Lecture 6 — January 21, 2013

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In the previous lecture, we discussed max flow problems. Today, we consider the Travelling Salesman Problem (TSP), and several restricted variants. Although these restrictions of TSP are also NP-hard, we will see a variant of TSP whose optimum solution we can approximate with at most logarithmic error.

1 Max Flow, Min Weight Flow

Max Flow:

We're given a directed graph $G = (V, A)$ and numbers $c_a \in \mathbb{R}^+ \quad \forall a \in A$

$$
\begin{aligned}
\max \quad & \sum_{a \in \delta^+(s)} x_a \\
\text{such that} \quad & \sum_{a \in \delta^+(v)} x_a - \sum_{a \in \delta^-(v)} x_a = 0 && \forall v \in V - \{s, t\} \\
& 0 \le x_a \le c_a && \forall a \in A
\end{aligned}
$$

Min Weight Flow:

Suppose further that the edges have weights $\{w_a : a \in A\}$, and f is a non-negative real number.

$$
\begin{aligned}
\min \quad & \sum_{a \in A} w_a x_a \\
\text{such that} \quad & \sum_{a \in \delta^+(s)} x_a = f \\
& \sum_{a \in \delta^+(v)} x_a - \sum_{a \in \delta^-(v)} x_a = 0 && \forall v \in V \backslash \{s, t\} \\
& 0 \le x_a \le c_a && \forall a
\end{aligned}
$$

If f and c_a $\forall a \in A$ are integral and the problems are feasible, both problems have integral optimal solutions.

2 Travelling Salesman Problem

1. Let G = (V, E) be a graph, $w_e \quad \forall e \in E$. Find a minimum weight cycle that visits each vertex exactly once. This is NP-hard (without weights, this is the Hamiltonian cycle problem).

2. Assume the input graph is the complete graph K_n. This is still NP-hard, by the following reduction. Given any G with $n = |V|$, put weights on K_n as follows:

$$w_e = \begin{cases} 1 & \text{if } e \in G \\ n^2 & \text{if } e \notin G \end{cases}$$

   Then the optimal solution has cost n if and only if G has a Hamiltonian cycle.

   Even approximations are still NP-hard. Say we're looking for a Hamiltonian cycle in K_n of cost $\leq \alpha *$ optimum. Again, given G with $n = |V|$, put weights on K_n as follows:

$$w_e = \begin{cases} 1 & \text{if } e \in G \\ n^2\alpha & \text{if } e \notin G \end{cases}$$

   Then the optimum solution is n if G has a Hamiltonian cycle, and $\geq \alpha n^2$ if G doesn't have a Hamiltonian cycle.

3. Assume the weights satisfy the following properties:

   Triangle inequality: $\forall a, b, c, w_{ab} \leq w_{ac} + w_{cb}$

   Non negativity: $\forall a, b, w_{ab} \geq 0$

   Symmetry: $w_{ab} = w_{ba}$

   This variant is still NP-hard, even if the metric is just distances for points in $\mathbb{R}^2$ [Garey, Graham, Johnson 1976], [Papadimitriou 1977].

   If the metric is $\mathbb{R}^d$ for some constant d, we can approximate the optimum solution to within any factor $1 + \varepsilon$ [Arora 1996], [Mitchell 1996]. For general metrics, we can approximate within a factor of 1.5 [Christofides 1976]. For a much more detailed history, see Vygen's recent survey at `http://www.or.uni-bonn.de/~vygen/files/optima.pdf`.

3 Directed Graphs

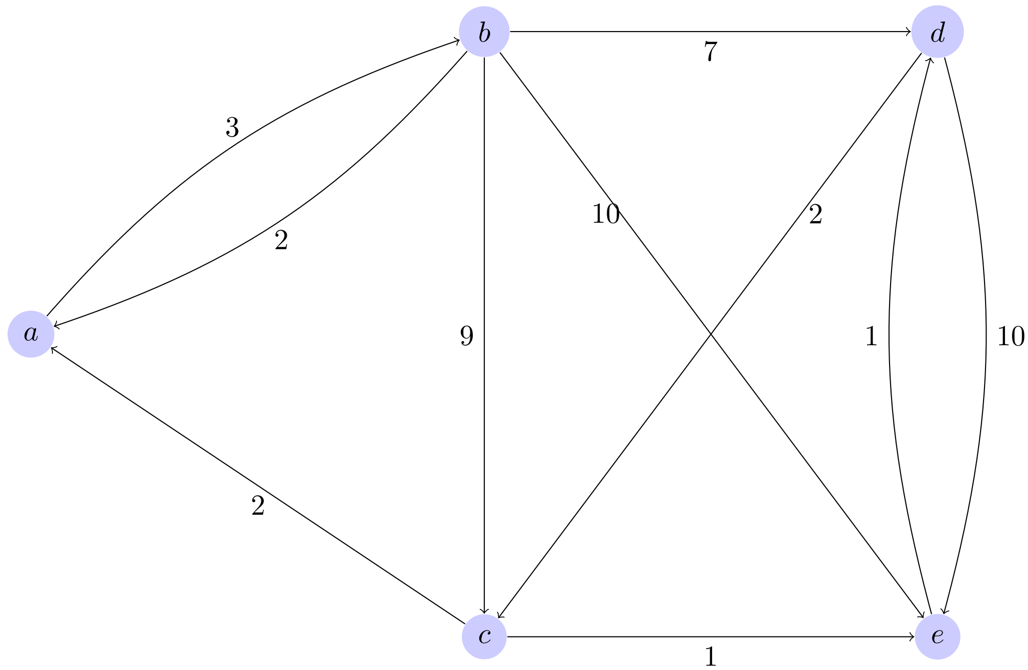
For the following, we're given the following data:

$$\begin{aligned} &V = \{1, 2, \ldots, n\} \\ &w : V \times V \to \mathbb{R}^+ \\ &\forall i, j \in V \quad w_{ij} \geq 0 \\ &\forall i, j, k \in V \quad w_{ij} \leq w_{ik} + w_{kj} \end{aligned}$$

Despite these restrictions, the optimum solution to the TSP problem is still hard to approximate to within 1.01.

We can approximate to within $\log_2(n) \cdot$ optimum [Frieze, Galbiati, Maffioli 1982]. Later: approximate to within $O\left(\frac{\log n}{\log \log n}\right)$ It is an open question if this can be improved to $O(1)$.

Consider the following example:



Note that this is not a complete example; there aren't edges between all vertices. We can use metric completion to compute weights to assign to edges between all vertices to complete this example.

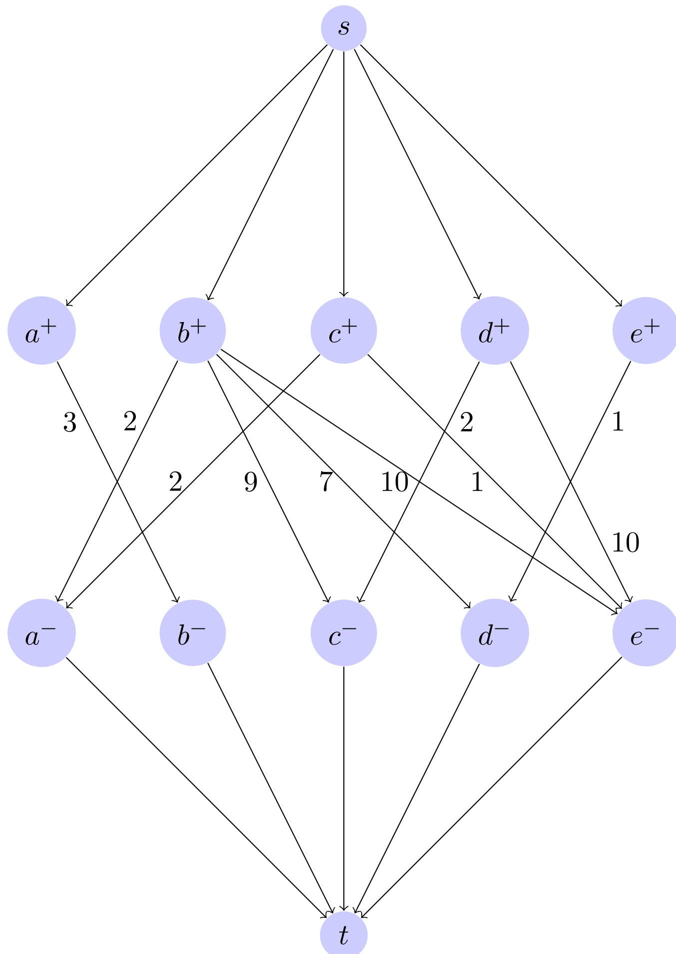
Instead of trying to solve TSP directly, let's find a collection of cycles that go through every vertex exactly once.

Note: minimum weight of a cycle cover $\leq$ minimum weight of a Hamiltonian cycle. We can reduce this problem to a minimum weight flow problem as follows:

For each vertex i, define new vertices i^+ and i^-. For each edge ij with weight w_{ij} in the original graph, define an edge i^+j^- in our new graph with weight w_{ij}. Let the desired flow value $f = |V|$, and $c_a = 1 \quad \forall a \in A$

Every vertex receives and sends 1 unit of flow. Because c and f are integral, there is an integral optimum solution: every arc sends either 1 or 0 units of flow.

For the example graph, the associated min weight graph is the following graph:

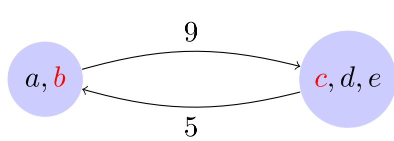


Given we've solved the min weight flow problem, let $C = \{ab : x_{ab} = 1, a, b \in V\}$. This is a cycle cover. Similarly, given a cycle cover we can extract a 0-1 flow.

This gives a bijection between cycle covers and 0-1 flows that preserves weights. So we can solve min cost cycle cover in polynomial time.

Idea for TSP:

1. Find a minimum weight cycle cover.
2. Pick a designated vertex from each cycle.
3. Make a new graph where each cycle has been shrunk to a supernode.
4. Weight of an arc between two supernodes is the weight of shortest path between two designated nodes in original graph



Claim 3.1. Minimum weight cycle cover $\leq$ minimum weight Hamiltonian cycle in the original graph

Proof. A Hamiltonian cycle is a sequence of paths between designated graph nodes. Every arc in the new graph has weight the shortest path between designated nodes. A Hamiltonian cycle visits every designated node, therefore $weight(HC) \geq minwt(CC)$.

□

Claim 3.2. The number of recursive steps is less than or equal to $\log_2(n)$.

Proof. Each cycle has length ≥ 2, so the number of vertices halves at each step. □

The CC in the shrunken graph gives paths in the original graph. Concatenate all these paths. The resulting walk is strongly connected and Eulerian (for every vertex, in-degree = out-degree). An Eulerian graph has a closed walk that visits every edge exactly once. To get a HC, look at the sequence of vertices in the closed walk, and keep the first occurence of each vertex.

Weight(new subsequence) $\leq$ weight(closed walk) by the triangle inequality.