UBC CPSC 536N: Sparse Approximations	Winter 2013
Lecture 13 — February 25, 2013	
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The main goal for this lecture is to obtain an $O\left(\frac{\log n}{\log\log n}\right)$ approximation algorithm for the ATSP problem. Outlining this algorithm will involve all of the disparate techniques and ideas developed in the past several lectures. First, by considering the Held-Karp LP relaxation, we will obtain a "fractional Hamiltonian cycle" x^* of minimum weight. Next we let x be a "scaled symmetrization" of x^*, and show that x is an (undirected) "fractional spanning tree", i.e., x lies in the spanning tree polytope. Next, applying pipage rounding to x, we obtain an integral, undirected spanning tree T. By orienting the edges of T, we obtain a digraph H to which we apply the transshipment process in order to obtain an Eulerian digraph H'. Finally, by taking shortcuts in H' (exploiting the triangle inequality), we obtain a Hamiltonian cycle with weight within a factor of $O\left(\frac{\log n}{\log\log n}\right)$ of the optimal value of the Held-Karp relaxtion.

1 An $O\left(\frac{\log n}{\log\log n}\right)$ Algorithm for ATSP

In the ATSP problem we are given a directed, complete graph $G = (V, A)$ with edge weights $w : A \to \mathbb{R}_{\geq 0}$, where the weight function satisfies the triangle inequality.

1.1 Held-Karp Relaxation

In Lecture 8, we defined the following linear program, called the the Held-Karp relaxation:

$$
\begin{aligned}
LP = \min\{w^T x : x(\delta^+(v)) = x(\delta^-(v)) &= 1 && \forall x \in V \\
x(\delta^+(U)) &\geq 1 && \forall U \subsetneq V, U \neq \varnothing \\
0 \leq x_a &\leq 1 && \forall a \in A\}
\end{aligned}
$$

Recall that despite the exponential number of constraints in this LP, we were able to obtain an optimal solution in polynomial time in n. Let x^* be an optimal solution to this LP. Now x^* represents a directed, fractional tree in G. We'd like to obtain an integral tree which approximates x^*, but our tool for this process, pipage rounding, works with fractional trees in the spanning tree polytope of undirected graphs. So we symmetrize x^* by replacing it with x, where $x_{\{u,v\}} = \left(\frac{n-1}{n}\right)(x^*_{uv} + x^*_{vu})$. We now show x is in the spanning tree polytope P of G, where here we consider G to be undirected by simply forgetting edge orientation and removing duplicate edges. Recall

$$
\begin{aligned}
P = \{x \in \mathbb{R}^E_{\geq 0} : x(E) = n-1, \quad & x(E[U]) \leq |U| - 1 \quad \forall U \subset V, U \neq \varnothing\}, \\
& \text{where } E[U] = \{\{u,v\} \in E : u, v \in U\}
\end{aligned}
$$

Claim 1.1. $\forall U \subset V, \quad x(\delta(U)) = \frac{2(n-1)}{n} x^*(\delta^+(U))$

Proof.

$$x(\delta(U)) = \sum_{\{u,v\}\in\delta(U)} x_{\{u,v\}} = \sum_{\{u,v\}\in\delta(U)} \left(\frac{n-1}{n}\right)(x^*_{uv} + x^*_{vu})$$
$$= \frac{n-1}{n}(x^*(\delta^+(U)) + x^*(\delta^-(U))) = \frac{2(n-1)}{n} x^*(\delta^+(U)),$$

where we used $x^*(\delta^+(U)) = x^*(\delta^-(U))$ in the last equality which was proved in Lecture 8, Claim 1.3. □

Claim 1.2. $x \in P$

Proof. We simply verify that x satisfies the constraints defining P.

First, note that $x_{\{u,v\}} = \left(\frac{n-1}{n}\right)(x^*_{uv} + x^*_{vu}) \geq 0$.

By claim 1, $x(\delta(v)) = \frac{2(n-1)}{n} \quad \forall v \in V$,

so $x(E) = \frac{1}{2}\sum_{v\in V} x(\delta(v)) = \frac{1}{2} n \frac{2(n-1)}{n} = n - 1$.

Since x^* satisfies the LP, we have $x^*(\delta^+(U)) \geq 1 \quad \forall U \subset V, U \neq \varnothing$.

Thus $x(\delta(U)) = \frac{2(n-1)}{n} x^*(\delta^+(U)) \geq \frac{2(n-1)}{n} \quad \forall U \subset V, U \neq \varnothing$.

Also, $\frac{2(n-1)}{n}|U| = \sum_{u\in U} x(\delta(u)) = 2x(E[U]) + x(\delta(U)) \geq 2x(E[U]) + \frac{2(n-1)}{n}$,

so $2x(E[U]) \leq \frac{2(n-1)}{n}|U| - \frac{2(n-1)}{n} = \frac{2(n-1)}{n}(|U| - 1)$

and thus $x(E[U]) < |U| - 1$. □

1.2 Eulerian Graph from a Fractional Tree

Our next step is to obtain an Eulerian graph.

In lecture 9, we used transshipments to get an Eulerian graph. Here is a generalized statement:

Claim 1.3. Suppose H is an integral (multi-) digraph with:

(1) weight$(H) \leq \alpha w^T x^*$

(2) $|\delta^+_H(U)| + |\delta^-_H(U)| \geq 1 \quad \forall U \subsetneq V, U \neq \varnothing$ (note this is equivalent to H being weakly connected)

(3) $|\delta^+_H(U)| \leq \alpha x^*(\delta^+(U)) \quad \forall U \subset V$

Then there is an integral, Eulerian, strongly-connected (multi-) digraph H' with:

(4) weight$(H') \leq O(\alpha) w^T x^*$

Proof. As in Lecture 9, we use transshipments to 'patch up' H in order to obtain an Eulerian graph. The arguments given there give us an integral, Eulerian, weakly-connected digraph, and the desired bound on its weight. Note that a Eulerian graph H' must be strongly connected:

given $U \subsetneq V, U \neq \varnothing$, if $|\delta^-_{H'}(U)| \geq 1$, walking along a Eulerian trail starting from a vertex in U we must be able to traverse the edges in $\delta^-_{H'}(U)$, and thus we must be able to get to $V \backslash U$, in particular $\delta^+_{H'}(U) \neq \varnothing$. If $|\delta^+_{H'}(U)| \geq 1$, we consider walking along a Eulerian trail starting from a vertex in $V \backslash U$, and apply the same argument. $\square$

We obtain such an H to apply Claim 1.3 to by applying pipage rounding to x. This produces an undirected tree T, and by the proof of Claim 1.11 of Lecture 12 with probability $\geq 1 - \frac{3}{n}$, for any $U \subset V$ we have

$|\delta_T(U)| \leq \alpha x(\delta(U)), \text{where } \alpha := \frac{6 \log n}{\log \log n}$, and thus

$$|\delta_T(U)| \leq \alpha x(\delta(U)) = \alpha \frac{2(n-1)}{n} x^*(\delta^+(U)) < 2\alpha x^*(\delta^+(U))$$

Next, let $w_{\{u,v\}} = \min\{w_{uv}, w_{vu}\}$. We calculate the expectation of the weight of T, in order to use Markov's inequality:

$$\begin{aligned}
\mathbb{E}[\text{weight of T}] &= \sum_e w_e \Pr[e \in T] \\
&= \sum_e w_e x_e \\
&= \frac{n-1}{n} \sum_{\{u,v\} \in E} w_{\{u,v\}}(x^*_{uv} + x^*_{vu}) \\
&\leq \frac{n-1}{n} \sum_{uv \in A} w_{uv} x^*_{uv} \\
&< w^T x^*,
\end{aligned}$$

so applying Markov's inequality we get:

$$\Pr[\text{weight of T} > 2w^T x^*] \leq \frac{\mathbb{E}[\text{weight of T}]}{2w^T x^*} < \frac{1}{2}.$$

Collecting the above, we have that with probability $\geq \frac{1}{3}$ (assume $n \geq 5$) that:

1. $\text{weight}(T) \leq 2w^T x^*$
2. $|\delta_T(U)| < 2\alpha x^*(\delta^+(U)) \quad \forall U \subset V$

To get H, add arc uv to H if $\{u,v\} \in T$ and $w_{uv} \leq w_{vu}$, where we break ties arbitrarily. So:

1. $\text{weight}(H) = \text{weight}(T) \leq 2w^T x^*$
2. H is weakly connected, since T is a tree.
3. $|\delta^+_H(U)| \leq |\delta_T(U)| \leq 2\alpha x^*(\delta^+(U))$

So by Claim 1.3, we get an integral, Eulerian, strongly-connected multi-digraph H' with $\text{weight}(H') = O(\alpha) w^T x^*$

Taking advantage of the triangle inequality and applying a shortcutting argument (see the end of Lecture 6), we can obtain a Hamiltonian cycle from H' with weight bounded above by $\text{weight}(H)$.

2 Integrality Gaps

Consider an integer program $IP = \{\min w^T x : x \in P, x \in \mathbb{Z}^n\}$ and suppose $LP = \{\min w^T x : x \in P'\}$ and $P' \supset P \cap \mathbb{Z}^n$. This LP is called an LP-relaxation of the IP.

Remark: the optimum value of the LP is less than or equal to the optimum value of the IP.

We want an integer solution $y \in P$ such that

$$w^T y \leq \alpha \cdot (\text{IP-optimum}) \tag{*}$$

However, typically we cannot compute IP-optimum. Instead, we show that

$$w^T y \leq \alpha \cdot (\text{LP-optimum}) \tag{**}$$

Note (**) implies (*). If we can prove (**) for all instances, we say that the LP-relaxation has integrality gap $\leq \alpha$.

We showed that the Held-Karp LP has integrality gap $O\left(\frac{\log n}{\log\log n}\right)$, assuming the triangle inequality. This result is due to [Asadpour, Goemans, Madry, Gharan, Saberi 2010].

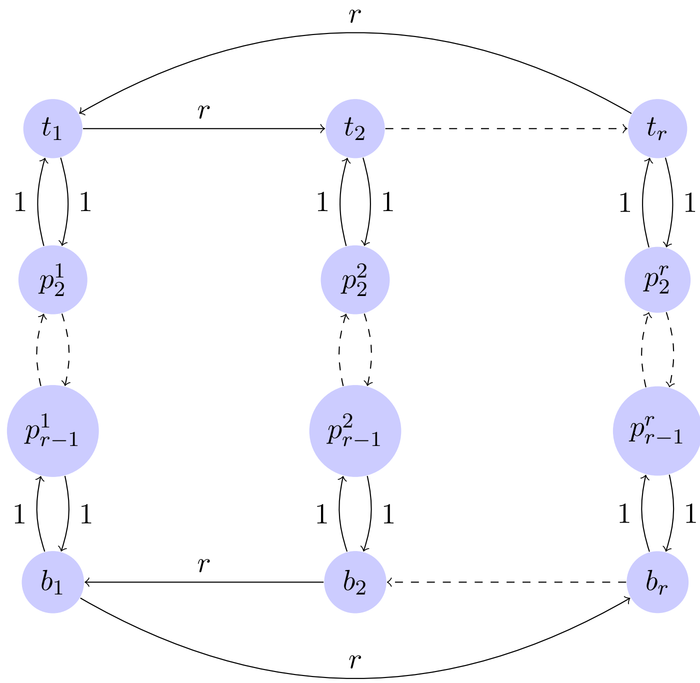
2.1 Lower Bounds for Held-Karp Integrality Gap

In [Charikar, Goemans, Karloff 2006], the authors obtained a constant asymptotic lower bound for the Held-Karp integrality gap. More precisely they showed that for all $r \geq 3$ and $k \geq 2$ there exists an instance of ATSP on $O(r^k)$ vertices such that

$\frac{\text{IP-opt}}{\text{LP-opt}} \geq \frac{r-1}{r+1} \cdot \frac{2k-1}{k}$

Since $\frac{r-1}{r+1} \cdot \frac{2k-1}{k} \to 2$ as $(r,k) \to (\infty, \infty)$, we obtain the asymptotic bound.

For simplicity, we'll just briefly describe their construction for $k = 2$, $r \geq 3$, which gives a gap $\frac{3(r-1)}{2(r+1)} \to 3/2$.



There are two cycles, the top and bottom cycles. Each of these cycles has r vertices, and each edge in either cycle has weight r. We choose a bijection between the vertices of these two cycles, and for each pair of corresponding vertices we have a two way path containing r vertices. Each edge on these paths has weight 1. As there are r paths, every path has r vertices, and each pair of paths is disjoint, the graph has $r \cdot r = r^2$ vertices.

Claim 2.1. The Held-Karp LP has optimum value at most $2r^2 + r$.

Proof. (Sketch) The fractional solution that puts weight 1/2 on every arc is feasible. $\square$

Claim 2.2. Every integral solution has value at least $3(r-1)r$.

Proof. By case analysis. $\square$