UBC CPSC 536N: Expander Graphs Winter 2013

Lecture 17 — March 13, 2013

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This lecture is about the expanders graphs. Before going into detail, first we give a brief introduction about these graphs:

- Originally studied in information and coding theory [Gallager 1963, Pinsker 1973, Margulis 1973].
- An important concept for early days of Theoretical Computer Science (TCS). Some related works are: complexity lower bounds [Valiant 1976, Abelson 1979], parallel algorithms [Ajtai, Komlos, and Szemeredi 1983], communication networks [Pippenger 1977], and recycling randomness [Ajtai, Komlos, and Szemeredi 1987].
- Used often as bad examples, such as metric geometry [London, Linial, and Rabinovich 1995], Nick Harvey's work: complicated submodular functions [Balcan and Harvey 2011].
- Used also in many positive results, including:
  1. Linear time error correcting codes [Sipser and Spielman 1994, Spielman 1995].
  2. Proof of Probabilistically Checkable Proofs (PCP) theorem [Dinur 2006].

After describing the importance of expander graphs, we now explore them in more detail:

1 What is an expander?

First let us be informal. Being sparse (i.e., n vertices and $\Theta(n)$ edges) is a key property of expander graphs. The other key property can be described in four roughly-equivalent ways. Expander graphs:

- Are very well connected.
- Behave a lot like a random d-regular graph, or in another words, have pseudo-randomness hiding inside.
- Have eigenvalues very similar to the complete graphs' ones.
- Have random walks which "mix rapidly" (i.e., you will get almost no information about the starting point given the position at step t of the random walk).

2 How to construct expanders?

There are many approaches for constructing the expander graphs, including:

1. Showing that a random d-regular graph works with high probability which, depending on quality of parameters, could be easy [Pinsker 1973] or hard [Friedman 2003].

2. Using explicit construction. "Ramanujan graphs" are key examples for this kind of construction. They essentially are optimal and give the best possible parameters of any expanders [Lubotzky, Philips, and Sarnak 1986]. The issue with this approach is that although it is usually easy to write down the graphs, the proofs are usually difficult and needs some sophisticated math (Fourier analysis, number theory, representation theory, etc.)
   As a particular example (which is not a Ramanujan graph), let $V = \mathbb{Z}_p$ and p be a prime. Connect vertex $x \in V$ to $x-1$, $x+1$, and x^{-1} (taking $0^{-1} = 0$). This is an expander.

3. Using combinatorial construction. Zigzag graphs are considered as good examples for the combinatorial construction [Reingold, Vadhan, and Wigderson 2002]. This method is different from the explicit construction method, in a way that here, we can't *explicitly* identify the expander graph. In fact we should use some recursive construction to make the graph. Here, we want an algorithm running in $polylog(n)$ time to give all neighbours of a given vertex. This technique is very useful, since the proofs are much easier.

4. Using polynomial time algorithm [Ajtai 1994, $\cdots$] which is as follows: given n, in $poly(n)$ time, we want to write down an expander on n vertices. In this lecture, we are going to address this approach.

2.1 Constructing the Expanders with Polynomial Time Algorithm

Definition 2.1. Graph $G = (V, E)$ has edge-expansion c if

$$\underbrace{|\delta(U)|}_{=\chi(U)^T L_G \chi(U)} \geq c|U| \qquad \forall U \subseteq V, U \leq \frac{|V|}{2}.$$

Definition 2.2. Family $\{G_n\}$, $|V(G_n)| = n$, is a family of expanders if

1. G_n has edge-expansion c, which c is constant and independent of n.
2. number of edges of G_n is $O(n)$.

Let K be a complete graph on n vertices. Then $L_K = n \cdot I - J$ where J is all-one matrix. Now let look at J:

$J = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$ is a rank-1 matrix and since it has only one non-zero eigenvalue. So using trace, we have:

$$\begin{aligned} trace(J) = n \Rightarrow eig(J) &= \{n, 0, \dots, 0\} \\ eig(L_k) &= \{n - eig(J)\} = \{0, n, \dots, n\} \\ \Longrightarrow \quad L_k &= n \cdot I_{im(L_K)}, \text{ where } im(L_K) = span(\vec{1})^{\perp}. \end{aligned}$$

Now that we know the eigenvalues of the complete graph, we are equipped to define the expansion of an arbitrary graph.

Consider an arbitrary graph $G = (V, E)$, $|V| = n$, $|E| = m$, $trace(L_G) =$ sum of degrees $= 2m$. Assume G is connected, then there exist $n-1$ non-zero eigenvalues, so the average of the non-zero ones is $\frac{2m}{n-1}$. For example, the complete graph has $m = \frac{n(n-1)}{2}$ and the average of its non-zero eigenvalues is $\frac{2m}{n-1} = n$. Moreover, they are extremely concentrated: they equal n.

So this motivates our definition of an expander: first G should be sparse (i.e., $m = \Theta(n)$), and its non-zero eigenvalues should be very close to their average, which is $\frac{2 \cdot \Theta(n)}{n-1} = \Theta(1)$. Formally,

Definition 2.3. G is a **spectral expander** if

1. $m = \Theta(n)$
2. its minimum non-zero eigenvalue is $\Theta(1)$. Equivalently, $L_G \succeq c \cdot I_{im(L_G)}$ for some positive constant c. As another equivalent for this condition we can write

$$x^T L_G x \ \geq \ c \cdot x^T x \qquad \forall x \in \mathbb{R}^n,\ x^T \vec{1} = 0.$$

Note that this last condition is quite similar to our definition of edge-expansion, which was

$$x^T L_G x \ \geq \ c \cdot x^T x \qquad \forall x \in \{0,1\}^n,\ x^T \vec{1} \leq n/2.$$

At this point, we investigate the relationship between spectral expansion and edge-expansion. Suppose G is connected and has spectral expansion c, then:

$$\begin{aligned} kernel(L_G) &= span(\vec{1}) \\ I_{ker(L_G)} &= \frac{\vec{1}}{\sqrt{n}} \cdot \frac{\vec{1}^T}{\sqrt{n}} \quad \left(= \frac{J}{n}\right) \\ I_{im(L_G)} &= I - I_{ker(L_G)} = I - \frac{\vec{1} \cdot \vec{1}^T}{n} \end{aligned}$$

Fix any $U \subseteq V$, $|U| \leq \frac{n}{2}$.

$$\begin{array}{rcl} |\delta(U)| & \underset{\text{Def 2.3}}{=} & \chi(U)^T L_G \chi(U) \\ & \geq & \chi(U)^T [c \cdot I_{im(L_G)}] \chi(U) \\ & = & c \cdot \chi(U)^T \left[I - \vec{1} \cdot \vec{1}^T / n \right] \chi(U) \\ & = & c \cdot \chi(U)^T \chi(U) - \frac{c}{n} \cdot (\chi(U)^T \vec{1})^2 \\ & = & c \cdot |U| - c \cdot |U|^2 / n \\ & \geq & c \cdot |U| - c \cdot \left(|U| \cdot \frac{n}{2} \right) / n \\ & = & \frac{c}{2} \cdot |U| \\ & \Rightarrow & G \text{ has edge-expansion} \geq \frac{c}{2}. \end{array}$$

What about the converse? If G is an edge expander, is it also a spectral expander? The answer is yes. This is proven by "Cheeger's Inequality" which is more sophisticated, but unfortunately there is some loss in the expansion parameter.

In the next lecture, we talk about how to construct the spectral expanders.