$ is a uniformly random Hamilton cycle.*

Proof. Let us first assume that $A = \{xy\}$ is a singleton set, in which case the contraction steps in Algorithm 1 are trivial.

Let $H' = r_{xy}(H)$. We claim that for any edge xw,

$$\Pr[xw \in H'] = \frac{2}{n-1}. \tag{3}$$

Indeed, $xy \in H'$ if and only if u is a neighbour of x. Thus $\Pr[xy \in H'] = 2/(n-1)$. On the other hand, suppose $w \neq y$. Then

$$\begin{aligned}\Pr[xw \in H'] &= \Pr[xw \in H' \wedge xw \in H] + \Pr[xw \in H' \wedge xw \notin H] \\ &= \Pr[xw \in H' \mid xw \in H] \Pr[xw \in H] + \Pr[xw \in H' \mid xw \notin H] \Pr[xw \notin H].\end{aligned}$$

Recall that the assumption of the resampling oracle (condition 1 in Definition 6) is that H is uniformly random conditioned on $xy \in H$, we have $\Pr[xw \in H] = 1/(n-2)$. If $xw \in H$ then $xw \in H'$ (since xy is the only edge incident on x that is removed) so $\Pr[xw \in H' \mid xw \in H] = 1$. If $xw \notin H$ then $xw \in H'$ if and only if $u = w$, so $\Pr[xw \in H' \mid xw \in H] = 1/(n-1)$. Therefore

$$\Pr[xw \in H'] = \frac{1}{n-2} + \frac{n-3}{(n-1)(n-2)} = \frac{2}{n-1}$$

This completes the proof of (3).

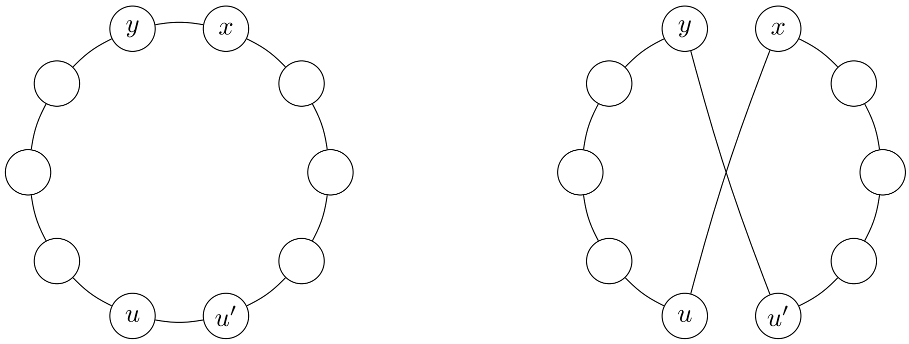

Figure 1: This figure shows a sample run of the resample oracle r_{xy}. In the graph on the left, the vertex u is chosen uniformly at random from $V(K_n) \setminus \{x\}$. Given u, the vertex u' is determined; u' is the unique neighbour of u in a path from u to x that avoids y. In the graph on the right, the edges xy and uu' have been removed and replaced with the edges xu and yu' (line 14 of the algorithm).

To show that H' is uniformly distributed over all Hamilton cycles in K_n, we fix an arbitrary Hamilton cycle $\tilde{H}$, then analyze $\Pr[H' = \tilde{H}]$. There are two cases.
Case 1: $xy \in \tilde{H}$: If $H' = \tilde{H}$ then $xy \in H'$ and $H' \setminus \{xy\} = \tilde{H} \setminus \{xy\}$. Therefore

$$\begin{aligned}\Pr[H' = \tilde{H}] &= \Pr[xy \in H' \wedge H' \setminus \{xy\} = \tilde{H} \setminus \{xy\}] \\ &= \Pr[xy \in H'] \Pr[H' \setminus \{xy\} = \tilde{H} \setminus \{xy\} \mid xy \in H']. \qquad (4)\end{aligned}$$

By assumption, H is uniformly random conditioned on $xy \in H$ and if $xy \in H'$ then $H' = H$. So $\Pr[H' \setminus \{xy\} = \tilde{H} \setminus \{xy\} \mid xy \in H'] = 1/(n-2)!$ since there are $(n-2)!$ Hamilton cycles containing xy. Hence, $\Pr[H' = \tilde{H}] = 2/(n-1)!$.
Case 2: $xy \notin \tilde{H}$: Let P, P' be the two paths from x to y in $\tilde{H}$. Let u be the unique neighbour of x in P and u' be the unique neighbour of y in P'. By equation (3)

$$\Pr[xu \in H'] = \frac{2}{n-1}. \qquad (5)$$

The event that $yu' \in H'$ given that $xu \in H'$ is the same as the event that in H, u' is the unique neighbour of u along the path from x to u that avoids y (see line 12 of Algorithm 1). Thus

$$\Pr[yu' \in H' \mid xu \in H'] = 1/(n-2). \qquad (6)$$

Finally, there are $2(n-3)!$ Hamilton cycles containing xy and uu' but only half of them have uu' in the orientation needed by line 12 of Algorithm 1. Since H is uniformly random conditioned on $xy \in H$, we have

$$\begin{aligned}\Pr[H' \setminus \{xu, yu'\}] &= \Pr[\tilde{H} \setminus \{xy, uu'\} \mid \{xu, yu'\} \subseteq H'] \\ &= 1/(n-3)!. \qquad (7)\end{aligned}$$

Multiplying (5), (6), and (7) together gives

$$\Pr[H' = \tilde{H}] = \frac{2}{(n-1)!}. \qquad (8)$$

So we conclude that H' is a uniformly random Hamilton cycle.

We now remove the assumption that A is a singleton. We argue that H' is always a uniformly random Hamilton cycle conditioned on $A' \subset H'$. Since each recursive call removes an edge from A', the lemma follows by iteratively applying this claim until $A' = \emptyset$.

Let $m = |A'| = |A| - 1$. Note that H_c, the Hamilton cycle after contracting the edges in A', is a uniformly random Hamilton cycle on K_{n-m} conditioned on $x_0y_0 \in E(H_c)$. Thus, by the singleton case, it follows that H'_c is a uniformly random Hamilton cycle on K_{n-m}. Suppose that A' forms k disjoint paths. Then there are exactly $2^{k-1}(n-m-1)!$ Hamilton cycles containing A'. Since our uncontraction step involves reversing each path with equal probability (line 16 of algorithm), this implies that if $\tilde{H}$ is any fixed Hamilton cycle such that $A \subseteq \tilde{H}$ then $\Pr[H' = \tilde{H}] = 2^{-k+1}/(n-m-1)!$, which is what we wanted to show. □

Claim 11. *The resampling oracle $r_A(H)$ does not cause any new events E_B if $B \cap A = \emptyset$.*

Proof. Observe that in Algorithm 1, any new edge that we add always contains an endpoint that intersects with some edge in A. Hence, if $B \cap A = \emptyset$ then the resampling oracle does not cause E_B. □

The previous two claims imply that Algorithm 1 is a resampling oracle for $\{E_A\}$ with respect to G. Lemma 7 imply that G is a lopsidependency graph for $\{E_A\}$. □

3. Rainbow Hamilton cycles in K_n

In this section we show that our new resampling oracles for Hamilton cycles imply constructive results for rainbow Hamilton cycles.

Theorem 12. *Fix an edge-colouring of K_n and suppose that each colour appears on at most $q = \gamma n$ edges where $\gamma = \frac{27}{1024}$. Then there exists a rainbow Hamilton cycle. Moreover, the rainbow Hamilton cycle can be found with $O(n^4)$ resampling oracle calls, in expectation.*

Proof. We will deal with the bad events E_{ef} where e, f are distinct edges of K_n with the same colour. Clearly, if all the bad events are avoided, then we have found a rainbow Hamilton cycle. Define the lopsidependency graph G where $E_{ef} \sim E_{e'f'}$ unless $(e \cup f) \cap (e' \cup f') \neq \emptyset$, i.e. unless $(e \cup f)$ and $(e' \cup f')$ are not vertex-disjoint.

For distinct edges e, f of K_n write $p_{ef} = \Pr[E_{ef}]$. If $e \cap f = \emptyset$ then $p_{ef} = \frac{4}{(n-1)(n-2)}$. If $e \cap f \neq \emptyset$ then $p_{ef} = \frac{2}{(n-1)(n-2)}$. Either way, $p_{ef} \leq \frac{4}{(n-1)(n-2)} =: p$ provided e, f are distinct edges.

For each $v \in e \cup f$, let $\mathcal{Q}_v = \{E_{e'f'} \in \Gamma^+(E_{ef}) : v \in e' \cup f'\}$. There are $(n-1)$ edges incident to v and for each edge, there are at most $q-1$ other edges that have the same colour. Hence, $|\mathcal{Q}_v| \leq (n-1)(q-1)$. Let $A_{ef} = \mathrm{Ind}(E_{ef})$ be the set of all independent sets contained in $\Gamma^+(E_{ef})$. We claim that

$$\sum_{I \in A_{ef}} \prod_{E_{e'f'} \in I} \mu_{e'f'} \leq \prod_{v \in e \cup f} \left(1 + \sum_{E_{e'f'} \in \mathcal{Q}_v} \mu_{e'f'}\right). \tag{9}$$

where $\mu_{e'f'}$ are any arbitrary nonnegative real numbers. To see this, observe that $I \subset \cup_{v \in e \cup f} \mathcal{Q}_v$ and that there is at most one event $E_{e'f'} \in I$ such that $E_{e'f'} \in \mathcal{Q}_v$. Hence, any independent set $I \subseteq \Gamma^+(E_{ef})$ can contain at most four events with at most one event in each $\mathcal{Q}_v$. Thus, any term on the left hand side of (9) also occurs on the right hand side of (9). Finally, all terms are positive, so (9) holds.

Set $\mu_{ef} = \mu = \beta p$ where β will be chosen later. Then from (9) and the fact that $|\mathcal{Q}_v| \le (q-1)(n-1)$, we have

$$\sum_{I \in A_{ef}} \prod_{E_{e'f'} \in I} \mu \le \left(1 + (q-1)(n-1)\beta \frac{4}{(n-1)(n-2)}\right)^4.$$

Since $\gamma < 1/2$, we have $\frac{q-1}{n-2} = \frac{\gamma n - 1}{n-2} = \frac{\gamma(n-1/\gamma)}{n-2} < \gamma$. Therefore, the previous expression is at most $(1+4\beta\gamma)^4$.

To finish off the proof, we can use the cluster expansion criterion of the local lemma. Set $\beta = \left(\frac{4}{3}\right)^4$. Then

$$\frac{\mu}{\sum_{I \in A_{ef}} \prod_{E_{e'f'} \in I} \mu} \ge \frac{\beta p}{(1+4\beta\gamma)^4} \ge p$$

and the cluster expansion criterion of the local lemma is satisfied.

Finally, the running time follows by Theorem 8 because $\log(1+\mu) \le \mu = O(1/n^2)$ and there are $O(n^4)$ events. □

Theorem 12 only guarantees the existence of a single rainbow C_n in K_n when no colour is used more than $\frac{27}{1024}n$ times. If we considered only a slightly larger complete graph, while maintaing the invariant that no colour is used more than $\frac{27}{1024}n$ times, then we can find many rainbow C_n.

Corollary 13. *Let $\alpha > 2$ be a constant. Suppose that an edge-colouring of $K_{\lceil \alpha n \rceil}$ uses each colour at most $\frac{27}{1024}n$ times. Let $\beta \in (2, \alpha)$. Then $K_{\lceil \alpha n \rceil}$ has at least $\left(\frac{\alpha}{\beta}\right)^n (\beta - 2)n$ rainbow C_n.*

Proof. Observe that $K_{\lfloor \beta n \rfloor}$ contains at least $(\beta-2)n$ rainbow C_n. To see this, pick a set S of n vertices from $K_{\lfloor \beta n \rfloor}$. Note that the complete subgraph induced by S contains each colour at most $\frac{27}{1024}n$ times so Theorem 12 implies that there exists a rainbow Hamilton cycle. Let $v_0 \in S$ and consider the smaller graph $K_{\lfloor \beta n \rfloor} \setminus \{v_0\}$. By the same argument, we can find a rainbow C_n in $K_{\lfloor \beta n \rfloor} \setminus \{v_0\}$. Let v_1 be a vertex in the C_n and now consider the graph $K_{\lfloor \beta n \rfloor} \setminus \{v_0, v_1\}$. Continuing this procedure gives us at least $\lfloor \beta n \rfloor - n \ge (\beta - 2)n$ rainbow C_n in $K_{\beta n}$.

Consider picking a random subset $S \subseteq V(K_{\lceil \alpha n \rceil})$ with $|S| = \lfloor \beta n \rfloor$. Let R be the number of rainbow C_n in $K_{\lceil \alpha n \rceil}$ and X be the number of rainbow C_n in K_S, the complete graph on S. As argued above $X \ge (\beta - 2)n$. The probability that a rainbow C_n from $K_{\lceil \alpha n \rceil}$ appears in K_S is bounded above by $(\beta/\alpha)^n$. By linearity of expectation, $\mathbb{E}X \le R(\beta/\alpha)^n$. Hence, $R(\beta/\alpha)^n \ge (\beta-2)n$ from which we obtain that $R \ge (\alpha/\beta)^n(\beta-2)n$. □

For example, if we set $\alpha = 6$, $\beta = 3$ in the previous corollary, then this means that K_{6n} contains $n2^n$ rainbow C_n.

As a final application, we show that it is also possible to find many *disjoint* rainbow Hamilton cycles in a bounded colouring of K_n.

Theorem 14. *Fix an edge-colouring of K_n and suppose that each colour appears on at most $q = \gamma n$ edges where $\gamma = \frac{27}{2048}$. Then there are at least $t = \gamma n$ disjoint rainbow Hamilton cycles. Moreover, the disjoint rainbow Hamilton cycles can be found with $O(n^6)$ resampling oracle calls, in expectation.*

Proof. We pick t Hamilton cycles uniformly and independently at random. We will consider events of the form E^i_{ef} and E^{ij}_e where E^i_{ef} is the event that Hamilton cycle i contains two edges e, f of the same colour and E^{ij}_e is the event that both Hamilton cycles i and j use edge e. Let $p = \frac{4}{(n-1)(n-2)}$. Note that $\Pr[E^i_{ef}] \leq p$ and $\Pr[E^{ij}_e] \leq p$. The former is proved in the proof of Theorem 12. To see the latter, observe that the probability that e is contained in a Hamilton cycle is exactly $\frac{2}{n-1}$. Since Hamilton cycles are chosen uniformly at random and independently of each other, this means that $\Pr\left[E^{ij}_e\right] \leq \frac{4}{(n-1)^2} \leq p$.

The dependency graph has three types of dependencies:

1. $E^i_{ef} \sim E^i_{e'f'}$ if $(e \cup f) \cap (e' \cup f') \neq \emptyset$;
2. $E^i_{ef} \sim E^{ij}_{e'}$ if $(e \cup f) \cap e' \neq \emptyset$; and
3. $E^{ij}_e \sim E^{i'j}_{e'}, E^{ij'}_{e'}$ if $e \cap e' \neq \emptyset$.

We first look at the neighbourhood of E^i_{ef}. Let $v \in e \cup f$. Define the sets

$$\mathcal{Q}^1_v = \{E^i_{e'f'} \in \Gamma^+(E^i_{ef}) : v \in e' \cup f'\} \text{ and}$$
$$\mathcal{Q}^2_v = \{E^{ij}_{e'} \in \Gamma^+(E^i_{ef}) : v \in e' \cup f'\}.$$

There are $n-1$ edges incident to v and for each edge, at most $q-1$ other edges can have the same colour. Hence, $|\mathcal{Q}^1_v| \leq (n-1)(q-1)$. Since there are at most t Hamilton cycles, we also have $|\mathcal{Q}^2_v| \leq (n-1)(t-1)$. Set $\mathcal{Q}_v = \mathcal{Q}^1_v \cup \mathcal{Q}^2_v$. Note that $\mathcal{Q}_v$ contains at most $(n-1)(q+t-2)$ events.

We now make two observations. The first is that $\Gamma^+(E^i_{ef}) = \cup_{v \in e \cup f} \mathcal{Q}_v$. The second is that $\mathcal{Q}_v$ induces a clique in G. This is because $\mathcal{Q}_v$ contains events of the form $E^i_{e'f'}$ or $E^{ij}_{e'}$ and v is contained in $e' \cup f'$ or e', respectively. Hence, for any two events $E, E' \in \mathcal{Q}_v$, it holds that $E \sim E'$.

Let A^i_{ef} be the set of all independent sets of $\Gamma^+(E^i_{ef})$. Then

$$\sum_{I \in A^i_{ef}} \prod_{E \in I} \mu_E \leq \prod_{v \in e \cup f} \left(1 + \sum_{E \in \mathcal{Q}_v} \mu_E\right), \tag{10}$$

where μ_E are arbitrary nonnegative real numbers and the events E are of the form $E^{i'}_{e'f'}$ or $E^{ij}_{e'}$. Set $\mu_E = \mu = \beta p$ for all events E, where $\beta > 0$ is a constant we will choose later. Then the inequality in (10) becomes

$$\sum_{I \in A^i_{ef}} \prod_{E \in I} \mu \leq \left(1 + (n-1)(q+t-2)\beta \frac{4}{(n-1)(n-2)}\right)^4.$$

Now set $t = q = \gamma n$. To satisfy the cluster expansion criterion, it suffices to choose β such that

$$\frac{\beta p}{(1 + 8\beta\gamma)^4} \geq p.$$

A calculation shows that $\beta = \left(\frac{4}{3}\right)^4$ satisfies the above inequality.

We now consider the neighbourhood of the events E_e^{ij} and show that the cluster expansion criterion remains satisfied with the above choice of β and μ. Let $v \in e$. Define the sets

$$\begin{aligned}
\mathcal{R}_{v,i}^1 &= \{E_{e'f'}^i \in \Gamma^+(E_e^{ij}) : v \in e' \cup f'\},\\
\mathcal{R}_{v,j}^1 &= \{E_{e'f'}^j \in \Gamma^+(E_e^{ij}) : v \in e' \cup f'\},\\
\mathcal{R}_{v,i}^2 &= \{E_{e'}^{ij'} \in \Gamma^+(E_e^{ij}) : v \in e'\}, \text{and}\\
\mathcal{R}_{v,j}^2 &= \{E_{e'}^{i'j} \in \Gamma^+(E_e^{ij}) : v \in e'\}.
\end{aligned}$$

Set $\mathcal{R}_{v,i} = \mathcal{R}_{v,i}^1 \cup \mathcal{R}_{v,i}^2$ and $\mathcal{R}_{v,j} = \mathcal{R}_{v,j}^1 \cup \mathcal{R}_{v,j}^2$. Using the same counting argument as before, we have $|\mathcal{R}_{v,i}| \le (n-1)(q+t-2)$ and $|\mathcal{R}_{v,j}| \le (n-1)(q+t-2)$. We also have $\cup_{v\in e}(\mathcal{R}_{v,i} \cup \mathcal{R}_{v,j}) = \Gamma^+(E_e^{ij})$. Finally, $\mathcal{R}_{v,i}$ and $\mathcal{R}_{v,j}$ induce a clique in G.

Let A_e^{ij} be the set of all independent sets of $\Gamma^+(E_e^{ij})$. The analog of (10) is then

$$\begin{aligned}
\sum_{I \in A_e^{ij}} \prod_{E \in I} \mu_E &\le \left(\prod_{v\in e}\left(1 + \sum_{E \in \mathcal{R}_{v,i}} \mu_E\right)\right)\left(\prod_{v\in e}\left(1 + \sum_{E \in \mathcal{R}_{v,j}} \mu_E\right)\right)\\
&\le \left(1 + (n-1)(q+t-2)\beta\frac{4}{(n-1)(n-2)}\right)^4.
\end{aligned}$$

The last inequality is because $\mu_E = \beta p$ for all events E. Continuing as before shows that the cluster expansion criterion is satisfied. This finishes the proof of the existential part of the claim.

Finally, the running time follows by Theorem 8 because $\log(1+\mu) \le \mu = O(1/n^2)$ and there are $O(n^5)$ events. □

4. Conclusion

We show that the set of Hamilton cycles, together with the events E_A, give a lopsidependency graph. In contrast, Albert, Frieze, and Reed [4] provided a weaker analysis of the dependencies between events involving Hamilton cycles. Our new analysis makes it simple to use the local lemma to prove theorems on rainbow Hamilton cycles in K_n. Furthermore, our efficient resampling oracles immediately lead to efficient algorithms to make these theorems constructive.

We will conclude with a few open problems. In [5], it was shown that perfect matchings also give rise to a lopsidependency graph. For what other graphs do there exist lopsidependency graphs? In other words, can we characterize a family of graphs $\mathcal{G}$ such that Lemma 9 holds with Hamilton cycles replaced with any $G \in \mathcal{G}$?

Dudek, Frieze, and Ruciński [30] extend the LLL to prove results for rainbow Hamilton cycles in complete hypergraphs. This was made algorithmic by Achlioptas and Iliopoulos [6]. However, they did not prove that the space of Hamilton cycles in complete hypergraphs gave a lopsidependency graph. Is there a lopsidependency graph? Are there resampling oracles in this case?

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Appendix A. A self-contained proof of Lemma 9

Our argument in Section 2 proved Lemma 9 by designing an efficient resampling oracle for the event E_A. A short argument in [11] then implies that the graph G satisfies the lopsidependency condition (in fact, the lopsided association condition, which is a stronger condition). In order to make this paper self-contained, we give an alternative proof of Lemma 9 which does not involve resampling oracles.

Proof of Lemma 9. Let $A, B \subseteq E(K_n)$ where $A \cap B = \emptyset$ and $\Pr[E_A], \Pr[E_B] \in (0,1)$. We will compute $\Pr[E_A \mid \overline{E_B}]$ exactly.

Suppose A contains m_A edges and consists of k_A disjoint paths. Define similar parameters m_B and k_B for the set B.

There are two things we need to count. We need to count the number of Hamilton cycles in K_n that do not contain B and the number of Hamilton cycles in K_n that do not contain B but contain A. We will split this task up into a few short claims.

Claim 15. *Let $A \subset E(K_n)$ be a set of m edges consisting of k disjoint paths. The number of Hamilton cycles that contain A is*

$$\frac{(n-m-1)!}{2} 2^k.$$

Proof. Begin by contracting A in K_n. The contracted graph has $(n-m-1)!/2$ Hamilton cycles. Each disjoint path in A has 2 orientations, so uncontracting A gives $\frac{(n-m-1)!}{2} \cdot 2^k$ Hamilton cycles containing A. □

Claim 16. *Let $A \subset E(K_n)$ be a set of m edges consisting of k disjoint paths. The number of Hamilton cycles that do not contain A is*

$$\frac{(n-m-1)!}{2} \left((n-1)_{(m-1)} - 2^k \right).$$

Here, $(n)_m$ denotes the falling factorial, i.e. $(n)_m = \frac{n!}{(n-m)!}$.

Proof. The number of Hamilton cycles in K_n is $\frac{(n-1)!}{2}$ and the number of Hamilton cycles in K_n containing A is given by Claim 16. Therefore, the number of Hamilton cycles in K_n that do not contain A is

$$\frac{(n-1)!}{2} - \frac{(n-m-1)!}{2} 2^k = \frac{(n-m-1)!}{2}\left((n-1)_{(m-1)} - 2^k\right).$$

□

Claim 17. *Let $A, B \subset E(K_n)$ be a set of m_A and m_B vertex disjoint edges consisting of k_A and k_B disjoint paths, respectively. The number of Hamilton cycles in K_n that contain A but avoid B is*

$$2^{k_A} \cdot \frac{(n-m_B-m_A-1)!}{2} \cdot \left((n-m_A-1)_{(m_B-1)} - 2^{k_B}\right).$$

Proof. Contracting A in K_n gives the complete graph K_{n-m_A} (recall that A and B are vertex disjoint). By Claim 16, the number of Hamilton cycles in K_{n-m_A} that do not contain B is

$$\frac{(n-m_A-m_B-1)!}{2}\left((n-m_A-1)_{(m_B-1)} - 2^{k_B}\right).$$

Uncontracting A gives

$$2^{k_A} \cdot \frac{(n-m_B-m_A-1)!}{2} \cdot \left((n-m_A-1)_{(m_B-1)} - 2^{k_B}\right)$$

Hamilton cycles in K_n that contain A but not B. □

Let N_1 be the number of Hamilton cycles in K_n that do not contain B and N_2 be the number of Hamilton cycles in K_n that contain A but not B. Claim 16 gives

$$N_1 = \frac{(n-m_B-1)!}{2}\left((n-1)_{(m_B-1)} - 2^{k_B}\right)$$

while Claim 17 gives

$$N_2 = 2^{k_A} \cdot \frac{(n-m_B-m_A-1)!}{2} \cdot \left((n-m_A-1)_{(m_B-1)} - 2^{k_B}\right).$$

Therefore

$$\Pr[E_A \mid \overline{E_B}] = N_2/N_1 \leq 2^{k_A}\frac{(n-m_A-1)!}{(n-1)!} = \Pr[E_A].$$

□