Polynomial Time Algorithms

- P = { computational problems that can be solved efficiently }
  i.e., solved in time $\leq \mathbf{n^c}$, for some constant $\mathbf{c}$, where $\mathbf{n}$=input size
- This is a bit vague
  - Consider an LP max { $c^T x$: $Ax \leq b$ } where A has size m x d
  - Input is a binary file containing the matrix A, vectors b and c
- Two ways to define "input size"
  A. # of bits used to store the binary input file
  B. # of numbers in input file, i.e., $m \cdot d + m + d$
- Leads to two definitions of "efficient algorithms"
  A. Running time $\leq \mathbf{n^c}$ where $\mathbf{n}$ = # bits in input file
  B. Running time $\leq \mathbf{n^c}$ where $\mathbf{n}$ = $m \cdot d + m + d$

"Polynomial Time Algorithm"

"**Strongly** Polynomial Time Algorithm"

Algorithms for Solving LPs

Name	Publication	Running Time	Practical?
Fourier-Motzkin Elimination	Fourier 1827, Motzkin 1936	Exponential	No
Simplex Method	Dantzig '47	Exponential	Yes
Perceptron Method	Agmon '54, Rosenblatt '62	Exponential	Sort of
Ellipsoid Method	Khachiyan '79	Polynomial	No
Interior Point Method	Karmarkar '84	Polynomial	Yes
Analytic Center Cutting Plane Method	Vaidya '89 & '96	Polynomial	No
Random Walk Method	Bertsimas & Vempala '02-'04	Polynomial	Probably not
Boosted Perceptron Method	Dunagan & Vempala '04	Polynomial	Probably not
Random Shadow-Vertex Method	Kelner & Spielman '06	Polynomial	Probably not

- Unsolved Problems:
  - Is there a **strongly polynomial time** algorithm?
  - Does some implementation of **simplex method** run in polynomial time?

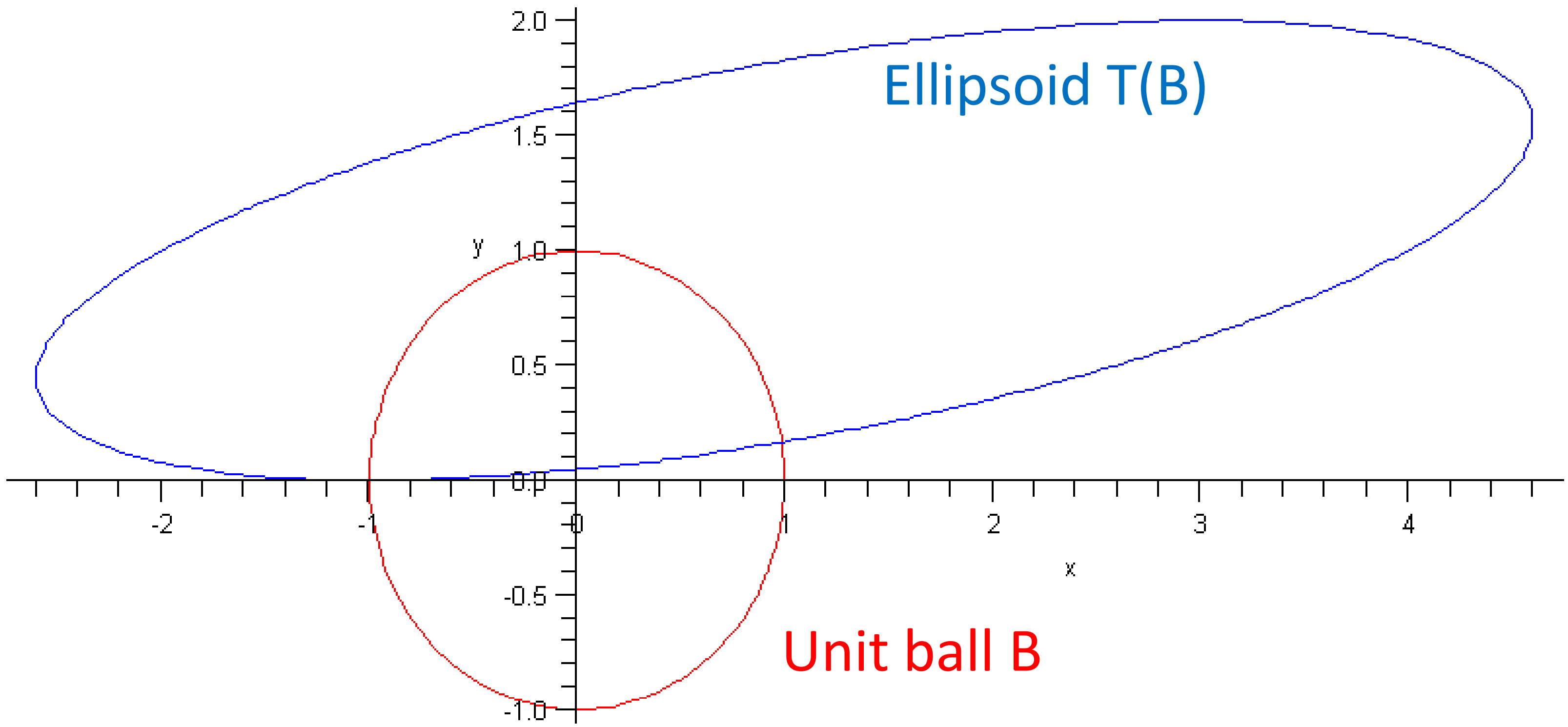
Ellipsoids

- **Def:** Let B = { x : $\|x\| \leq 1$ }. Let f : $\mathbb{R}^n \rightarrow \mathbb{R}^m$ be an affine map. Then f(B) is an **ellipsoid**.
- We restrict to the case n=m and f invertible, i.e., f(x) = Ax+b where A is square and non-singular
- **Claim 2:** f(B) = { $x \in \mathbb{R}^n$: $(x-b)^T A^{-T} A^{-1} (x-b) \leq 1$ }.

2D Example

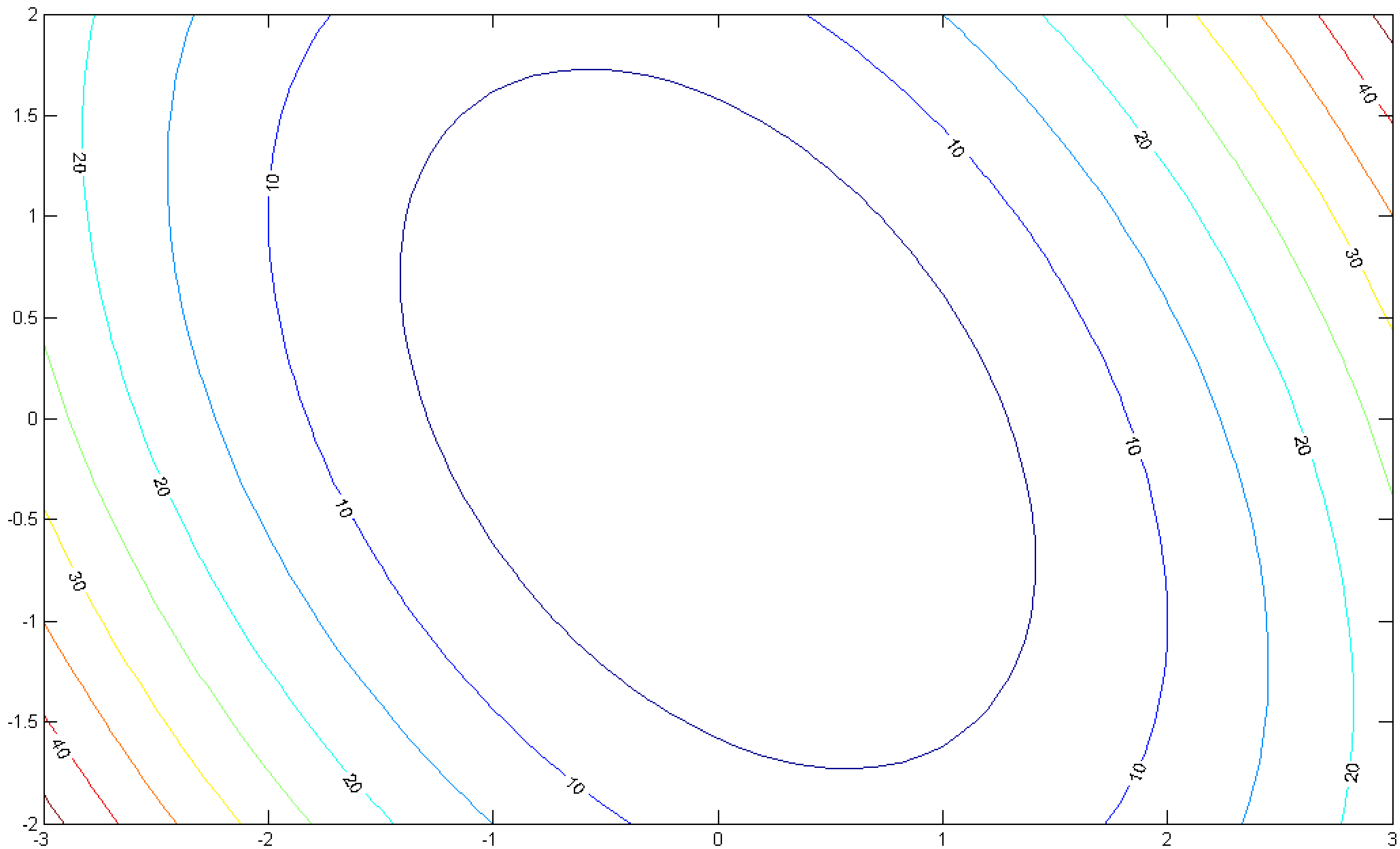
Define $T(x) = Ax + b$ where $A = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

```
implicitplot([x^2+y^2=1,(x-1)^2-4*(x-1)*(y-1)+13*(y-1)^2=9], x=-5..5, y=-5..5,
numpoints=10000, color=[red,blue] );
```



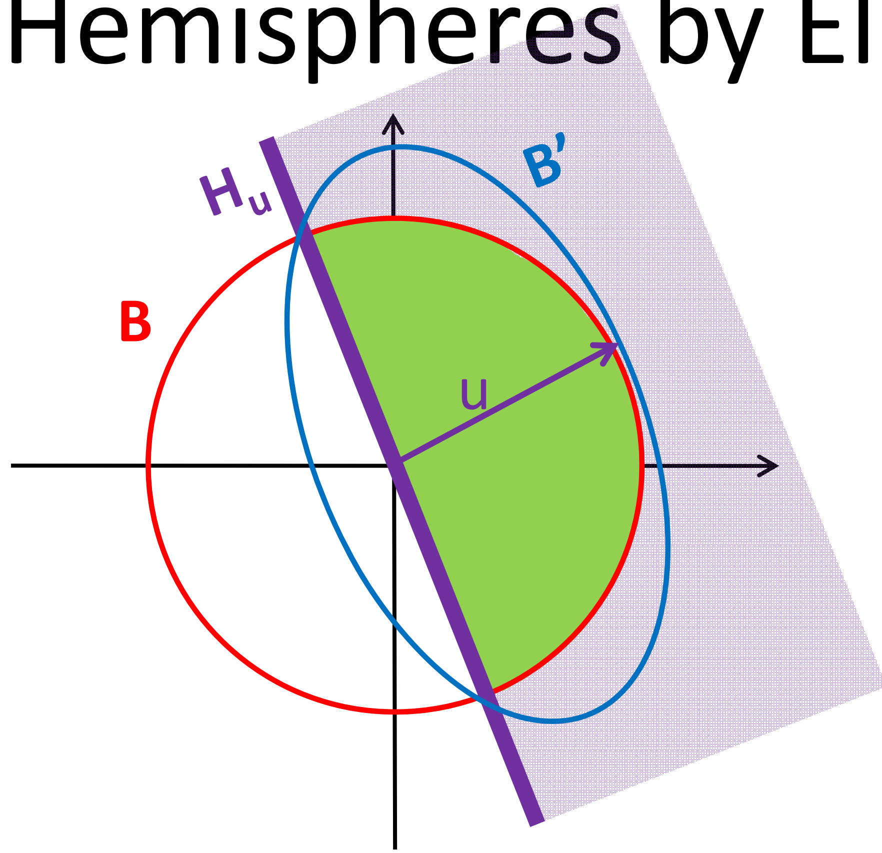
Ellipsoids

- **Def:** Let $B = \{ x : \|x\| \leq 1 \}$. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an affine map. Then $f(B)$ is an **ellipsoid**.
- We restrict to the case $n=m$ and f invertible, i.e., $f(x) = Ax+b$ where A is square and non-singular
- **Claim 2:** $f(B) = \{ x \in \mathbb{R}^n : (x-b)^T A^{-T} A^{-1} (x-b) \leq 1 \}$.
- This ellipsoid can also be denoted
  $$E(M,b) = \{ x \in \mathbb{R}^n : (x-b)^T M^{-1} (x-b) \leq 1 \},$$
  for the positive definite matrix $M=AA^T$ and vector b.
- Note that $E(aM^{-1},0)$ is a level set of $f(x) = x^T M x$:
  $$E(aM^{-1},0) = \{ x \in \mathbb{R}^n : x^T M x \leq a \}.$$



- Let $M = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$
- Plot of level sets of $x^T M x$.

Covering Hemispheres by Ellipsoids



- Let B = { unit ball }.
- Let H_u = { $x : x^T u \geq 0$ }, where $\|u\|=1$.
- Find a small ellipsoid B' that covers $B \cap H$.

Main Theorem:

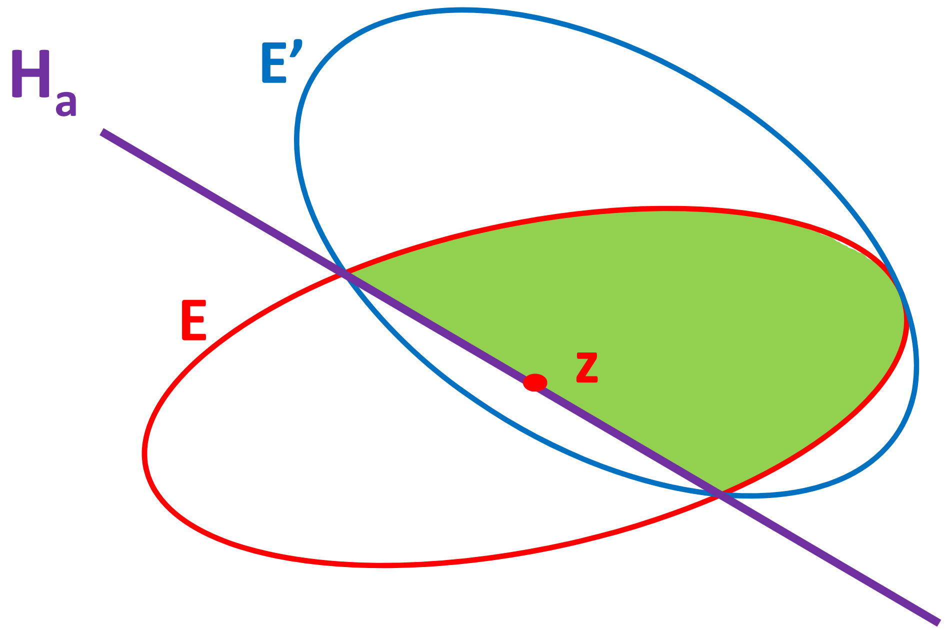
Let $B = \{ x : \|x\| \leq 1 \}$ and $H_u = \{ x : x^\mathsf{T} u \geq 0 \}$, where $\|u\|=1$.

Let $M = \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1} u u^\mathsf{T}\right)$ and $b = \frac{u}{n+1}$.

Let $B' = E(M, b)$. Then:

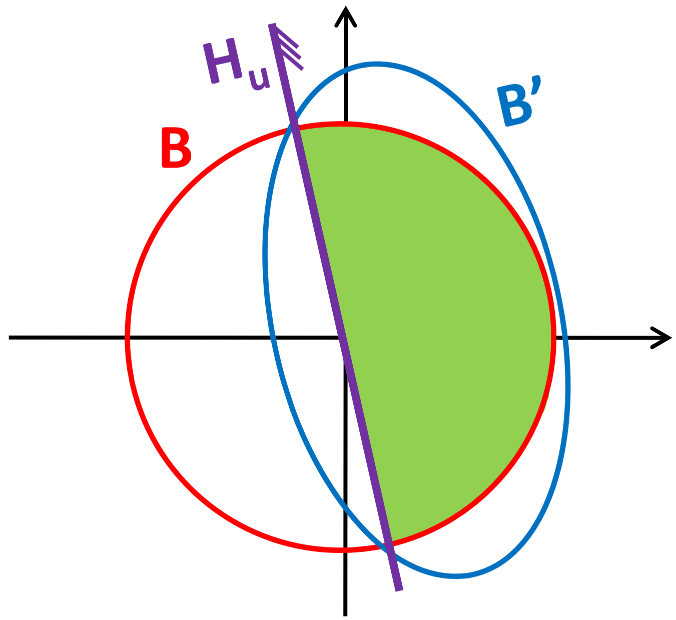
1) $B \cap H_u \subseteq B'$.
2) $\frac{\mathrm{vol}(B')}{\mathrm{vol}(B)} \leq e^{-1/4(n+1)} \leq 1 - \frac{1}{8(n+1)}$

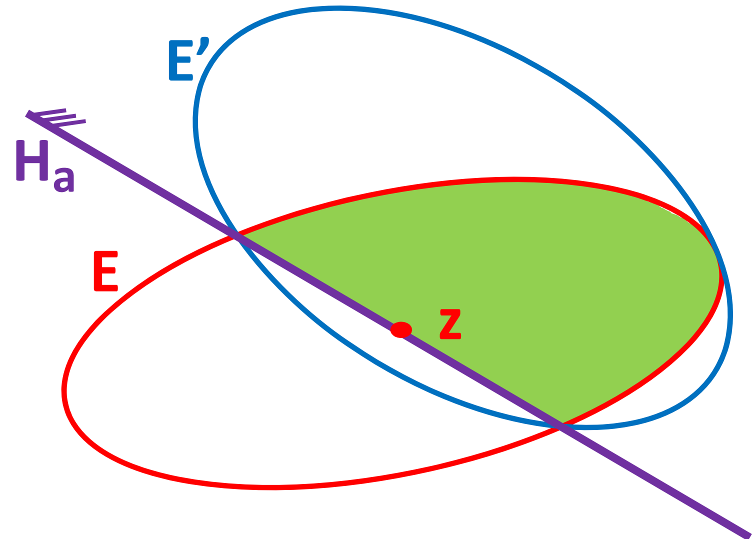
Covering Half-ellipsoids by Ellipsoids



- Let E be an ellipsoid centered at z
- Let $H_a = \{ x : a^T x \geq a^T z \}$
- Find a small ellipsoid E' that covers $E \cap H_a$

Use our solution for hemispheres!





Goal

Find an affine map f and choose u such that:

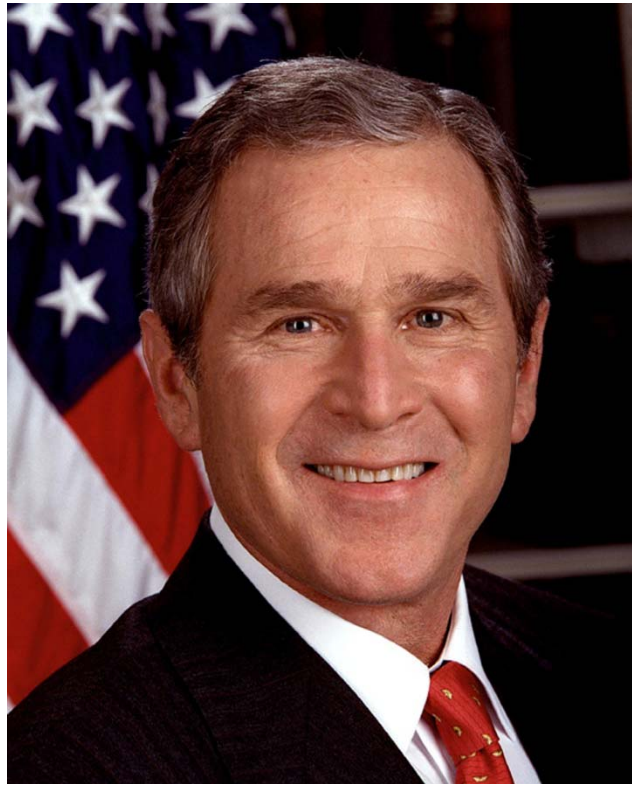
$$f(B) = E \quad \text{and} \quad f(H_u) = H_a$$

Define $E' = f(B')$.

Claim: E' is an ellipsoid.

Claim: $E \cap H_a \subseteq E'$.

The Genius behind the Ellipsoid Method



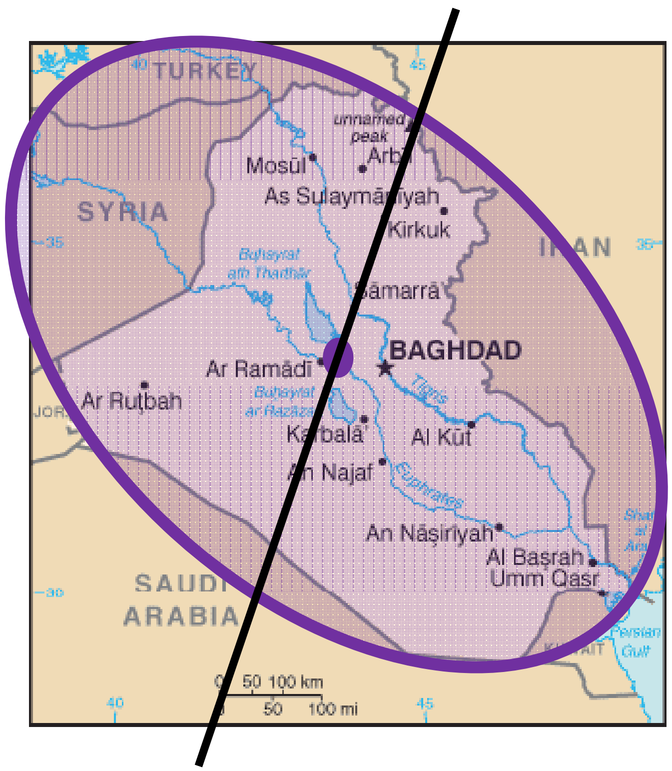
"Intelligence gathered by this and other governments leaves no doubt that the Iraq regime continues to possess and conceal some of the most lethal weapons ever devised"
George W. Bush, 3/18/2003

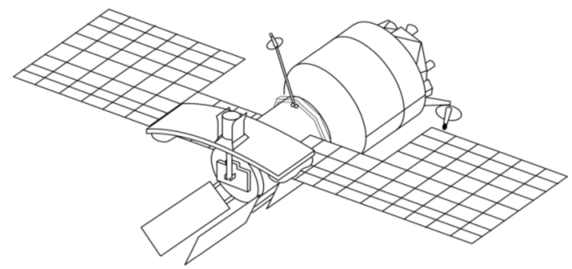
WMD in Iraq

"We are learning more as we interrogate or have discussions with Iraqi scientists and people within the Iraqi structure, that perhaps he destroyed some, perhaps he dispersed some. And so we will find them." George W. Bush, 4/24/2003

Finding WMD

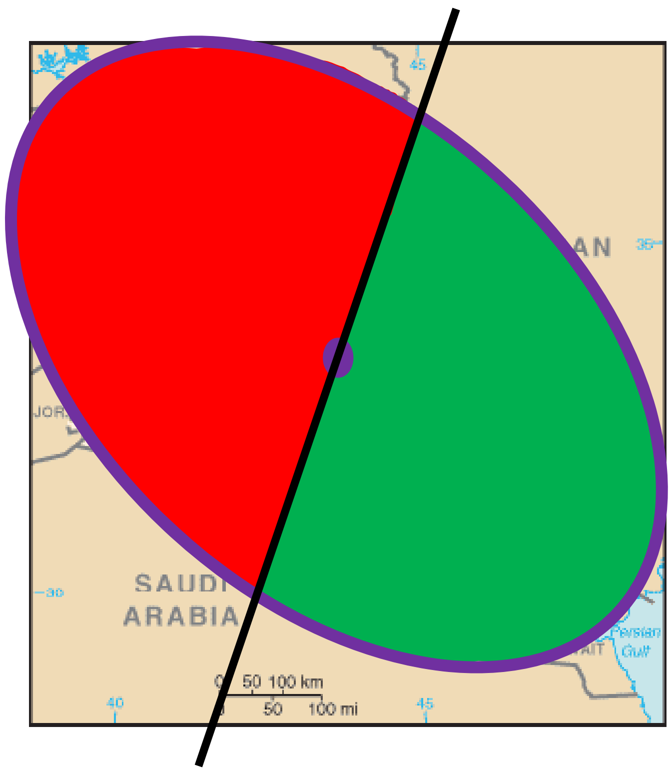
- USA have a satellite with a WMD detector
- The detector scans a round region of the earth
- It can compare two halves of the region, and decide which half is "more likely" to have WMD

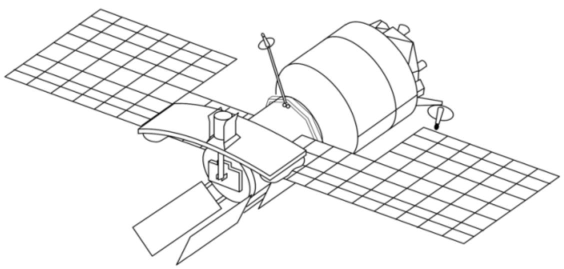




Finding WMD

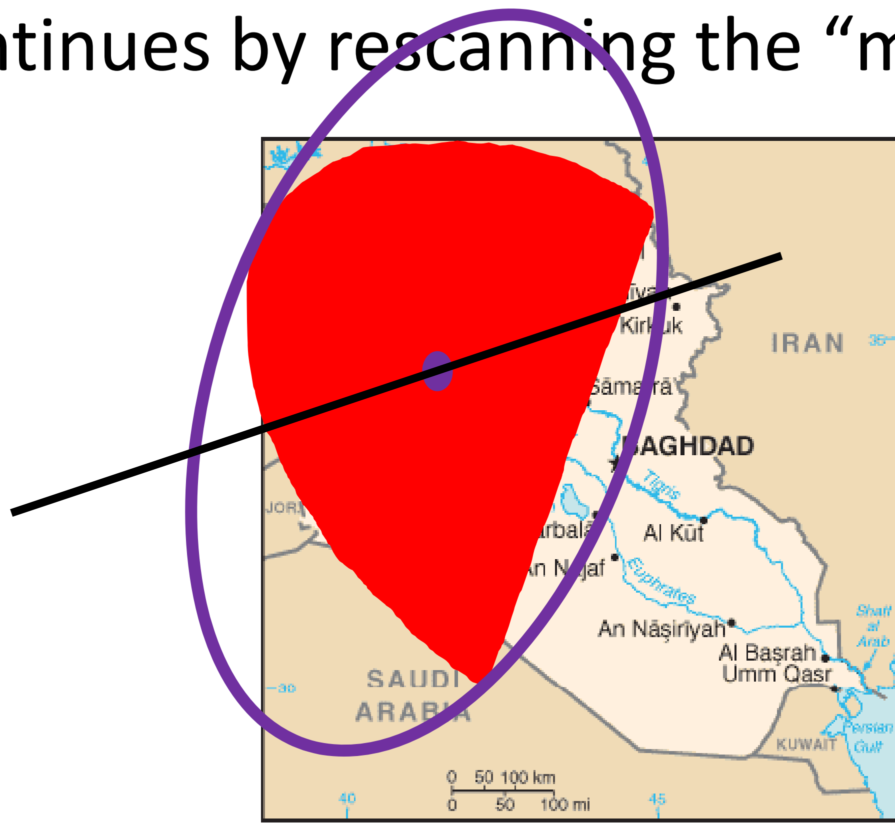
- USA have a satellite with a WMD detector
- The detector scans a round region of the earth
- It can compare two halves of the region, and decide which half is “more likely” to have WMD
- It continues by rescanning the “more likely” half

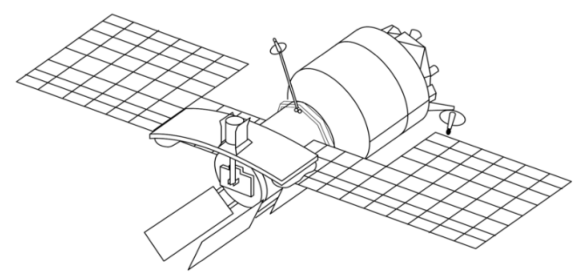




Finding WMD

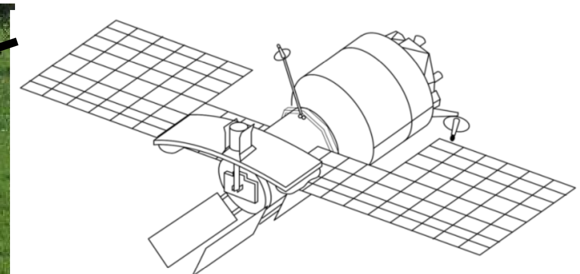
- USA have a satellite with a WMD detector
- The detector scans a round region of the earth
- It can compare two halves of the region, and decide which half is "more likely" to have WMD
- It continues by rescanning the "more likely" half





Finding WMD

- USA have a satellite with a WMD detector
- The detector scans a round region of the earth
- It can compare two halves of the region, and decide which half is "more likely" to have WMD
- It continues by rescanning the "more likely" half

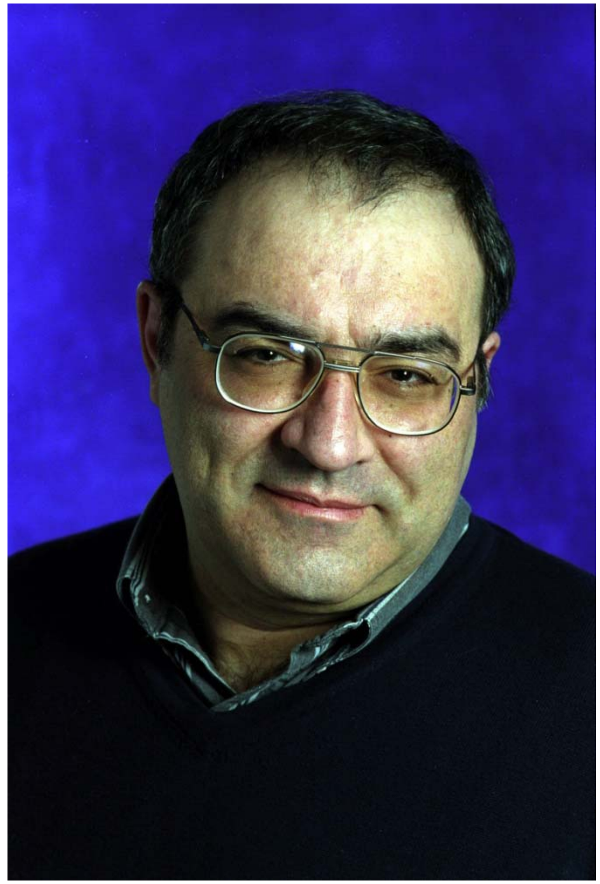


Finding WMD

- It continues by rescanning the "more likely" half
- If region is so small that it obviously contains no WMD, then conclude: Iraq has no WMD

"No one was more surprised than I that we didn't find [WMDs]."
U.S. General Tommy Franks, 12/2/2005

Generalization to Higher Dimensions

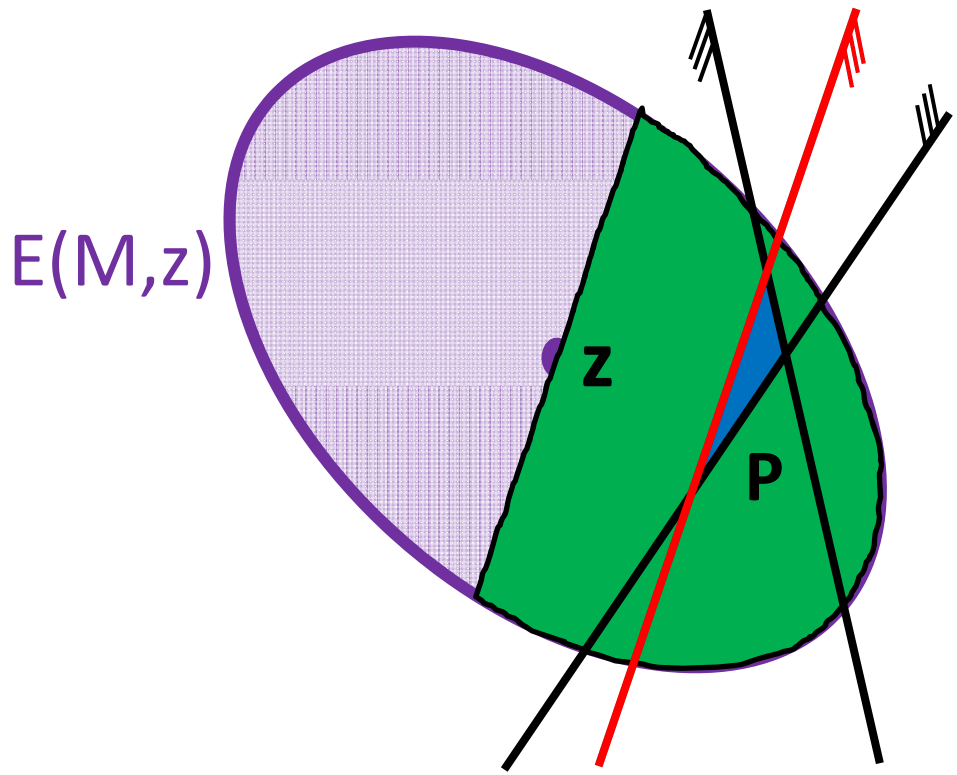


Leonid Khachiyan

Even smarter than George W. Bush!

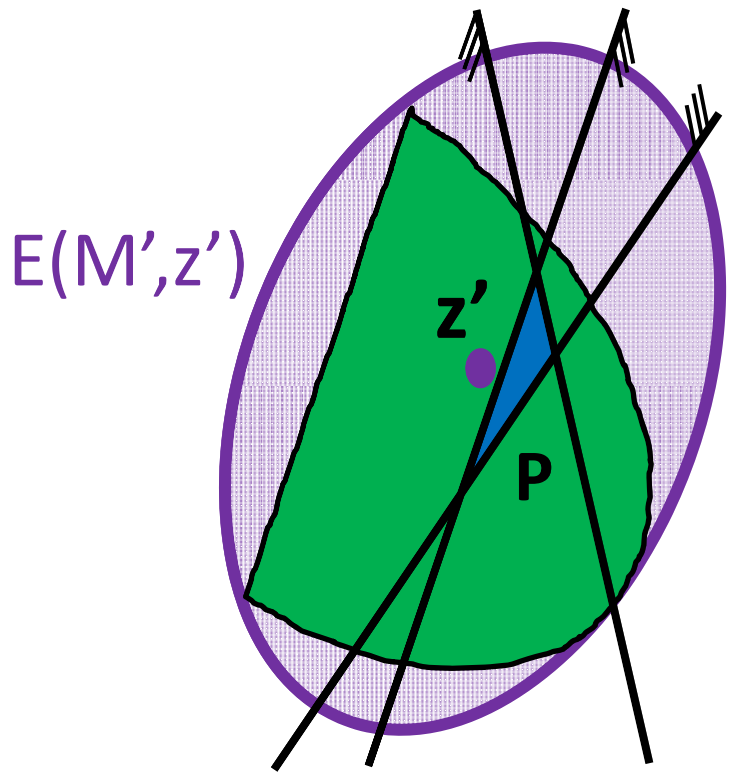
The Ellipsoid Method

- Want to find $x \in P$
- Have ellipsoid $E(M,z) \supseteq P$
- If $z \notin P$ then it violates a constraint "$a_i^T x \leq b_i$"
- So $P \subseteq \{ x : a_i^T x \leq a_i^T z \}$
- So $P \subseteq E(M,z) \cap \{ x : a_i^T x \leq a_i^T z \}$



The Ellipsoid Method

- Have ellipsoid $E(M,z) \supseteq P$
- If $z \notin P$ then it violates a constraint "$a_i^T x \leq b_i$"
- So $P \subseteq \{ x : a_i^T x \leq a_i^T z \}$
- So $P \subseteq E(M,z) \cap \{ x : a_i^T x \leq a_i^T z \}$
- Let $E(M',z')$ be ellipsoid covering $E(M,z) \cap \{x : a_i^T x \leq a_i^T z\}$
- Repeat…



The Ellipsoid Method

- **Input:** A polytope $P = \{ Ax \leq b \}$ and R and r. (e.g., P=WMD)
- **Output:** A point $x \in P$, or announce "P is empty"

> Let E(M,z) be an ellipsoid s.t. $P \subseteq E(M,z)$ (e.g., E(M,z)=B(0,R))
> If vol E(M,z) < vol B(0,r) then Halt: "P is empty"
> If $z \in P$, Halt: "$z \in P$"
> Else
>
> Let "$a_i^T x \leq b_i$" be a constraint of P violated by z (i.e., $a_i^T z > b_i$)
>
> Let $H = \{ x : a_i^T x \leq a_i^T z \}$ (so $P \subseteq E(M,z) \cap H$)
>
> Let E(M',z') be an ellipsoid covering $E(M,z) \cap H$
>
> Set $M \leftarrow M'$ and $z \leftarrow z'$ and go back to Start

How to find this?

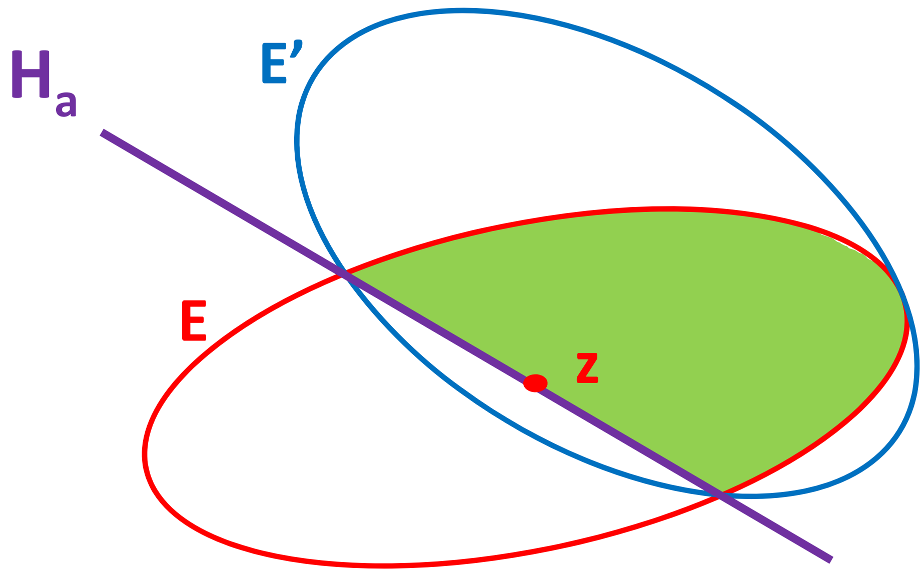
- **Notation:** Let B(z,r) = ball of radius r around point $z \in \mathbb{R}^n$
- **Assumptions:**

"The WMD is in Iraq": $\exists R>0$ such that $P \subseteq B(0,R)$

"WMD bigger than cow": If $P \neq \emptyset$ then $\exists r>0$, $z \in \mathbb{R}^n$ s.t. $B(z,r) \subseteq P$

Covering Half-ellipsoids by Ellipsoids

- Let E be an ellipsoid centered at z
- Let $H_a = \{ x : a^T x \leq a^T z \}$



Solution

As stated earlier, we can find an ellipsoid E' such that

- $E \cap H_a \subseteq E'$
- $\mathrm{vol}(E') \leq \mathrm{vol}(E) \cdot e^{-1/4(n+1)}$

How many iterations?

- E_i = ellipsoid in i^{th} iteration. Initially E_0 = B(0,R)
- **Claim 1:** $\mathrm{vol}(E_k) \leq \mathrm{vol}(E_0) \cdot e^{-k/4(n+1)}$.
- **Proof:** We showed $\mathrm{vol}(E_{i+1}) \leq \mathrm{vol}(E_i) \cdot e^{-1/4(n+1)}$.

  So $\mathrm{vol}(E_k) \leq \mathrm{vol}(E_0) \prod_{i=1}^{k} e^{-1/4(n+1)} = \mathrm{vol}(E_0) \cdot e^{-k/4(n+1)}$. ∎

- **Claim 2:** Number of iterations $\leq$ 4 n(n+1) log(R/r).
- **Proof:** Suppose $k > 4n(n+1)\log(R/r)$

  Then $-k/4(n+1) < n\log(r/R)$

  So $e^{-k/4(n+1)} < (r/R)^n = \dfrac{\mathrm{vol}\, B(0,r)}{\mathrm{vol}\, B(0,R)}$

  By Claim 1, $\mathrm{vol}\, E_k < \mathrm{vol}\, E_0 \cdot \dfrac{\mathrm{vol}\, B(0,r)}{\mathrm{vol}\, B(0,R)} = \mathrm{vol}\, B(0,r)$

  So the algorithm stops. ∎

Ellipsoid Method for Solving LPs

- Ellipsoid method finds feasible point in $P = \{ x : Ax \leq b \}$
  i.e., it can solve a system of inequalities

- But we want to **optimize**, i.e., solve $\max \{ c^T x : x \in P \}$

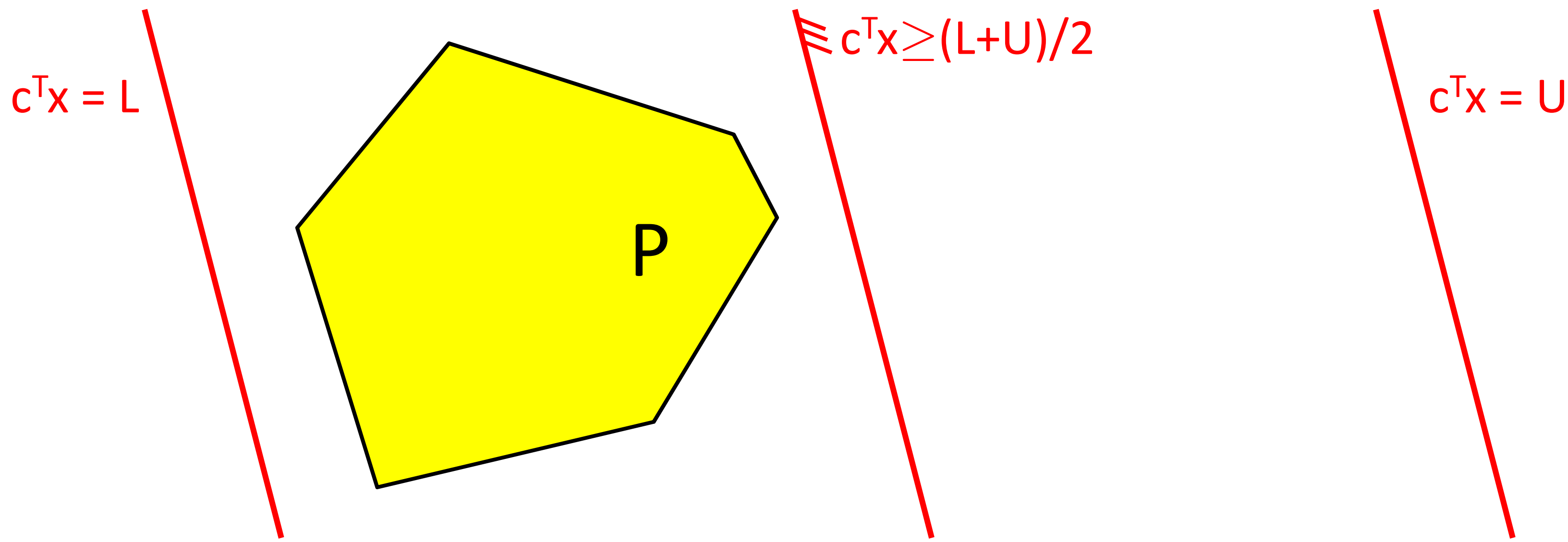
- **Restatement of Strong Duality Theorem:** (from Lecture 3)
  Primal has optimal solution $\Leftrightarrow$ Dual has optimal solution
  $\Leftrightarrow$ the following system is solvable:

$$Ax \leq b \qquad A^{\mathsf{T}} y = c \qquad y \geq 0 \qquad c^{\mathsf{T}} x \geq b^{\mathsf{T}} y$$

- **Important Point**
  Solving an LP is equivalent to solving a system of inequalities
  $\Rightarrow$ Ellipsoid method can be used to solve LPs

Ellipsoid Method for Solving LPs

- Ellipsoid method finds feasible point in $P = \{ x : Ax \leq b \}$ i.e., it can solve a system of inequalities
- But we want to **optimize**, i.e., solve $\max \{ c^T x : x \in P \}$
- **Alternative approach:** Binary search for optimal value
  - Suppose we know optimal value is in interval [L,U]
  - Add a new constraint $c^T x \geq (L+U)/2$
  - If LP still feasible, replace L with (L+U)/2 and repeat
  - If LP not feasible, replace U with (L+U)/2 and repeat



Issues with Ellipsoid Method

1. It needs to compute square roots, so it must work with irrational numbers
   - **Solution:** Approximate irrational numbers by rationals. Approximations proliferate, and it gets messy.
2. Can only work with bounded polyhedra P
   - **Solution:** If P non-empty, there exists a feasible x s.t. $|x_i| \leq U \ \forall i$, where U is a bound based on numbers in A and b. So we can assume that $-U \leq x_i \leq U$ for all i.
3. Polyhedron P needs to contain a small ball B(z,k)
   - **Solution:** If $P = \{ x : Ax \leq b \}$ then we can perturb b by a tiny amount. The perturbed polyhedron is feasible iff P is, and if it is feasible, it contains a small ball.

Ellipsoid Method in Polynomial Time

- **Input:** A polyhedron $P = \{ x : Ax \leq b \}$ where A has size m x d. This is given as a binary file containing matrix A and vector b.
- **Input size:** **n** = # of bits used to store this binary file
- **Output:** A point $x \in P$, or announce "P is empty"
- **Boundedness:** Can add constraints $-U \leq x_i \leq U$, where $U = 16^{d^2 n}$. The new P is contained in a ball B(0,R), where $R < n \cdot U$.
- **Contains ball:** Add ϵ to b_i, for every i, where $\epsilon = 1/U^2$. The new P contains a ball of radius $r = \epsilon \cdot 2^{-dn} > 1/U^3$.
- **Iterations:** We proved that:
  # iterations $\leq 4d(d+1)\log(R/r)$, and this is $< 40d^6n^2$
- Each iteration does only basic matrix operations and can be implemented in polynomial time.
- **Conclusion:** Overall running time is polynomial in n (and d)!

What Does Ellipsoid Method Need?

- The algorithm uses no properties of polyhedra
- It just needs to (repeatedly) answer the question:
  Is $z \in P$?
  If not, give me a constraint "$a^T x \leq b$" of P violated by z

Let $E(M,z)$ be an ellipsoid s.t. $P \subseteq E(M,z)$
If vol $E(M,z)$ < vol $B(0,r)$ then Halt: "P is empty"
If $z \in P$, Halt: "$z \in P$"
Else
    Let "$a_i^T x \leq b_i$" be a constraint of P violated by z   (i.e., $a_i^T z > b_i$)
    Let $H = \{ x : a_i^T x \leq a_i^T z \}$ (so $P \subseteq E(M,z) \cap H$)
    Let $E(M',z')$ be an ellipsoid covering $E(M,z) \cap H$
    Set $M \leftarrow M'$ and $z \leftarrow z'$ and go back to Start

- **Input:** A polytope $P = \{ Ax \leq b \}$
- **Output:** A point $x \in P$, or announce "P is empty"

The Ellipsoid Method

- The algorithm uses almost nothing about polyhedra (basic feasible solutions, etc.)
- It just needs to (repeatedly) answer the question:

Separation Oracle

Is $z \in P$?

If not, find a vector a s.t. $a^T x < a^T z \;\; \forall x \in P$

- The algorithm works for **any convex set** P, as long as you can give a separation oracle.
  - P still needs to be bounded and contain a small ball.
- **Remarkable Theorem:** [Grotschel-Lovasz-Schijver '81]
  For any convex set $P \subseteq \mathbb{R}^n$ with a separation oracle, you can find a feasible point efficiently.
- **Caveats:**
  - "Efficiently" depends on size of ball containing P and inside P.
  - Errors approximating irrational numbers means we get "approximately feasible point"

Martin Grotschel

Laszlo Lovasz

Alexander Schrijver

The Ellipsoid Method For Convex Sets

Separation Oracle

Is $z \in P$?

If not, find a vector a s.t. $a^T x < a^T z \quad \forall x \in P$

- **Feasibility Theorem:** [Grotschel-Lovasz-Schijver '81]
  For any convex set $P \subseteq \mathbb{R}^n$ with a separation oracle, you can find a feasible point efficiently.
  - Ignoring (many, technical) details, this follows from ellipsoid algorithm
- **Optimization Theorem:** [Grotschel-Lovasz-Schijver '81]
  For any convex set $P \subseteq \mathbb{R}^n$ with a separation oracle, you can solve optimization problem $\max \{ c^T x : x \in P \}$.
  - How?
  - Follows from previous theorem and binary search on objective value.
- This can be generalized to minimizing non-linear (convex) objective functions.

Separation Oracle for Ball

- Let's design a separation oracle for the convex set $P = \{ x : \|x\| \leq 1 \}$ = unit ball $B(0,1)$.

Separation Oracle

Is $z \in P$?

If not, find a vector a s.t. $a^T x < a^T z \;\; \forall x \in P$

- **Input:** a point $z \in \mathbb{R}^n$
- If $\|z\| \leq 1$, return "Yes"
- If $\|z\| > 1$, return $a = z / \|z\|$
  - For all $x \in P$ we have
    $a^T x = z^T x / \|z\| \leq \|x\|$ **Why?** Cauchy-Schwarz
  - For z we have
    $a^T z = z^T z / \|z\| = \|z\| > 1 \geq \|x\| \quad \Rightarrow \; a^T x < a^T z$

Separation Oracle for Ball

- Let's design a separation oracle for the convex set $P = \{ x : \|x\| \leq 1 \}$ = unit ball B(0,1).

> **Separation Oracle**
>
> Is $z \in P$?
>
> If not, find a vector a s.t. $a^T x < a^T z \ \ \forall x \in P$

- **Conclusion:** Since we were able to give a separation oracle for P, we can optimize a linear function over it.
- **Note:** $\max \{ c^T x : x \in P \}$ is a **non-linear program**. (Actually, it's a **convex program**.)