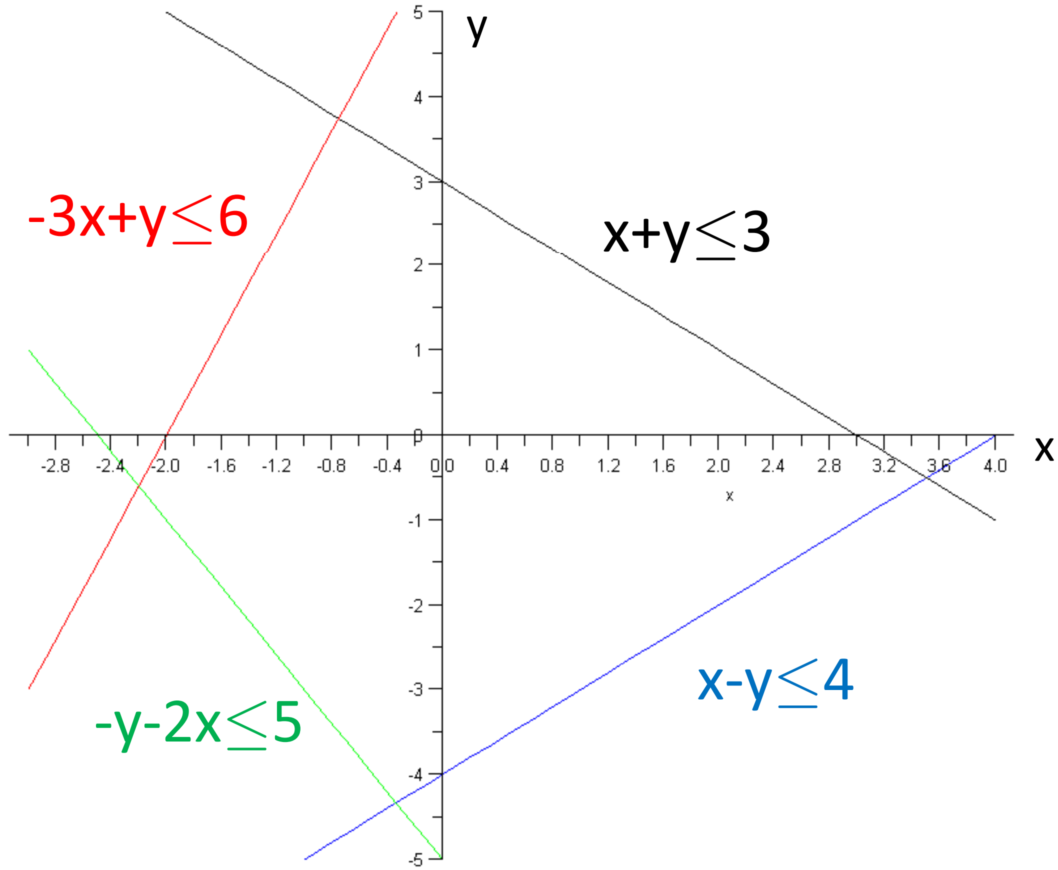
2D System of Inequalities

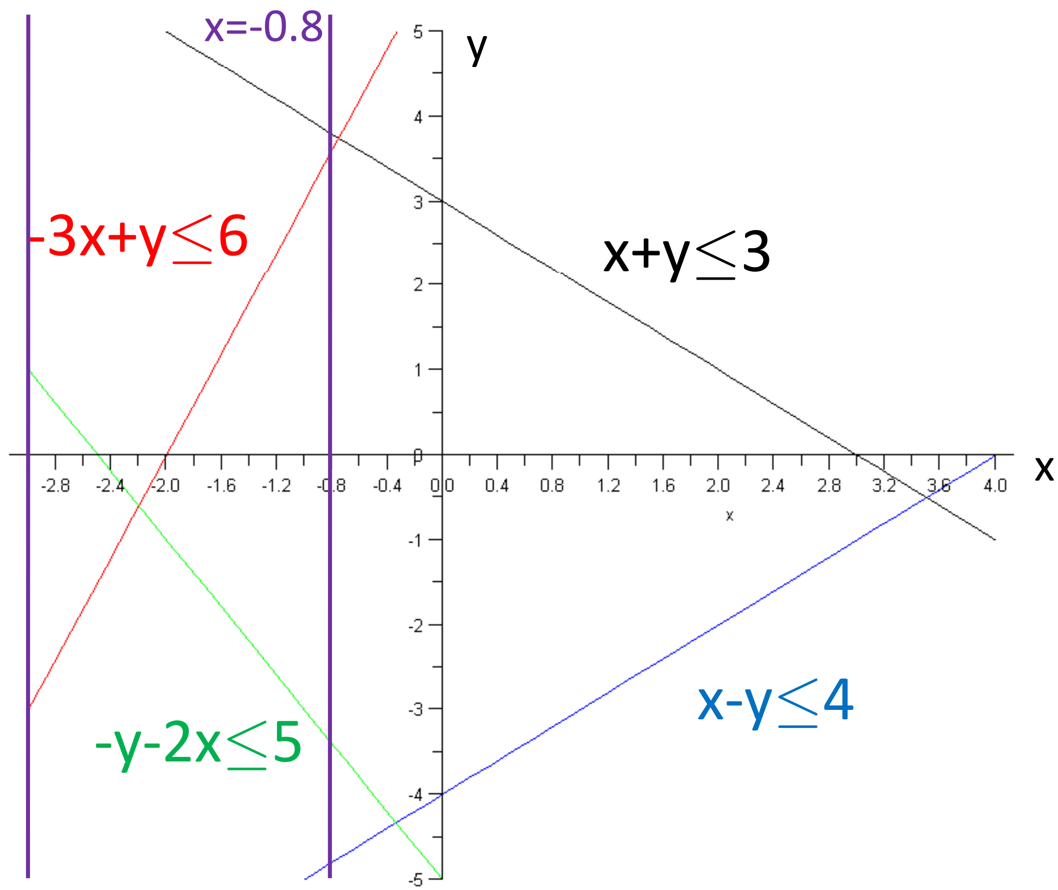
Consider the polyhedron

$Q = \{ (x,y) : -3x+y \leq 6,$
$x+y \leq 3,$
$-y-2x \leq 5,$
$x-y \leq 4 \}$



- **Given x, for what values of y is (x,y) feasible?**
  - Need: $y \leq 3x+6$, $y \leq -x+3$, $y \geq -2x-5$, and $y \geq x-4$

2D System of Inequalities



Consider the polyhedron
$Q = \{ (x,y) : -3x+y \leq 6,$
$x+y \leq 3,$
$-y-2x \leq 5,$
$x-y \leq 4 \}$

- **Given x, for what values of y is (x,y) feasible?**
  - i.e., $y \leq \min\{3x+6, -x+3\}$ and $y \geq \max\{ -2x-5, x-4 \}$
  - For x=-0.8, (x,y) feasible if $y \leq \min\{\mathbf{3.6}, 3.8\}$ and $y \geq \max\{\mathbf{-3.4}, -4.8\}$
  - For x=-3, (x,y) feasible if $y \leq \min\{\mathbf{-3}, 6\}$ and $y \geq \max\{\mathbf{1}, -7\}$ **Impossible!**

2D System of Inequalities

- Consider the set
  $Q = \{ (x,y) : -3x+y \leq 6,\ x+y \leq 3,\ -y-2x \leq 5,\ x-y \leq 4 \}$
- Given x, for what values of y is (x,y) feasible?
  - i.e., $y \leq \min\{3x+6, -x+3\}$ and $y \geq \max\{ -2x-5, x-4 \}$
  - Such a y exists $\Leftrightarrow$ $\max\{-2x-5, x-4\} \leq \min\{3x+6, -x+3\}$

    $\Leftrightarrow$ the following inequalities are solvable

Every "lower" constraint is $\leq$ every "upper" constraint

$$\begin{aligned} -2x-5 &\leq 3x+6 \\ x-4 &\leq 3x+6 \\ -2x-5 &\leq -x+3 \\ x-4 &\leq -x+3 \end{aligned} \equiv Q' = \left\{ x : \begin{array}{c} -5x \leq 11 \\ -2x \leq 10 \\ -x \leq 8 \\ 2x \leq 7 \end{array} \right\} = \left\{ x : \begin{array}{l} x \geq -11/5 \\ x \geq -5 \\ x \geq -8 \\ x \leq 7/2 \end{array} \right\}$$

- **Conclusion:** Q is non-empty $\Leftrightarrow$ Q' is non-empty.
- This is easy to decide because Q' involves only 1 variable!

Joseph Fourier

Fourier-Motzkin Elimination

Theodore Motzkin

- **Generalization:** given a set $Q = \{ (x_1,\ldots,x_n) : Ax \leq b \}$, we want to find set $Q' = \{ (x'_1,\ldots,x'_{n-1}) : A'x' \leq b' \}$ satisfying

$$(x_1,\ldots,x_{n-1}) \in Q' \quad \Leftrightarrow \quad \exists x_n \text{ s.t. } (x_1,\ldots,x_{n-1},x_n) \in Q$$

- Q' is called a **projection** of Q (onto the first n-1 coordinates)
- Fourier-Motzkin Elimination is a procedure for producing Q' from Q
- **Consequences:**
  - An (inefficient!) algorithm for solving systems of inequalities, and hence for solving LPs too
  - A way of proving Farkas' Lemma by induction

- **Lemma:** Let $Q = \{ (x_1,\ldots,x_n) : Ax \leq b \}$. We can construct $Q' = \{ (x'_1,\ldots,x'_{n-1}) : A'x' \leq b' \}$ satisfying
  (1) $(x_1,\ldots,x_{n-1}) \in Q' \quad \Leftrightarrow \quad \exists x_n$ s.t. $(x_1,\ldots,x_{n-1},x_n) \in Q$
  (2) Every inequality defining Q' is a non-negative linear combination of the inequalities defining Q.
- **Proof:** Put inequalities of Q in three groups: (a_i = i^{th} row of A)
  $Z = \{ i : a_{i,n} = 0 \}$ $P = \{ j : a_{j,n} > 0 \}$ $N = \{ k : a_{k,n} < 0 \}$
- WLOG, $a_{j,n} = 1 \ \forall j \in P$ and $a_{k,n} = -1 \ \forall k \in N$
- For any $x \in \mathbb{R}^n$, let $x' \in \mathbb{R}^{n-1}$ be vector obtained by deleting coordinate x_n
- The constraints defining Q' are:
  - $a_i'x' \leq b_i \quad \forall i \in Z$
  - $a_j'x' + a_k'x' \leq b_j + b_k \quad \forall j \in P, \forall k \in N$
- This proves (2).

This is sum of j^{th} and k^{th} constraints of Q, because n^{th} coordinate of $a_j + a_k$ is zero!

- In fact, (2) implies the "$\Leftarrow$ direction" of (1):
  For every $x \in Q$, x' satisfies all inequalities defining Q'.
- Why? Because every constraint of Q' is a non-negative lin. comb. of constraints from Q, with n^{th} coordinate equal to 0.

- **Lemma:** Let $Q = \{ (x_1,\ldots,x_n) : Ax \leq b \}$. We can construct $Q' = \{ (x'_1,\ldots,x'_{n-1}) : A'x' \leq b' \}$ satisfying
  (1) $(x_1,\ldots,x_{n-1}) \in Q' \iff \exists x_n$ s.t. $(x_1,\ldots,x_{n-1},x_n) \in Q$
  (2) Every inequality defining Q' is a non-negative linear combination of the inequalities defining Q.
- **Proof:** Put inequalities of Q in three groups:
  $Z = \{ i : a_{i,n} = 0 \}$ $P = \{ j : a_{j,n} = 1 \}$ $N = \{ k : a_{k,n} = -1 \}$
- The constraints defining Q' are:
  - $a_i'x' \leq b_i \quad \forall i \in Z$
  - $a_j'x' + a_k'x' \leq b_j + b_k \quad \forall j \in P, \forall k \in N$
- It remains to prove the "$\Rightarrow$ direction" of (1).
- Note that: $a_k'x' - b_k \leq b_j - a_j'x' \quad \forall j \in P, \forall k \in N.$
  $\Rightarrow \underbrace{\max_{k \in N} \{ a_k'x' - b_k \}} \leq \min_{j \in P} \{ b_j - a_j'x' \}$

Let x_n be this value, and let $x = (x'_1,\ldots,x'_{n-1},x_n)$.

By definition of x, and since $a_{k,n} = -1$

By definition of x_n, $a_k'x' - b_k \leq x_n$

Then:
$$\left.\begin{aligned} a_k x - b_k &= a_k'x' - x_n - b_k \leq 0 \quad \forall k \in N \\ b_j - a_j x &= b_j - a_j'x' - x_n \geq 0 \quad \forall j \in P \\ a_i x &= a_i'x' \leq b_i \quad \forall i \in Z \end{aligned}\right\} \Rightarrow x \in Q$$

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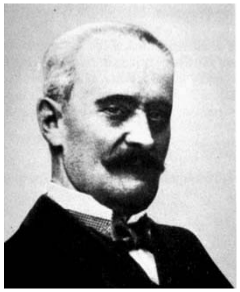
Variants of Farkas' Lemma



Gyula Farkas

The System	$Ax \leq b$	$Ax = b$
has **no** solution $x \geq \mathbf{0}$ iff	$\exists y \geq 0$, $A^T y \geq \mathbf{0}$, $b^T y < 0$	$\exists y \in \mathbb{R}^n$, $A^T y \geq \mathbf{0}$, $b^T y < 0$
has **no** solution $x \in \mathbb{R}^n$ iff	$\exists y \geq 0$, $A^T y = \mathbf{0}$, $b^T y < 0$	$\exists y \in \mathbb{R}^n$, $A^T y = \mathbf{0}$, $b^T y < 0$

We'll prove this one

- **Lemma:** Exactly one of the following holds:
  - There exists $x \in \mathbb{R}^n$ satisfying $Ax \leq b$
  - There exists $y \geq 0$ satisfying $y^T A = 0$ and $y^T b < 0$
- **Proof:** Suppose x exists. Need to show y cannot exist.

  Suppose y also exists. Then:

$$0 = 0x = y^{\mathsf{T}} Ax \leq y^{\mathsf{T}} b < 0$$

  Contradiction! y cannot exist.

- **Lemma:** Exactly one of the following holds:
  - There exists $x \in \mathbb{R}^n$ satisfying $Ax \leq b$
  - There exists $y \geq 0$ satisfying $y^T A = 0$ and $y^T b < 0$
- **Proof:** Suppose no solution x exists.

  We use induction. Trivial for n=0, so let $n \geq 1$.

  We use Fourier-Motzkin Elimination.

  Get an **equivalent** system $A'x' \leq b'$ where

  $$(A' | 0) = MA \qquad b' = Mb$$

  for some **non-negative matrix** M.

Lemma: Let $Q = \{ (x_1, \ldots, x_n) : Ax \leq b \}$. We can construct $Q' = \{ (x_1, \ldots, x_{n-1}) : A'x' \leq b' \}$ satisfying

1) Q is non-empty $\Leftrightarrow$ Q' is non-empty
2) Every inequality defining Q' is a **non-negative linear combination** of the inequalities defining Q.

(This statement is slightly simpler than our previous lemma)

- **Lemma:** Exactly one of the following holds:
  - There exists $x \in \mathbb{R}^n$ satisfying $Ax \leq b$
  - There exists $y \geq 0$ satisfying $y^T A = 0$ and $y^T b < 0$
- **Proof:**

Get an equivalent system $A'x' \leq b'$ where

$$(A' | 0) = MA \qquad b' = Mb$$

for some non-negative matrix M.

We assume $Ax \leq b$ has no solution, so $A'x' \leq b'$ has no solution.

By induction, $\exists y' \geq 0$ s.t. $y'^T A' = 0$ and $y'^T b' < 0$.

Define $y = M^T y'$.

Then: $y \geq 0$, because $y' \geq 0$ and M non-negative

$$y^T A = y'^T MA = y'^T (A' | 0) = 0$$

$$y^T b = y'^T Mb = y'^T b' < 0$$

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