UBC CPSC 536N: Sparse Approximations	Winter 2013

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1 Primal and Dual LPs

We consider linear programs of the form

$$\max \left\{ c^\mathsf{T} x \,:\, Ax \le b \right\}.$$

The dual is

$$\min \left\{ b^\mathsf{T} y \,:\, A^\mathsf{T} y = c,\ y \ge 0 \right\}.$$

Theorem 1.1 (Weak Duality). Let x be feasible for the primal and let y be feasible for the dual. Then:

- $c^\mathsf{T} x \le b^\mathsf{T} y$, and
- if $c^\mathsf{T} x = b^\mathsf{T} y$ then both x and y are optimal.

Theorem 1.2 (Strong Duality part 1). Assume that primal and dual are both feasible. Let z^* = optimal value of primal and w^* = optimal value of dual then $z^* = w^*$

Proof. Suppose toward contradiction that $z^* < w^*$ then this system of inequalities is unfeasible:

$$\begin{cases} \begin{array}{rcrcr} Ax & & & \le & b \\ & & A^\mathsf{T} y & \le & c \\ & - & A^\mathsf{T} y & \le & -c \\ & - & Iy & \le & 0 \\ -c^\mathsf{T} x & + & b^\mathsf{T} y & \le & 0 \end{array} \end{cases}$$

By Farkas lemma the system $Ax \le b$ is infeasible iff $\exists y \ge 0$ such that $A^\mathsf{T} y = 0$, $b^\mathsf{T} y < 0$. Let $y = [s, t^+, t^-, u, v]$. Define $t = t^+ - t^-$:

$$\begin{cases} A^\mathsf{T} s - vc = 0 \\ At - u + vb = 0 \\ b^\mathsf{T} s + c^\mathsf{T} t < 0 \end{cases}$$

- **Case 1**: $v = 0, A^\mathsf{T} s = 0$. Let y^* be any feasible dual solution. Let x be any feasible primal solution, then $\forall \alpha \ge 0$ $y^* + \alpha s$ is dual feasible, Also $At = u \ge 0$ so $x^* - \alpha t$ is primal feasible $A(x^* - \alpha t) = Ax^* - \alpha At \le b$.

  $$\forall \alpha \ge 0 : c^\mathsf{T}(x^* - \alpha t) \le b^\mathsf{T}(y^* + \alpha s) \iff c^\mathsf{T} x^* - b^\mathsf{T} y^* \le \alpha(b^\mathsf{T} s + c^\mathsf{T} t) \overset{\alpha \to \infty}{\longrightarrow} -\infty$$

  Contradiction.

- **Case 2**: $v > 0$. Replace $s \leftarrow s/v$, $t \leftarrow t/v$, $u \leftarrow u/v$. Then $\exists\ s, u \geq 0$:

$$\begin{cases} A^\mathsf{T} s = c \\ At - u = -b \\ b^\mathsf{T} s + c^\mathsf{T} t < 0 \\ b^\mathsf{T} s < c^\mathsf{T}(-t) \end{cases} \Rightarrow \begin{cases} At \geq -b \\ A(-t) \leq b \end{cases} \Rightarrow -t \text{ is primal feasible and } s \text{ is dual feasible.}$$

Contradicts weak duality.

$\square$

Theorem 1.3 (Strong Duality, part 2). If primal has an optimal solution, so does dual.

Proof. By Weak Duality and Fundamental Theorem of Linear Programming, dual either is infeasible or has optimal solution. Suppose that it is infeasible and $\{A^\mathsf{T} y = c,\ y \geq 0\}$ has no solution. By Farkas lemma, $\exists u$ s.t. $Au \geq 0$, $c^\mathsf{T} u \leq 0$. Let x^* be an optimal solution of primal. so $x^* - \alpha u$ is feasible for primal. $\forall \alpha \geq 0$, $c^\mathsf{T}(\alpha^* - \alpha u) \overset{\alpha\to\infty}{\longrightarrow} \infty$. Contradicts that the primal has optimal solution so dual cannot be infeasible . $\square$

2 Variants of Farkas' Lemma

The System	$Ax \leq b$	$Ax = b$
has no solution $x \geq 0$ iff	$\exists y \geq 0,\ A^\mathsf{T} y \geq 0,\ b^\mathsf{T} y \leq 0$	$\exists y \in \mathbb{R}^n, A^\mathsf{T} y \geq 0,\ b^\mathsf{T} y < 0$
has no solution $x \in \mathbb{R}^n$ iff	$\exists y \geq 0,\ A^\mathsf{T} y = 0,\ b^\mathsf{T} y \leq 0$	$\exists y \in \mathbb{R}^n, A^\mathsf{T} y = 0,\ b^\mathsf{T} y < 0$

We will prove that the system $Ax \leq b$ has no solution $x \in \mathbb{R}^n$ iff $\exists y \geq 0,\ A^\mathsf{T} y = 0,\ b^\mathsf{T} y \leq 0$.

Lemma 2.1. Exactly one of the following holds:

- There exists $x \in \mathbb{R}^n$ satisfying $Ax \leq b$
- There exists $y \geq 0$ satisfying $y^\mathsf{T} A = 0$ and $y^\mathsf{T} b < 0$

To prove this lemma, we require the following result which we prove later.

Lemma 2.2. Let $Q = \{\ (x_1, \cdots, x_n)\ :\ Ax \leq b\ \}$. There exists a polyhedron $Q' = \{\ ({x'}_1, \cdots, {x'}_{n-1})\ :\ A'x' \leq b'\ \}$ satisfying:

- Q is non-empty $\iff$ Q' is non-empty.
- Every inequality defining Q' is a non-negative linear combination of the inequalities defining Q.

Proof. First case, suppose x exists, we show that y cannot exists. By contradiction assume that both x and y exist, then:

$$0 = 0x = y^\mathsf{T} Ax \leq y^\mathsf{T} b < 0$$

which is a contradiction.

Second case, suppose that no solution x exists. By induction we construct the solution y for the second equation. It is trivial for $n = 0$ so let $n \geq 1$. Using the Lemma **??** we can get an equivalent system $A'x' \leq b'$ where $(A'|0) = MA$ and $b' = Mb$ for some non-negative matrix M. We assume that $Ax \leq b$ has no solution, so $A'x' \leq b'$ has no solution. By induction $\exists y' \geq 0$ such that ${y'}^{\mathsf{T}}b' < 0$. Define $y = M^{\mathsf{T}}y'$:

$$\begin{cases} y \geq 0 \text{ since } y' \geq 0 \text{ and } M \text{ is non-negative.} \\ y^{\mathsf{T}}A = {y'}^{\mathsf{T}}(A'|0) = 0 \\ y^{\mathsf{T}}b = {y'}^{\mathsf{T}}Mb = {y'}^{\mathsf{T}}b' < 0 \end{cases}$$

□

3 Fourier-Motzkin Elimination

3.1 2D system of inequalities

Consider the polyhedron

$$Q = \{ \ (x, y) \ : \ -3x + y \leq 6, x + y \leq 3, -y - 2x \leq 5, x - y \leq 4 \ \}.$$

Given x, for what values of y is (x, y) feasible?

- Need $y \leq 3x + 6$, $y \leq -x + 3$, $y \geq -2x - 5$ and $y \geq x - 4$
- *i.e*, $y \leq \min\{3x + 6, -x + 3\}$ and $y \geq \{-2x - 5, x - 4\}$
  · For $x = 0.8$, (x, y) is feasible if $y \leq \min\{3.6, 3.8\}$ and $y \geq \max\{-3.4, -4.8\}$
  · For $x = -3$, (x, y) is feasible if $y \leq \min\{-3, 6\}$ and $y \geq \max\{1, -7\}$ which is impossible.
- such y exists $\iff$ $\max\{-2x - 5, x - 4\} \leq \min\{3x + 6, -x + 3\}$ $\iff$ the following inequalities are solvable:

$$Q' = \left\{ \begin{array}{rl} -2x - 5 & \leq 3x + 6 \\ x - 4 & \leq 3x + 6 \\ -2x - 5 & \leq -x + 3 \\ x - 4 & \leq -x + 3 \end{array} \right\} \equiv \left\{ \begin{array}{rl} -5x & \leq 11 \\ -2x & \leq 10 \\ -x & \leq 8 \\ 2x & \leq 7 \end{array} \right\} \equiv \left\{ \begin{array}{ll} x & \geq -11/5 \\ x & \geq -5 \\ x & \geq -8 \\ x & \leq 7/2 \end{array} \right\}$$

In this example it is easy to see that Q is non-empty $\iff$ Q' is non-empty.

For a generalization of this suppose that we are given a set $Q = \{ \ (x_1, x_2, \cdots \ x_n) \ : \ Ax \leq b \ \}$ and we want to find set $Q' = \{ \ ({x'}_1, \cdots, {x'}_{n-1}) \ : \ A'x' \leq b' \ \}$ satisfying $(x_1, \cdots, x_{n-1}) \in Q \iff \exists x_n$ s.t. $(x_1, \cdots, x_{n-1}, x_n) \in Q$. Q' is called projection of Q. Fourier-Motzkin Elimination is a procedure for producing Q' from Q. This gives us an (inefficient) algorithm for solving systems of inequalities and hence for solving LPs too.

3.2 Fourier-Motzkin Elimination

We now prove the main result on Fourier-Motzkin elimination. This is a strengthening of Lemma **??**.

Lemma 3.1. Let $Q = \{\ (x_1, \cdots, x_n)\ :\ Ax \le b\ \}$. We can construct $Q' = \{\ ({x'}_1, \cdots, {x'}_{n-1})\ :\ A'x' \le b'\ \}$ satisfying:

- $(x_1, \cdots, x_{n-1}) \in Q' \iff \exists x_n\ s.t\ (x_1, \cdots, x_{n-1}, x_n) \in Q$
- Every inequality defining Q' is a non-negative linear combination of the inequalities defining Q.

It is clear from the first condition that Q is non-empty iff Q' is non-empty

Proof. Put inequalities of Q in three groups $Z = \{\ i : a_{i,n} = 0\ \}$, $P = \{\ j : a_{j,n} > 0\ \}$ and $N = \{\ k : a_{k,n} < 0\ \}$. Without loss of generality we assume that $\forall j \in P,\ a_{j,n} = 1$ and $\forall k \in N,\ a_{k,n} = -1$. For any $x \in \mathbb{R}^n$, let $x' \in \mathbb{R}^{n-1}$ denote the vector obtained by deleting coordinate x_n.

The polyhedron Q' is defined as follows:

$$Q\ := \left\{ \begin{array}{rcll} {a_i}'x' & \le & b_i & \forall i \in Z \\ {a_j}'x' + {a_k}'x' & \le & b_j + b_k & \forall j \in P,\ k \in N \end{array} \right\}.$$

This proves the second part of the lemma and $\Leftarrow$ direction from the first part: every constraint of Q' is a non-negative linear combination of constraints from Q with n^{th} coordinate to 0 thus for every $x \in Q$, x' satisfies all inequalities defining Q'.

To prove the $\Rightarrow$ direction of the first part note that $\forall j \in P$, $\forall k \in N$ ${a_k}'x' - b_k \le b_j - {a_j}'x' \Rightarrow \max_{k \in N}\{{a_k}'x' - b_k\} \le \min_{j \in P}\{b_j - {a_j}'x'\}$. Let $x_n = \max_{k \in N}\{{a_k}'x' - b_k\}$ and $x = (x'_1, \cdots, x'_{n-1}, x_n)$ then by definition of x and since $a_{k,n} = -1$ we have that $\forall k \in N$ $a_k x - b_k = {a_k}'x' - x_n - b_k$. Also by definition of x_n and since ${a_k}'x' - b_k \le x_n$ we have that $\forall k \in N$ ${a_k}'x' - x_n - b_k \le 0$. Then:

$$\left. \begin{array}{l} \forall k \in N : a_k x - b_k = {a_k}'x' - x_n - b_k \le 0 \\ \forall j \in P : b_j - a_j x = b_j - {a_j}'x' - x_n \ge 0 \\ \forall i \in Z : a_i x = {a_i}'x' \le b_i \end{array} \right\} \Rightarrow x \in Q$$

□