A GENERALIZATION OF THE CAUCHY-SCHWARZ INEQUALITY INVOLVING FOUR VECTORS

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ABSTRACT. We generalize the well-known Cauchy-Schwarz inequality to an inequality involving four vectors. Although the statement is very simple and the proof is short, it does not seem to appear elsewhere in the literature.

The well-known Cauchy-Schwarz inequality is

$$a^\top a b^\top b \;\geq\; (a^\top b)^2 \quad \forall a, b \in \mathbb{R}^n. \tag{CS}$$

Numerous variants and generalizations of this inequality are known; see for example the survey of Dragomir [1] and the book of Steele [2].

In this note we consider generalizations to four vectors. For example, the following inequalities are straightforward.

$$a^\top a b^\top b + c^\top c d^\top d \;\geq\; 2a^\top b c^\top d \qquad \forall a, b, c, d \in \mathbb{R}^n \tag{1}$$

$$a^\top a b^\top b + c^\top c d^\top d \;\geq\; 2a^\top c b^\top d \qquad \forall a, b, c, d \in \mathbb{R}^n \tag{2}$$

$$a^\top a b^\top b + c^\top c d^\top d \;\geq\; 2a^\top c b^\top d + (a^\top b)^2 + (c^\top d)^2 - (a^\top c)^2 - (b^\top d)^2 \qquad \forall a, b, c, d \in \mathbb{R}^n. \tag{3}$$

Inequality (1) follows by applying (CS) separately to a, b and c, d, then deriving $(a^\top b)^2 + (c^\top d)^2 \geq 2a^\top b c^\top d$ from the arithmetic-mean geometric-mean inequality (AMGM). Inequality (2), which appears in Dragomir's survey [1] as Theorem 6, follows by deriving $a_i^2 b_j^2 + c_i^2 d_j^2 \geq 2a_i b_j c_i d_j$ from AMGM, then summing over all i and j. Inequality (3) follows by applying (CS) to a, b and c, d, then deriving $(a^\top c)^2 + (b^\top d)^2 \geq 2a^\top c b^\top d$ from AMGM.

The purpose of this note is to prove the following inequality that superficially appears to be quite similar, but whose derivation is not as obvious.

Theorem 1.

$$a^\top a b^\top b + c^\top c d^\top d \;\geq\; 2a^\top c\, b^\top d + (a^\top b)^2 + (c^\top d)^2 - (a^\top d)^2 - (b^\top c)^2 \qquad \forall a, b, c, d \in \mathbb{R}^n. \tag{4}$$

Simple special cases.

- Setting $c = d = 0$ recovers the Cauchy-Schwarz inequality.
- Setting $d = b$, we obtain

$$a^\top a\, b^\top b \;+\; c^\top c\, b^\top b \;\geq\; 2a^\top c\, b^\top b \quad \forall a, b, c \in \mathbb{R}^n$$
$$\implies \frac{a^\top a \;+\; c^\top c}{2} \;\geq\; a^\top c \quad \forall a, c \in \mathbb{R}^n.$$

This is the "additive" form of the Cauchy-Schwarz inequality, which appears in Steele's book [2] as Eq. (1.6) and is the special case $p_i = 1$, $q_i = 0$ of Dragomir's survey [1], Theorem 6.

Now let us turn to the proof of (4). The proof unfortunately does not follow by the same method as (3) because it is not necessarily true that $(a^\mathsf{T} d)^2 + (b^\mathsf{T} c)^2 \geq 2a^\mathsf{T} c b^\mathsf{T} d$.

Proof. The Lagrange identity, which appears in Dragomir's survey [1] as Eq. (1.3) and Steele's book [2] as Eq. (3.4), states that

$$a^\mathsf{T} a b^\mathsf{T} b - (a^\mathsf{T} b)^2 \;=\; \frac{1}{2}\sum_i \sum_j (a_i b_j - a_j b_i)^2.$$

So our desired inequality is equivalent to proving the non-negativity of

$$\frac{1}{2}\sum_{i,j}(a_i b_j - a_j b_i)^2 \;+\; \frac{1}{2}\sum_{i,j}(c_i d_j - c_j d_i)^2 \;-\; 2\sum_i a_i c_i \sum_j b_j d_j \;+\; \Big(\sum_i a_i d_i\Big)^2 \;+\; \Big(\sum_i b_i c_i\Big)^2. \tag{5}$$

Note that

$$\sum_i a_i c_i \sum_j b_j d_j \;=\; \sum_{i,j} a_i c_i b_j d_j \;=\; \sum_{i,j} a_j c_j b_i d_i. \tag{6}$$

So multiplying (5) by 2, gathering the summations and using the identity (6), our goal is to prove the non-negativity of

$$\sum_{i,j}\Big((a_i b_j - a_j b_i)^2 + (c_i d_j - c_j d_i)^2 - 2a_i c_i b_j d_j - 2a_j c_j b_i d_i + 2a_i d_i a_j d_j + 2b_i c_i b_j c_j\Big). \tag{7}$$

The key to the proof is the following manipulation, which can be easily verified. For any i and j we have

$$\begin{aligned}
&(a_i b_j - a_j b_i)^2 + (c_i d_j - c_j d_i)^2 - 2a_i c_i b_j d_j - 2a_j c_j b_i d_i + 2a_i d_i a_j d_j + 2b_i c_i b_j c_j \\
&\;=\; a_i^2 b_j^2 + a_j^2 b_i^2 + c_i^2 d_j^2 + c_j^2 d_i^2 - 2a_i b_j a_j b_i - 2c_i d_j c_j d_i - 2a_i c_i b_j d_j - 2a_j c_j b_i d_i + 2a_i d_i a_j d_j + 2b_i c_i b_j c_j \\
&\;=\; (a_i b_j - a_j b_i - c_i d_j + c_j d_i)^2 + 2(a_i d_i - b_i c_i)(a_j d_j - b_j c_j).
\end{aligned}$$

Therefore (7) equals

$$\begin{aligned}
&\sum_{i,j}\Big((a_i b_j - a_j b_i - c_i d_j + c_j d_i)^2 + 2(a_i d_i - b_i c_i)(a_j d_j - b_j c_j)\Big) \\
&\sum_{i,j}(a_i b_j - a_j b_i - c_i d_j + c_j d_i)^2 \;+\; 2\cdot\Big(\sum_i (a_i d_i - b_i c_i)\Big)^2.
\end{aligned}$$

This is a sum-of-squares, and hence non-negative. $\square$

References

[1] Sever Silvestru Dragomir. A Survey on Cauchy-Buniakowsky-Schwartz Type Discrete Inequalities. *J. Inequal. Pure & Appl. Math.*, 4(3), 2003.

[2] J. Michael Steele. *The Cauchy-Schwarz Master Class.* Cambridge University Press, 2004.