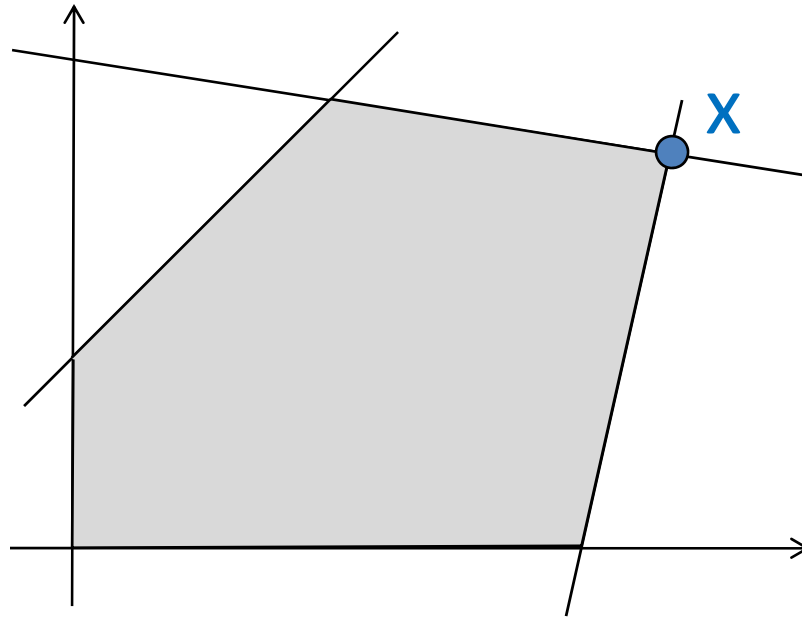


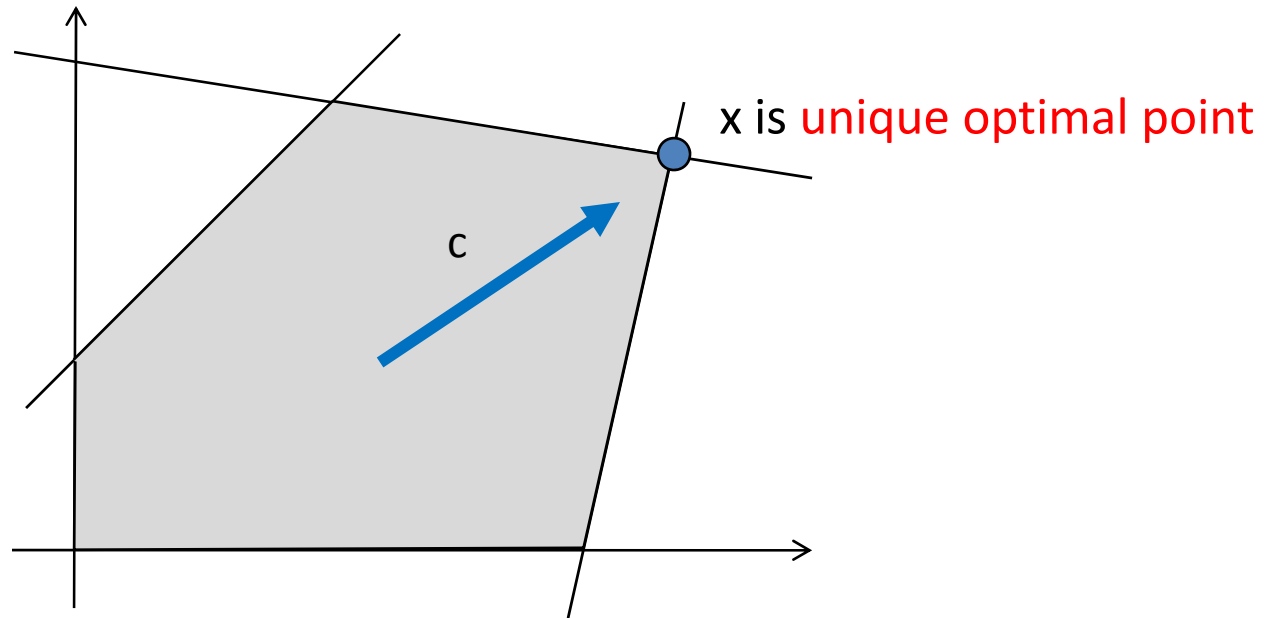
# What is a corner point?

- How should we define corner points?
- Under any reasonable definition, point  $x$  should be considered a corner point



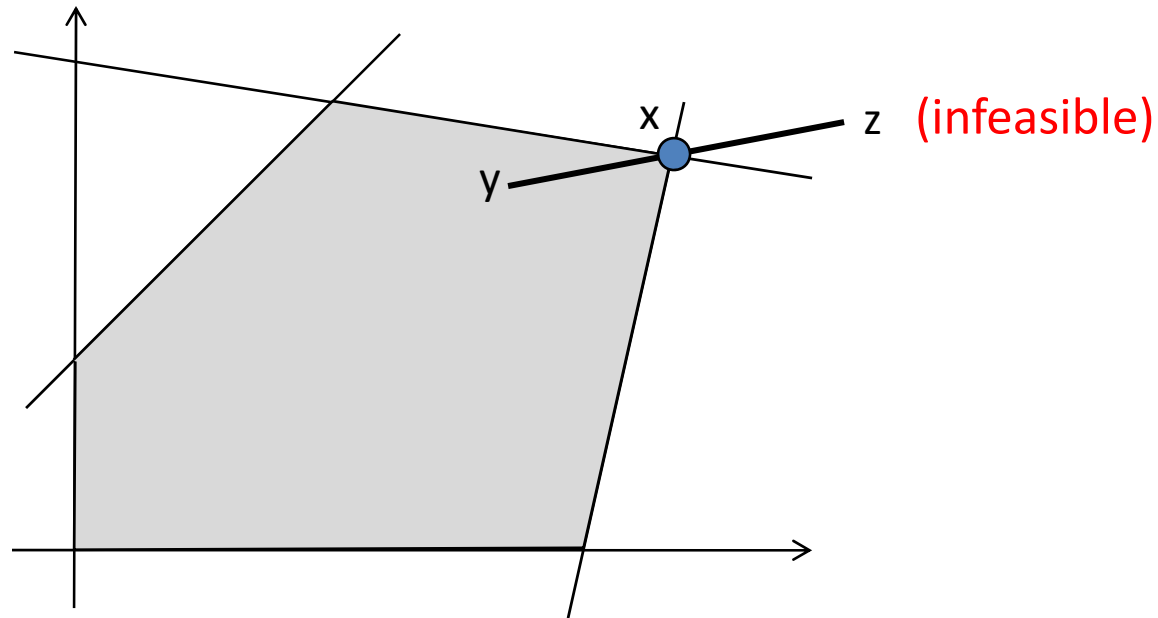
# What is a corner point?

- Attempt #1: “x is the ‘farthest point’ in some direction”
- Let  $P = \{ \text{feasible region} \}$
- There exists  $c \in \mathbb{R}^n$  s.t.  $c^T x > c^T y$  for all  $y \in P \setminus \{x\}$
- “For some objective function, x is the unique optimal point when maximizing over P”
- Such a point x is called a “**vertex**”



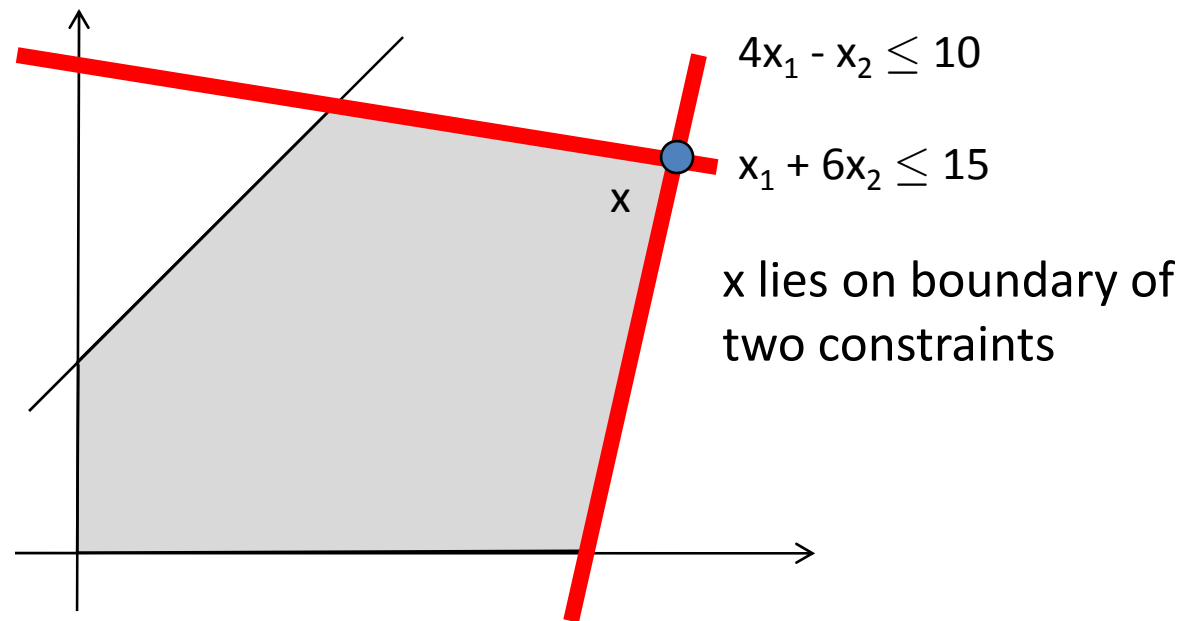
# What is a corner point?

- Attempt #2: “There is no feasible line-segment that goes through  $x$  in both directions”
- Whenever  $x = \alpha y + (1 - \alpha)z$  with  $y, z \neq x$  and  $\alpha \in (0, 1)$ , then either  $y$  or  $z$  must be infeasible.
- “If you write  $x$  as a convex combination of two feasible points  $y$  and  $z$ , the only possibility is  $x = y = z$ ”
- Such a point  $x$  is called an “**extreme point**”



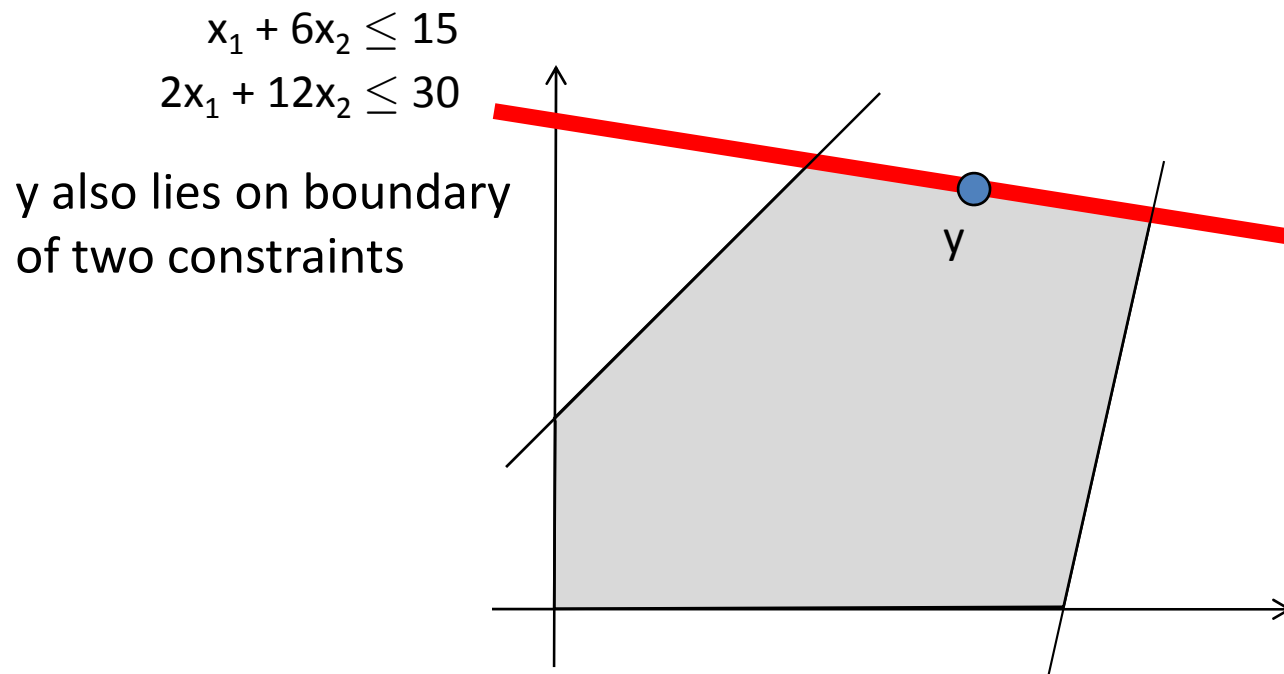
# What is a corner point?

- Attempt #3: “x lies on the boundary of many constraints”



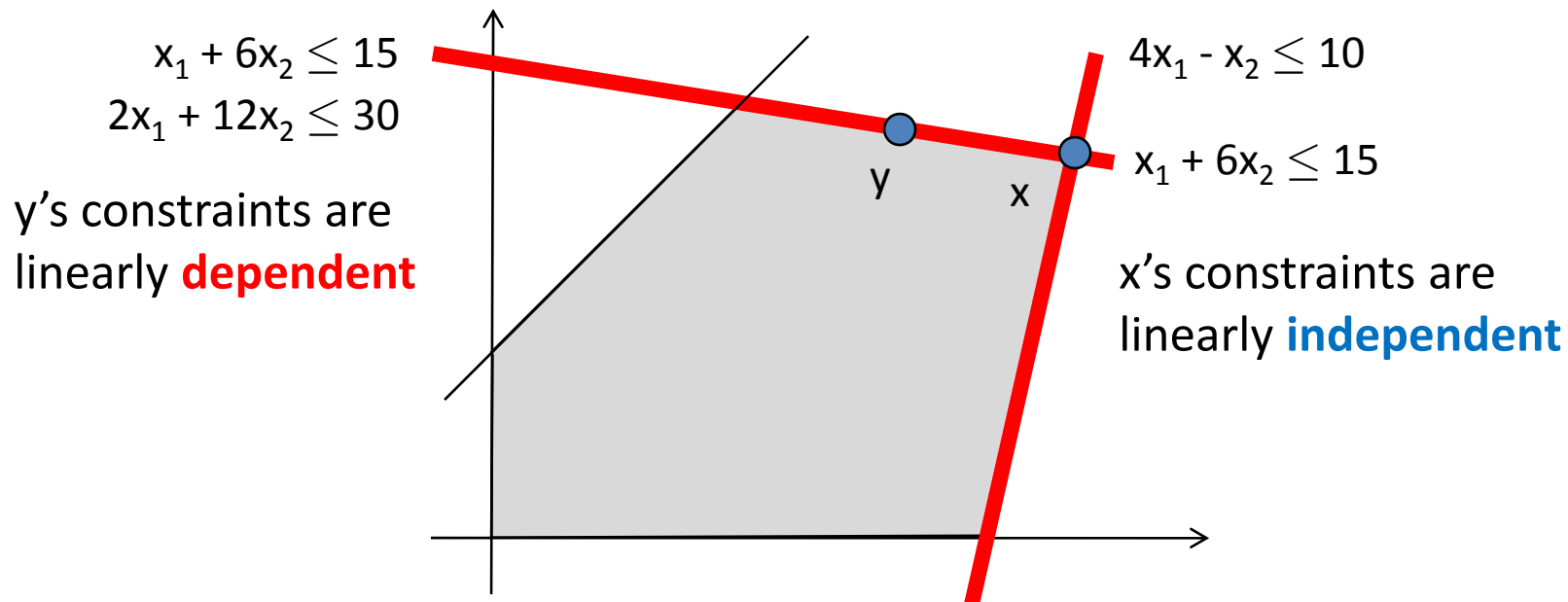
# What is a corner point?

- Attempt #3: “x lies on the boundary of **many** constraints”
- What if I introduce **redundant** constraints? **Not the right condition**



# What is a corner point?

- Revised Attempt #3: “ $x$  lies on the boundary of many **linearly independent constraints**”
- Feasible region:  $P = \{ x : a_i^T x \leq b_i \ \forall i \} \subset \mathbb{R}^n$
- Let  $\mathcal{I}_x = \{ i : a_i^T x = b_i \}$  and  $\mathcal{A}_x = \{ a_i : i \in \mathcal{I}_x \}$ . (“**Tight constraints**”)
- $x$  is a “**basic feasible solution (BFS)**” if  $\text{rank } \mathcal{A}_x = n$



**Lemma:** Let  $P$  be a polyhedron. The following are equivalent.

- i.  $x$  is a vertex (unique maximizer)
- ii.  $x$  is an extreme point (not convex combination of other points)
- iii.  $x$  is a basic feasible solution (BFS) (tight constraints have rank  $n$ )

**Proof** of (i)  $\Rightarrow$  (ii):

$x$  is a vertex  $\Rightarrow \exists c$  s.t.  $x$  is unique maximizer of  $c^T x$  over  $P$

Suppose  $x = \alpha y + (1-\alpha)z$  where  $y, z \in P$  and  $\alpha \in (0, 1)$ .

Suppose  $y \neq x$ . Then

$$c^T x = \alpha \underbrace{c^T y}_{< c^T x} + (1-\alpha) \underbrace{c^T z}_{\leq c^T x} \quad \begin{array}{l} \text{(since } c^T x \text{ is optimal value)} \\ \text{(since } x \text{ is unique optimizer)} \end{array}$$

$$\Rightarrow c^T x < \alpha c^T x + (1-\alpha) c^T x = c^T x \quad \textbf{Contradiction!}$$

So  $y=x$ . Symmetrically,  $z=x$ .

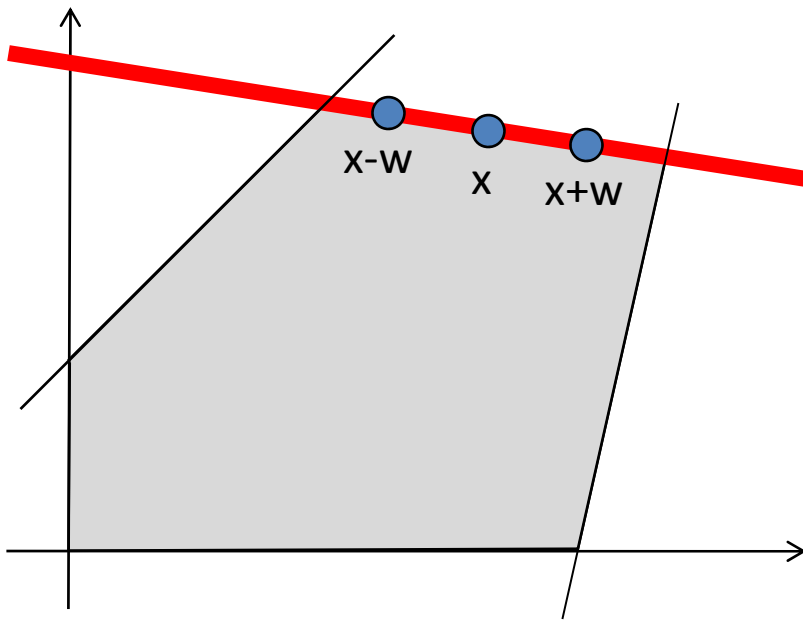
So  $x$  is an extreme point of  $P$ . ■

**Lemma:** Let  $P = \{ x : a_i^T x \leq b_i, \forall i \} \subset \mathbb{R}^n$ . The following are equivalent.

- i.  $x$  is a vertex (unique maximizer)
- ii.  $x$  is an extreme point (not convex combination of other points)
- iii.  $x$  is a basic feasible solution (BFS) (tight constraints have rank  $n$ )

**Proof Idea** of (ii)  $\Rightarrow$  (iii):

$x$  not a BFS  $\Rightarrow \text{rank } \mathcal{A}_x \leq n-1$



- Each tight constraint removes one degree of freedom
- At least one degree of freedom remains
- So  $x$  can “wobble” while staying on all the tight constraints
- Then  $x$  is a convex combination of two points obtained by “wiggling”.
- So  $x$  is not an extreme point.



**Lemma:** Let  $P = \{ x : a_i^T x \leq b_i, \forall i \} \subset \mathbb{R}^n$ . The following are equivalent.

- i.  $x$  is a vertex (unique maximizer)
- ii.  $x$  is an extreme point (not convex combination of other points)
- iii.  $x$  is a basic feasible solution (BFS) (tight constraints have rank  $n$ )

**Proof of (ii)  $\Rightarrow$  (iii):**  $x$  **not** a BFS  $\Rightarrow \text{rank } \mathcal{A}_x < n$  (Recall  $\mathcal{A}_x = \{ a_i : a_i^T x = b_i \}$ )

**Claim:**  $\exists w \in \mathbb{R}^n, w \neq 0$ , s.t.  $a_i^T w = 0 \forall a_i \in \mathcal{A}_x$  ( $w$  orthogonal to all of  $\mathcal{A}_x$ )

**Proof:** Let  $M$  be matrix whose rows are the  $a_i$ 's in  $\mathcal{A}_x$ .

$\dim \text{row-space}(M) + \dim \text{null-space}(M) = n$

But  $\dim \text{row-space}(M) < n \Rightarrow \exists w \neq 0$  in the null space.  $\square$

**Lemma:** Let  $P = \{ x : a_i^T x \leq b_i, \forall i \} \subset \mathbb{R}^n$ . The following are equivalent.

- i.  $x$  is a vertex (unique maximizer)
- ii.  $x$  is an extreme point (not convex combination of other points)
- iii.  $x$  is a basic feasible solution (BFS) (tight constraints have rank  $n$ )

**Proof of (ii)  $\Rightarrow$  (iii):**  $x$  **not** a BFS  $\Rightarrow \text{rank } \mathcal{A}_x < n$  (Recall  $\mathcal{A}_x = \{ a_i : a_i^T x = b_i \}$ )

**Claim:**  $\exists w \in \mathbb{R}^n, w \neq 0, \text{ s.t. } a_i^T w = 0 \ \forall a_i \in \mathcal{A}_x$  ( $w$  orthogonal to all of  $\mathcal{A}_x$ )

Let  $y = x + \epsilon w$  and  $z = x - \epsilon w$ , where  $\epsilon > 0$ .

**Claim:** If  $\epsilon$  very small then  $y, z \in P$ .

**Proof:** First consider tight constraints at  $x$ . (i.e., those in  $\mathcal{I}_x$ )

$$a_i^T y = a_i^T x + \epsilon a_i^T w = b_i + 0$$

So  $y$  satisfies this constraint. Similarly for  $z$ .

Next consider the loose constraints at  $x$ . (i.e., those not in  $\mathcal{I}_x$ )

$$b_i - a_i^T y = \underbrace{b_i - a_i^T x}_{\text{Positive}} - \underbrace{\epsilon a_i^T w}_{\text{As small as we like}} \geq 0$$

So  $y$  satisfies these constraints. Similarly for  $z$ .  $\square$

Then  $x = \alpha y + (1 - \alpha)z$ , where  $y, z \in P, y, z \neq x$ , and  $\alpha = 1/2$ .

So  $x$  is **not** an extreme point.  $\blacksquare$

**Lemma:** Let  $P = \{ x : a_i^T x \leq b_i \forall i \} \subset \mathbb{R}^n$ . The following are equivalent.

- i.  $x$  is a vertex (unique maximizer)
- ii.  $x$  is an extreme point (not convex combination of other points)
- iii.  $x$  is a basic feasible solution (BFS) (tight constraints have rank  $n$ )

**Proof of (iii)  $\Rightarrow$  (i):** Let  $x$  be a BFS  $\Rightarrow \text{rank } \mathcal{A}_x = n$  (Recall  $\mathcal{A}_x = \{ a_i : a_i^T x = b_i \}$ )

Let  $c = \sum_{i \in \mathcal{I}_x} a_i$ .

**Claim:**  $c^T x = \sum_{i \in \mathcal{I}_x} b_i$

**Proof:**  $c^T x = \sum_{i \in \mathcal{I}_x} a_i^T x = \sum_{i \in \mathcal{I}_x} b_i$ .  $\square$

**Claim:**  $x$  is an optimal point of  $\max \{ c^T x : x \in P \}$ .

**Proof:**  $y \in P \Rightarrow a_i^T y \leq b_i$  for all  $i$  If one of these is strict, then this is strict.  
 $\Rightarrow c^T y = \sum_{i \in \mathcal{I}_x} a_i^T y \leq \sum_{i \in \mathcal{I}_x} b_i = c^T x$ .  $\square$

**Claim:**  $x$  is the **unique** optimal point of  $\max \{ c^T x : x \in P \}$ .

**Proof:** If for any  $i \in \mathcal{I}_x$  we have  $a_i^T y < b_i$  then  $c^T y < c^T x$ .

So every optimal point  $y$  has  $a_i^T y = b_i$  for all  $i \in \mathcal{I}_x$ .

Since  $\text{rank } \mathcal{A}_x = n$ , there is only one solution:  $y = x$ !  $\square$

So  $x$  is a vertex.  $\blacksquare$

**Lemma:** Let  $P = \{ x : a_i^T x \leq b_i, \forall i \} \subset \mathbb{R}^n$ . The following are equivalent.

- i.  $x$  is a vertex (unique maximizer)
- ii.  $x$  is an extreme point (not convex combination of other points)
- iii.  $x$  is a basic feasible solution (BFS) (tight constraints have rank  $n$ )

## Interesting Corollary

**Corollary:** Any polyhedron has finitely many extreme points.

**Proof:** Suppose the polyhedron is defined by  $m$  inequalities.

Each extreme point is a BFS, so it corresponds to a choice of  $n$  linearly independent tight constraints.

There are  $\leq \binom{m}{n}$  ways to choose these tight constraints. ■

# Optimal solutions at extreme points

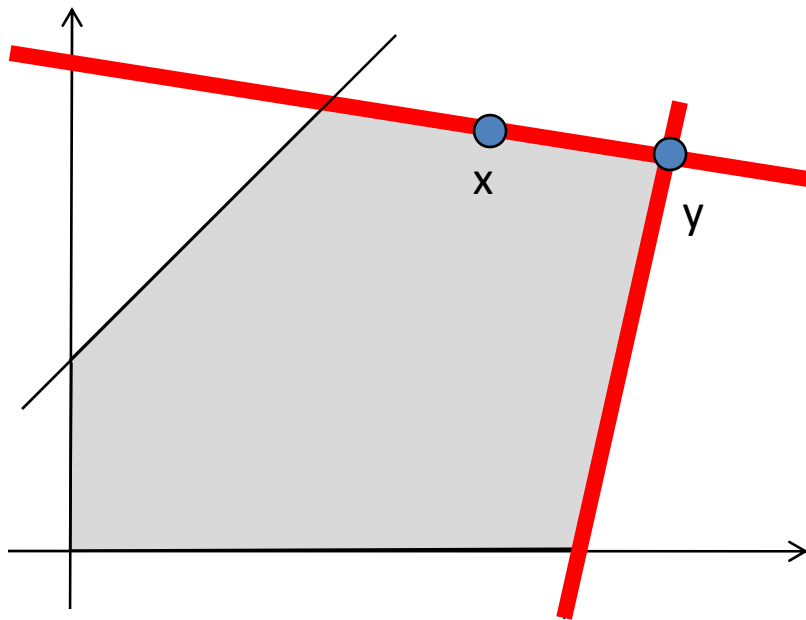
**Definition:** A **line** is a set  $L = \{ r + \lambda s : \lambda \in \mathbb{R} \}$  where  $r, s \in \mathbb{R}^n$  and  $s \neq 0$ .

**Lemma:** Let  $P = \{ x : a_i^T x \leq b_i \ \forall i \}$ . Suppose  $P$  does not contain any line.

Suppose the LP  $\max \{ c^T x : x \in P \}$  has an optimal solution.

Then some extreme point is an optimal solution.

**Proof Idea:** Let  $x$  be optimal. Suppose  $x$  not a BFS.



- At least one degree of freedom remains at  $x$
- So  $x$  can “wobble” while staying on all the tight constraints
- $x$  cannot wiggle off to infinity in both directions because  $P$  contains no line
- So when  $x$  wiggles, it hits a constraint
- When it hits first constraint, it is still feasible.
- So we have found a point  $y$  which has a new tight constraint.
- Repeat until we get a BFS.

**Lemma:** Let  $P = \{ x : a_i^T x \leq b_i \ \forall i \}$ . Suppose  $P$  does not contain any line.

Suppose the LP  $\max \{ c^T x : x \in P \}$  has an optimal solution.

Then some extreme point is an optimal solution.

**Proof:** Let  $x$  be optimal, with maximal number of tight constraints.

Suppose  $x$  not a BFS.

**Claim:**  $\exists w \in \mathbb{R}^n, w \neq 0, \text{ s.t. } a_i^T w = 0 \ \forall i \in \mathcal{I}_x$  (We saw this before)

Let  $y(\epsilon) = x + \epsilon w$ . Suppose  $c^T w = 0$ .

**Claim:**  $\exists \delta \text{ s.t. } y(\delta) \notin P$ . WLOG  $\delta > 0$ . (Otherwise  $P$  contains a line)

Set  $\delta = 0$  and gradually increase  $\delta$ . What is largest  $\delta$  s.t.  $y(\delta) \in P$ ?

$$y(\delta) \in P \Leftrightarrow a_i^T y(\delta) \leq b_i \ \forall i$$

$$\Leftrightarrow a_i^T x + \delta a_i^T w \leq b_i \ \forall i \quad (\text{Always satisfied if } a_i^T w \leq 0)$$

$$\Leftrightarrow \delta \leq (b_i - a_i^T x) / a_i^T w \ \forall i \text{ s.t. } a_i^T w > 0$$

Let  $h$  be the  $i$  that minimizes **this**. So  $\delta = (b_h - a_h^T x) / a_h^T w$ .

$y(\delta)$  is also optimal because  $c^T y(\delta) = c^T (x + \delta w) = c^T x$ .

But  $y(\delta)$  has one more tight constraint than  $x$ . Contradiction!

**Lemma:** Let  $P = \{ x : a_i^T x \leq b_i \ \forall i \}$ . Suppose  $P$  does not contain any line.  
Suppose the LP  $\max \{ c^T x : x \in P \}$  has an optimal solution.  
Then some extreme point is an optimal solution.

**Proof:** Let  $x$  be optimal, with maximal number of tight constraints.  
Suppose  $x$  not a BFS.

**Claim:**  $\exists w \in \mathbb{R}^n, w \neq 0, \text{ s.t. } a_i^T w = 0 \ \forall i \in \mathcal{I}_x$  (We saw this before)

Let  $y(\epsilon) = x + \epsilon w$ . Suppose  $c^T w > 0$ .

**Claim:**  $\exists \delta > 0 \text{ s.t. } y(\delta) \in P$ . (Same argument as before)

But then  $c^T y(\delta) = c^T (x + \delta w) > c^T x$ .

This contradicts optimality of  $x$ . ■

**Lemma:** Let  $P = \{ x : a_i^T x \leq b_i, \forall i \}$ . Suppose  $P$  does not contain any line.  
Suppose the LP  $\max \{ c^T x : x \in P \}$  has an optimal solution.  
Then some extreme point is an optimal solution.

## Interesting Consequence

### A simple but finite algorithm for solving LPs

**Input:** An LP  $\max \{ c^T x : x \in P \}$  where  $P = \{ x : a_i^T x \leq b_i, \forall i = 1 \dots m \}$ .

Caveat: We assume  $P$  contains no line, and the LP has an optimal solution.

**Output:** An optimal solution.

For every choice of  $n$  of the constraints

    If these constraints are linearly independent

        Find the unique point  $x$  for which these constraints are tight

        If  $x$  is feasible, add it to a list of all extreme points.

    End

End

Output the extreme point that maximizes  $c^T x$



# Dimension of Sets

- **Def:** An **affine space**  $A$  is a set  $A = \{ x+z : x \in L \}$ , where  $L$  is a linear space and  $z$  is any vector. The **dimension** of  $A$  is  $\dim L$ .
- Let's say  $\dim \emptyset = -1$ .
- **Def:** Let  $C \subseteq \mathbb{R}^n$  be non-empty. The **dimension** of  $C$  is  $\min \{ \dim A : A \text{ is an affine space with } C \subseteq A \}$ .

# Faces

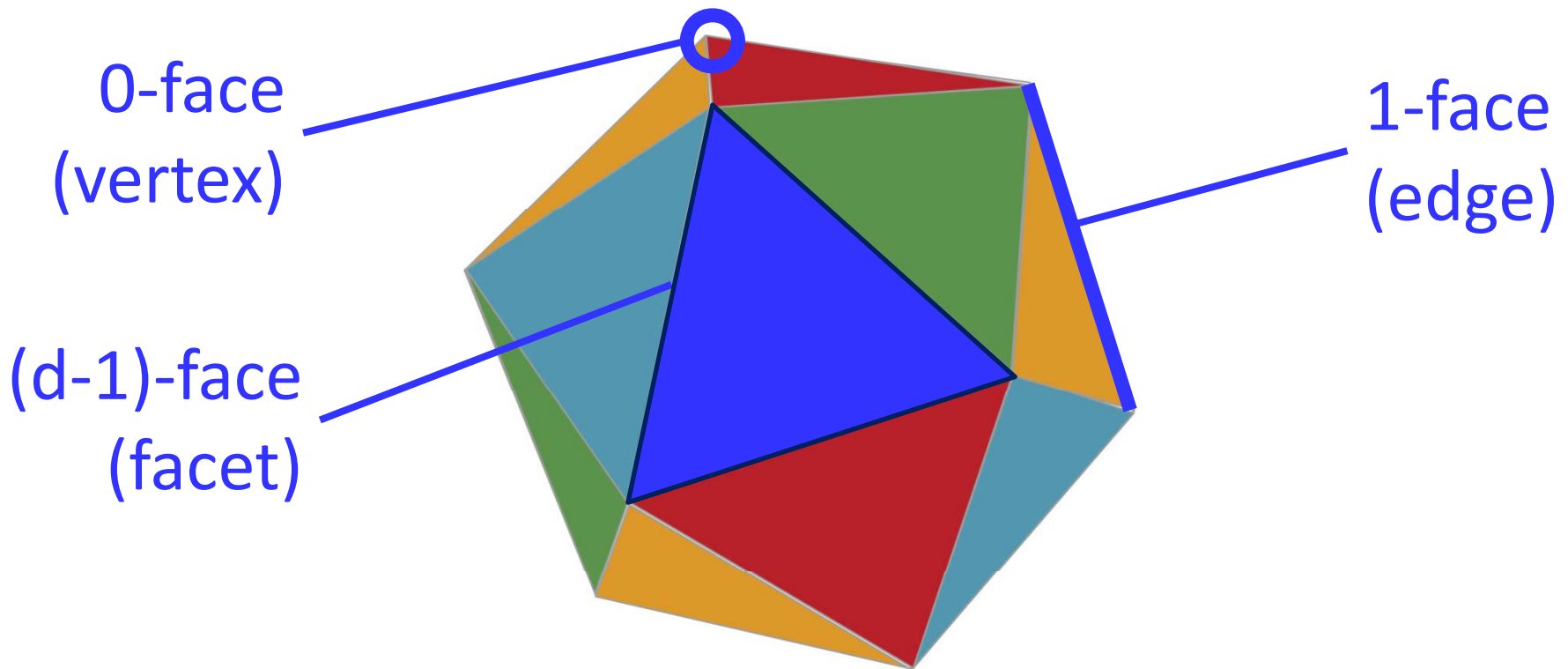
- **Def:** Let  $C \subseteq \mathbb{R}^n$  be any convex set. An inequality “ $a^T x \leq b$ ” is called **valid for C** if  $a^T x \leq b \quad \forall x \in C$ .
- **Def:** Let  $P \subseteq \mathbb{R}^n$  be a polyhedron. A **face** of  $P$  is a set
$$F = P \cap \{x : a^T x = b\}$$
where “ $a^T x \leq b$ ” is a valid inequality for  $P$ .
- Clearly every face of  $P$  is also a polyhedron.
- **Claim:**  $P$  is a face of  $P$ .
- **Proof:** Take  $a=0$  and  $b=0$ .
- **Claim:**  $\emptyset$  is a face of  $P$ .
- **Proof:** Take  $a=0$  and  $b=1$ .

# k-Faces

- **Def:** Let  $P \subseteq \mathbb{R}^n$  be a polyhedron. A **face** of  $P$  is a set
$$F = P \cap \{x : a^T x = b\}$$
where “ $a^T x \leq b$ ” is a valid inequality for  $P$ .
- **Def:** A face  $F$  with  $\dim F = k$  is called a **k-face**.
- Suppose  $\dim P = d$ 
  - A  $(d-1)$ -face is called a **facet**.
  - A 1-face is called an **edge**.
  - A 0-face  $F$  has the form  $F = \{v\}$  where  $v \in P$ .
- **Claim:** If  $F = \{v\}$  is a 0-face then  $v$  is a vertex of  $P$ .

# k-Faces

- **Def:** Let  $P \subseteq \mathbb{R}^n$  be a polyhedron. A **face** of  $P$  is a set
$$F = P \cap \{x : a^T x = b\}$$
where “ $a^T x \leq b$ ” is a valid inequality for  $P$ .
- **Def:** A face  $F$  with  $\dim F = k$  is called a **k-face**.



# The Simplex Method

- “The obvious idea of moving along edges from one vertex of a convex polygon to the next” [Dantzig, 1963]

## Algorithm

Let  $x$  be any vertex (we assume LP is feasible)

For each edge containing  $x$

If moving along the edge increases the objective function

If the edge is infinitely long,

**Halt:** LP is unbounded

Else

Set  $x$  to be other vertex in the edge

Restart loop

Halt:  $x$  is optimal

- In practice, very efficient.
- In theory, very hard to analyze.
- How many edges must we traverse in the worst case?

# Why is analyzing the simplex method hard?

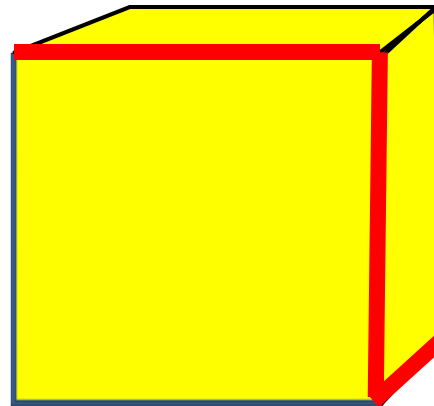
- For any polyhedron, and for any two vertices, are they connected by a path of few edges?
- [The Hirsch Conjecture](#) (1957)  
Let  $P = \{ x : Ax \leq b \}$  where  $A$  has size  $m \times n$ . Then any two vertices are connected by a path of  $\leq m-n$  edges.

**Example:** A cube.

Dimension  $n=3$ .

# constraints  $m=6$ .

Connected by a length-3 path?



Yes!

# Why is analyzing the simplex method hard?

- For any polyhedron, and for any two vertices, are they connected by a path of few edges?
- [The Hirsch Conjecture](#) (1957)  
Let  $P = \{ x : Ax \leq b \}$  where  $A$  has size  $m \times n$ . Then any two vertices are connected by a path of  $\leq m-n$  edges.
- We have no idea how to prove this.
- **Disproved!** There is a polytope with  $n=43$ ,  $m=86$ , and two vertices with no path of length  $\leq 43$  [[Santos, 2010](#)].
- **Theorem:** [Kalai-Kleitman 1992] There is always a path with  $\leq m^{\log n+2}$  edges.