UBC CPSC 536N: Sparse Approximations	Winter 2013

Lecture 1 — January 2, 2013

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This lecture has only abbreviated scribe notes as most of the material is in the slides and the notes of Goemans.

1 Primal and Dual LPs

We consider linear programs of the form

$$\max \left\{ c^{\mathsf{T}} x \,:\, Ax \leq b \right\}.$$

The dual is

$$\min \left\{ b^{\mathsf{T}} y \,:\, A^{\mathsf{T}} y = c,\ y \geq 0 \right\}.$$

Theorem 1.1 (Weak Duality). Let x be feasible for the primal and let y be feasible for the dual. Then:

- $c^{\mathsf{T}} x \leq b^{\mathsf{T}} y$, and
- if $c^{\mathsf{T}} x = b^{\mathsf{T}} y$ then both x and y are optimal.

2 Fundamental Theorem of Linear Programming

Theorem 2.1. Every linear program has exactly one of the following properties.

- It is infeasible,
- It is unbounded,
- It has an optimal solution.

Proof. The key point of this theorem is that if $\sup \left\{ c^{\mathsf{T}} x \,:\, Ax \leq b \right\}$ is some finite value v then the supremum must be achieved. Suppose otherwise; we will show a contradiction.

Let the matrix A has size $m \times n$. If the supremum is not achieved then the system

$$\begin{pmatrix} A \\ -c^{\mathsf{T}} \end{pmatrix} x \;\leq\; \begin{pmatrix} b \\ -v \end{pmatrix}$$

has no solution. By Farkas' lemma, there exists a vector $w \geq 0$ such that

$$w^{\mathsf{T}} \begin{pmatrix} A \\ -c^{\mathsf{T}} \end{pmatrix} x \;=\; 0 \qquad \text{and} \qquad w^{\mathsf{T}} \begin{pmatrix} b \\ -v \end{pmatrix} \;<\; 0.$$

Let us write $w = \binom{u}{\alpha}$ where $u \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$. Then we have

$$\begin{aligned} u &\geq 0 \\ \alpha &\geq 0 \\ A^\mathsf{T} u &= \alpha c \\ u^\mathsf{T} b &< \alpha v \end{aligned}$$

Case 1: Suppose $\alpha > 0$. Let $y = u/\alpha$. Then $A^\mathsf{T} y = c$, $y \geq 0$ so y is feasible for the dual LP. Also $b^\mathsf{T} y < v$ so there exists feasible x with $c^\mathsf{T} x > b^\mathsf{T} y$. This is a contradiction because x and y violate the weak duality theorem (Theorem 1.1).

Case 2: Suppose $\alpha = 0$. Then $u \geq 0$ satisfies $A^\mathsf{T} u = 0$ and $u^\mathsf{T} b < 0$. By Farkas' lemma again, the system $Ax \leq b$ has no solution, which contradicts our assumption that the primal LP is feasible. $\square$