

Notes on Convexity Inequalities

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In the analysis of randomized algorithms, we frequently use various inequalities to simplify expressions or make them easier to work with. These inequalities are usually obvious by plotting them, and proving them just uses basic calculus.

1 Facts from Convex Analysis

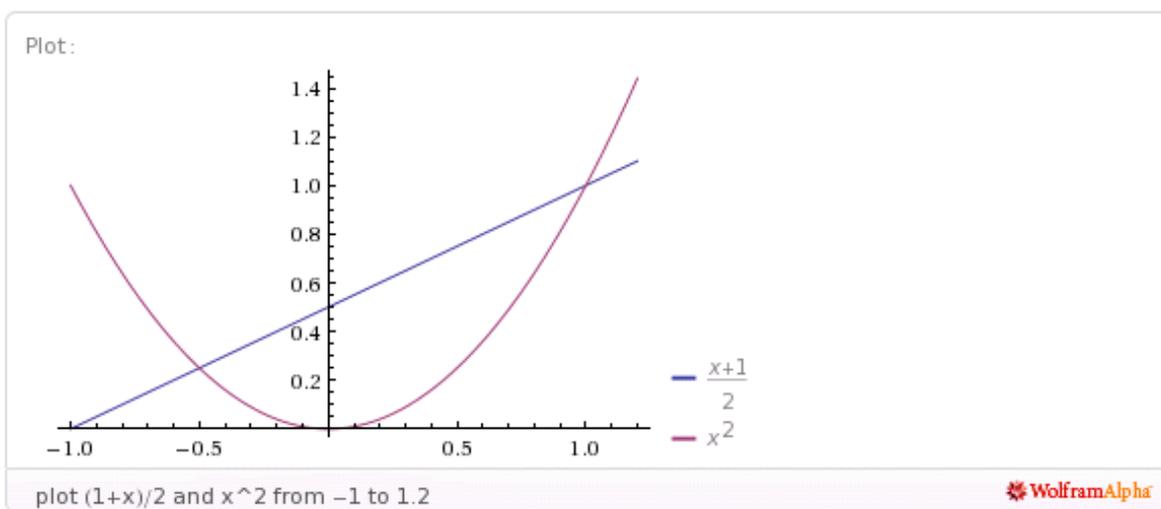
Let $f : S \rightarrow \mathbb{R}$ be a function defined on an interval $S \subseteq \mathbb{R}$.

Definition 1 We say that f is convex on S if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in S$ and all $\lambda \in [0, 1]$.

Geometrically, this says that the line segment between the points $(x, f(x))$ and $(y, f(y))$ lies above the function f . The following example illustrates that $f(x) = x^2$ is convex.



An equivalent statement of this definition is as follows.

Fact 2 Suppose $f : S \rightarrow \mathbb{R}$ is convex. Then, for any points $x, y \in S$ with $x < y$, we have

$$f(z) \leq \frac{f(y) - f(x)}{y - x} \cdot (z - x) + f(x)$$

for all $z \in [x, y]$.

For sufficiently nice functions, one can easily determine convexity by looking at its second derivative.

Fact 3 Suppose $f : S \rightarrow \mathbb{R}$ is twice differentiable. Then f is convex if and only if the second derivative f'' is non-negative on the interior of S .

In our example above we used $f(x) = x^2$. Its second derivative is $f''(x) = 2$, which is non-negative, so f is convex.

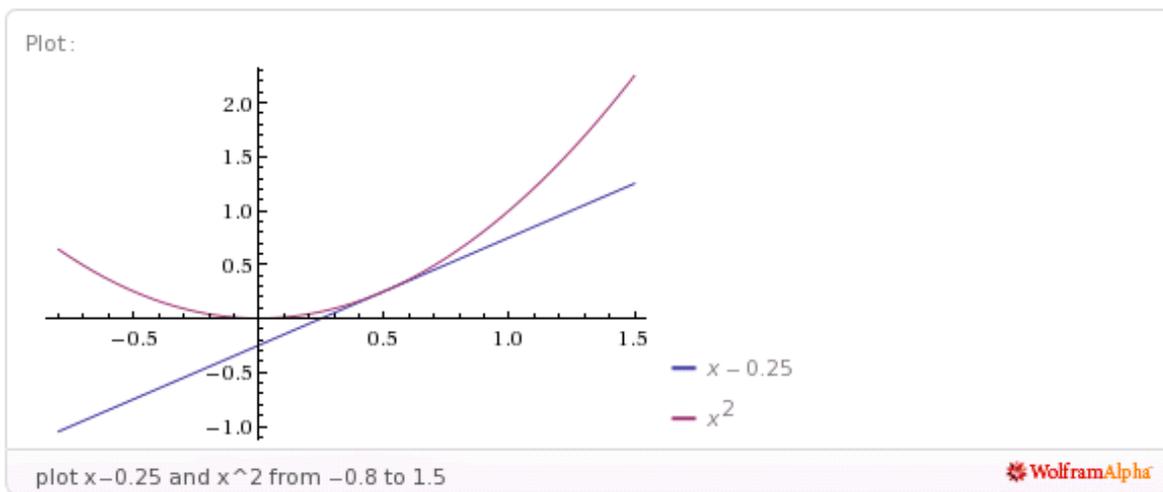
The next fact says that, for convex functions, the linear approximation at any point lies beneath the function.

Fact 4 (Subgradient Inequality) Suppose $f : S \rightarrow \mathbb{R}$ is convex, and f is differentiable at a point $x \in S$. Then

$$f(y) \geq f(x) + f'(x)(y - x),$$

for any point $y \in S$.

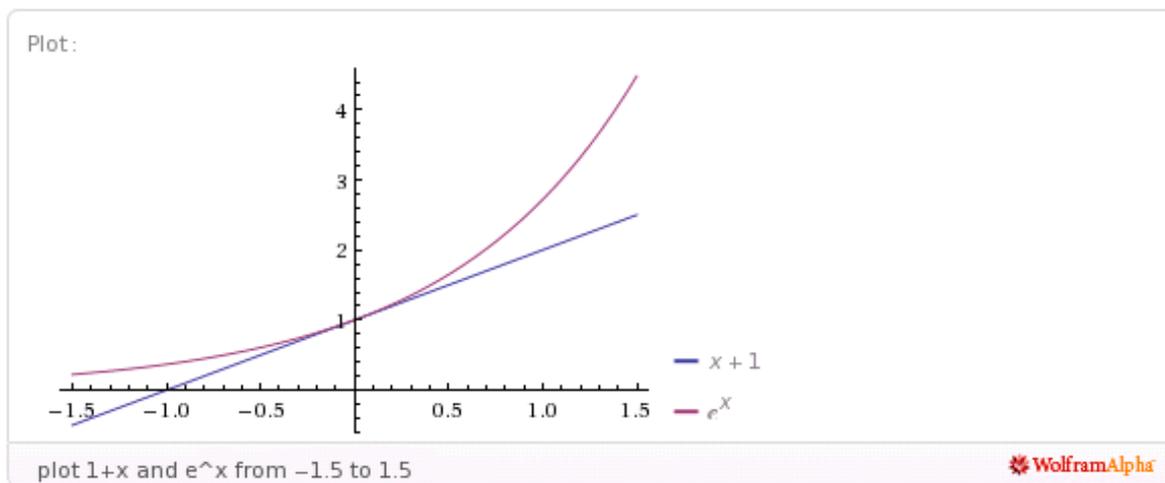
The following example illustrates this for $f(x) = x^2$ at the point $x = 0.5$.



2 The Inequalities

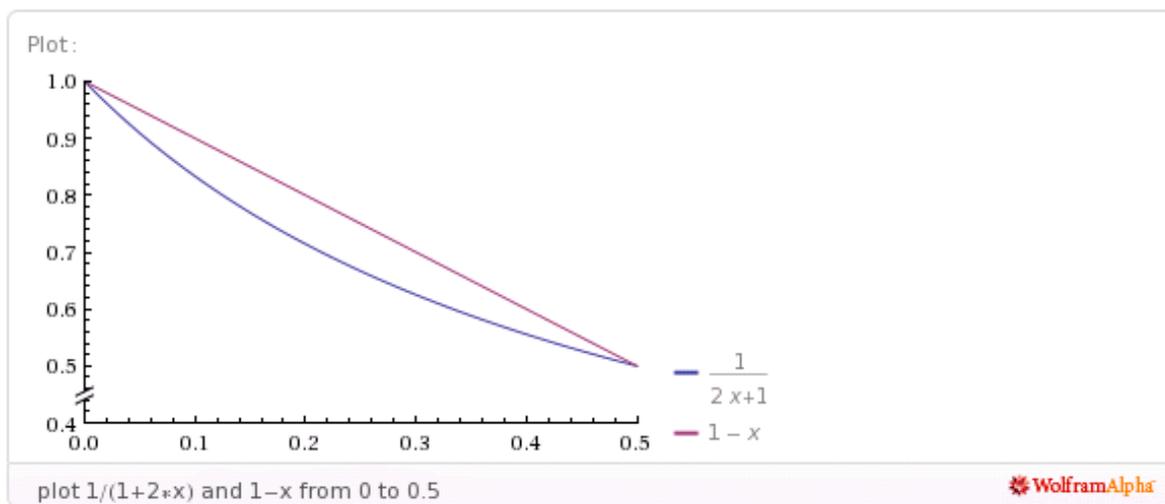
- **Inequality 1.** $1 + y \leq e^y$.

Convexity of e^y follows from Fact 3 since its second derivative is e^y , which is non-negative. The inequality follows by applying Fact 4 at the point $x = 0$.



- **Inequality 2.** $1/(1 + 2z) \leq 1 - z$ for all $z \in [0, 1/2]$.

Convexity of $1/(1 + 2z)$ on the set $S = (0, \infty)$ follows from Fact 3 since its second derivative is $8/(1 + 2z)^3$, which is non-negative. The inequality follows by applying Fact 2 at the points $x = 0$ and $y = 0.5$.



- **Inequality 3.** Fix any $c > 0$. Then $c^z \leq 1 + (c - 1)z$ for all $z \in [0, 1]$.

The second derivative of c^z is $c^z \cdot \ln^2(c)$, which is non-negative, so c^z is convex. The inequality follows by applying Fact 2 at the points $x = 0$ and $y = 1$.

