CPSC 536N
Sparse Approximations
Winter 2013
Lecture 1

[N. Harvey]

Linear Program

- General definition
  - Parameters: $c, a_1, \ldots, a_m \in \mathbb{R}^n$, $b_1, \ldots, b_m \in \mathbb{R}$
  - Variables: $x \in \mathbb{R}^n$

$$\begin{array}{llll} \min & c^\top x & & \qquad \textbf{Objective function} \\ \text{s.t.} & a_i^\top x & \le b_i & \forall i = 1, \ldots, m \qquad \textbf{Constraints} \end{array}$$

- Terminology
  - **Feasible point**: any x satisfying constraints
  - **Optimal point**: any feasible x that minimizes obj. func
  - **Optimal value**: value of obj. func for any optimal point

Linear Program

- General definition
  - Parameters: $c, a_1,\ldots,a_m \in \mathbb{R}^n$, $b_1,\ldots,b_m \in \mathbb{R}$
  - Variables: $x \in \mathbb{R}^n$

$$\begin{aligned} \min \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & a_i^\mathsf{T} x \leq b_i \qquad \forall i = 1, \ldots, m \end{aligned}$$

- Matrix form

$$\begin{aligned} \min \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

- Parameters: $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$
- Variables: $x \in \mathbb{R}^n$

Simple LP Manipulations

- "max" instead of "min"

  $\max c^T x \quad \equiv \quad \min -c^T x$

- "$\geq$" instead of "$\leq$"

  $a^T x \geq b \quad \Leftrightarrow \quad -a^T x \leq -b$

- "=" instead of "$\leq$"

  $a^T x = b \quad \Leftrightarrow \quad a^T x \leq b$ and $a^T x \geq b$

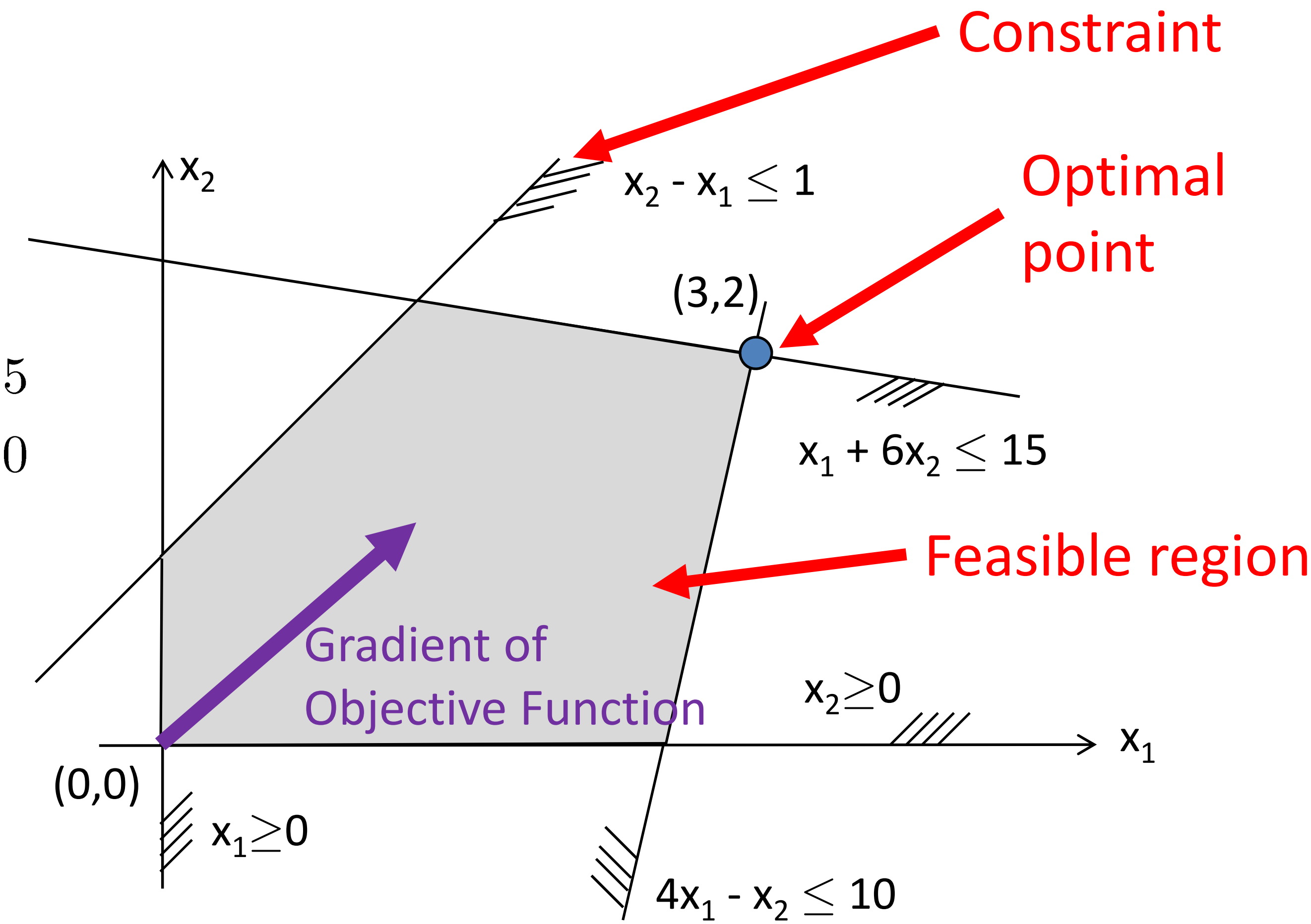
- **Note:** "<" and ">" are not allowed in constraints

  Because we want the feasible region to be **closed**, in the topological sense.

2D Example

Objective Function

$$
\begin{aligned}
\max \quad & \boxed{x_1 + x_2} \\
\text{s.t.} \quad & x_2 - x_1 && \leq 1 \\
& x_1 + 6x_2 && \leq 15 \\
& 4x_1 - x_2 && \leq 10 \\
& x_1 && \geq 0 \\
& x_2 && \geq 0
\end{aligned}
$$

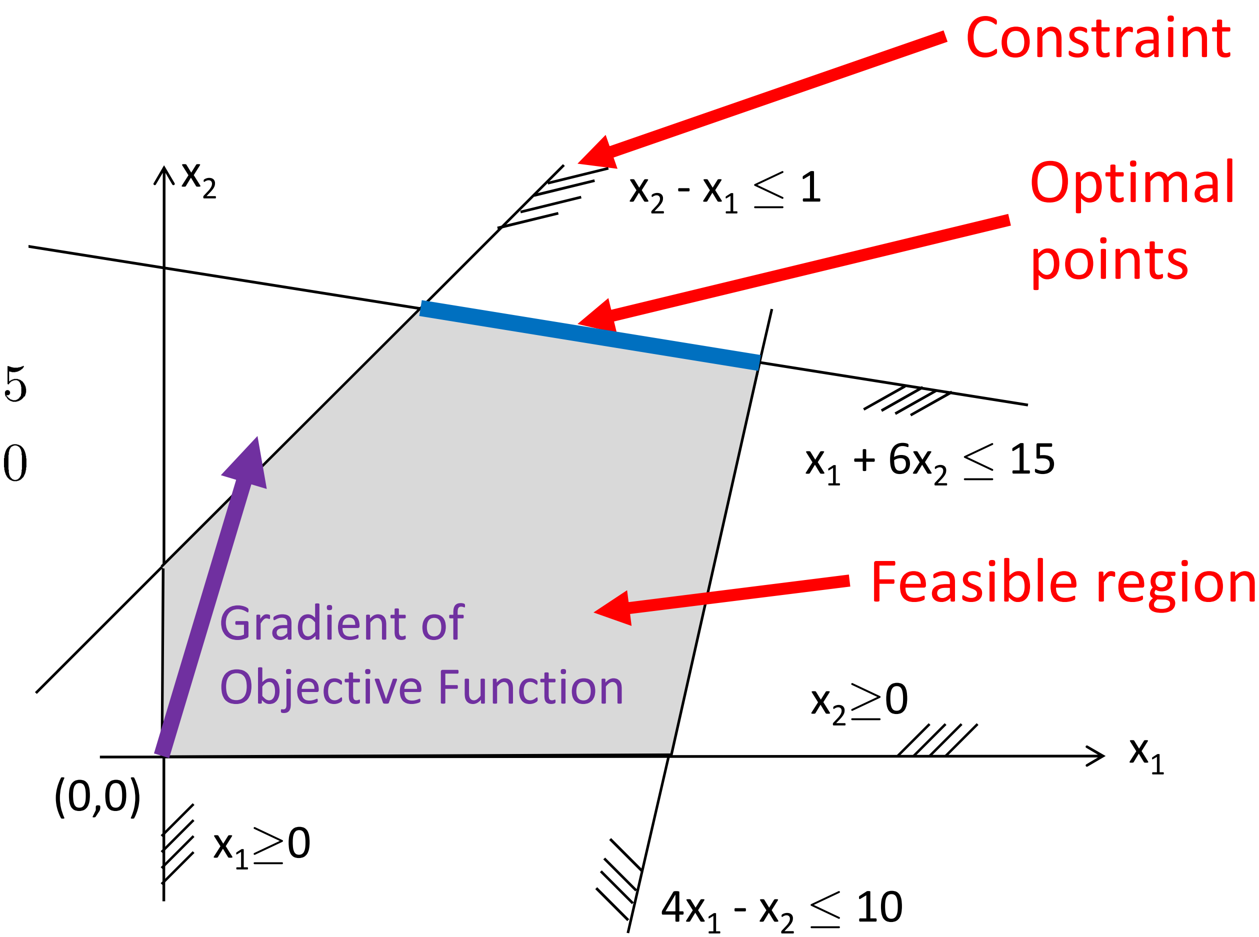


Unique optimal solution exists

2D Example

Objective Function

$$
\begin{array}{lll}
\max & \boxed{x_1/6 + x_2} & \\
\text{s.t.} & x_2 - x_1 & \leq 1 \\
 & x_1 + 6x_2 & \leq 15 \\
 & 4x_1 - x_2 & \leq 10 \\
 & x_1 & \geq 0 \\
 & x_2 & \geq 0
\end{array}
$$



Optimal solutions exist:
Infinitely many!

2D Example

$$
\begin{array}{lll}
\max & x_1 + x_2 & \\
\text{s.t.} & x_2 - x_1 & \geq 1 \\
& x_1 + 6x_2 & \leq 15 \\
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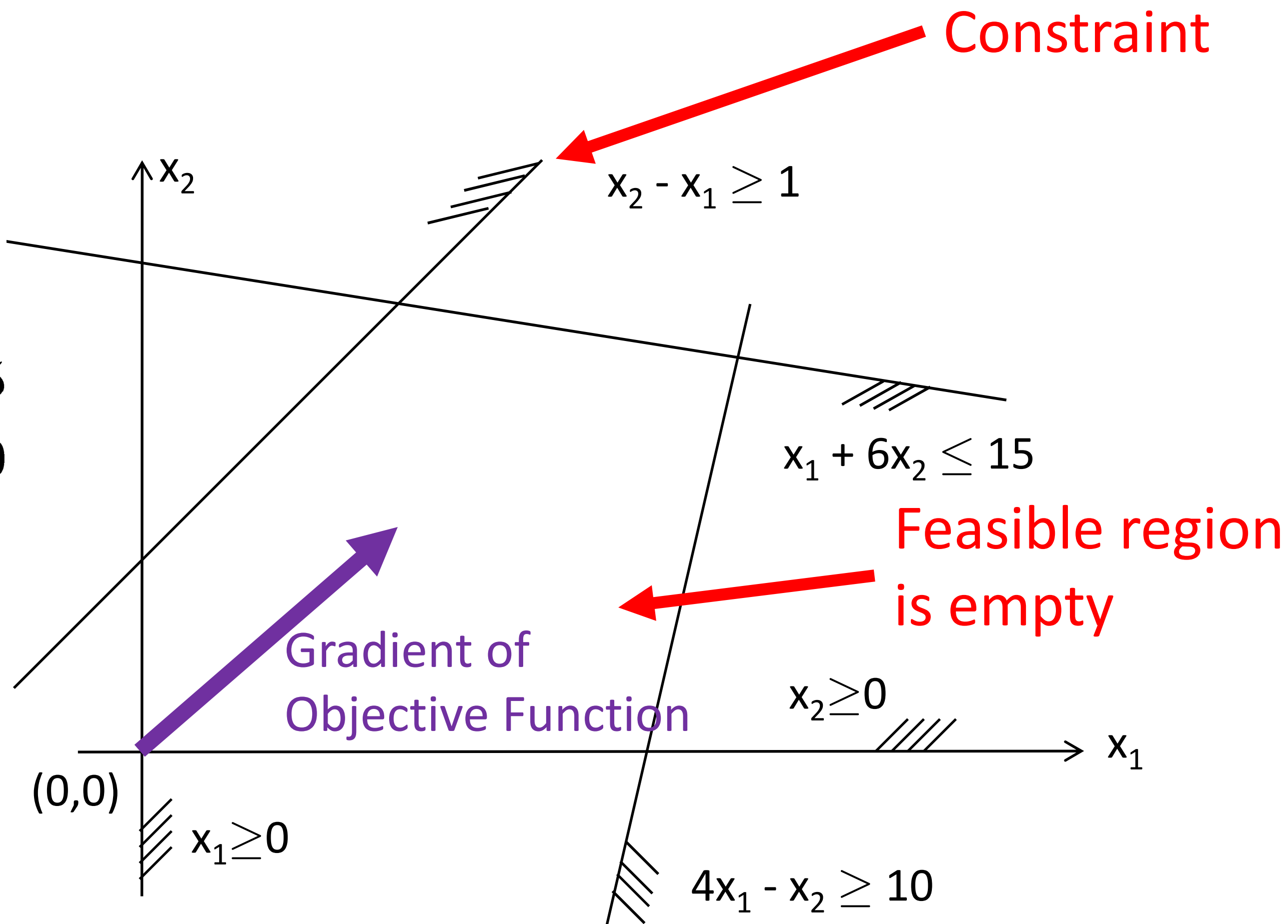


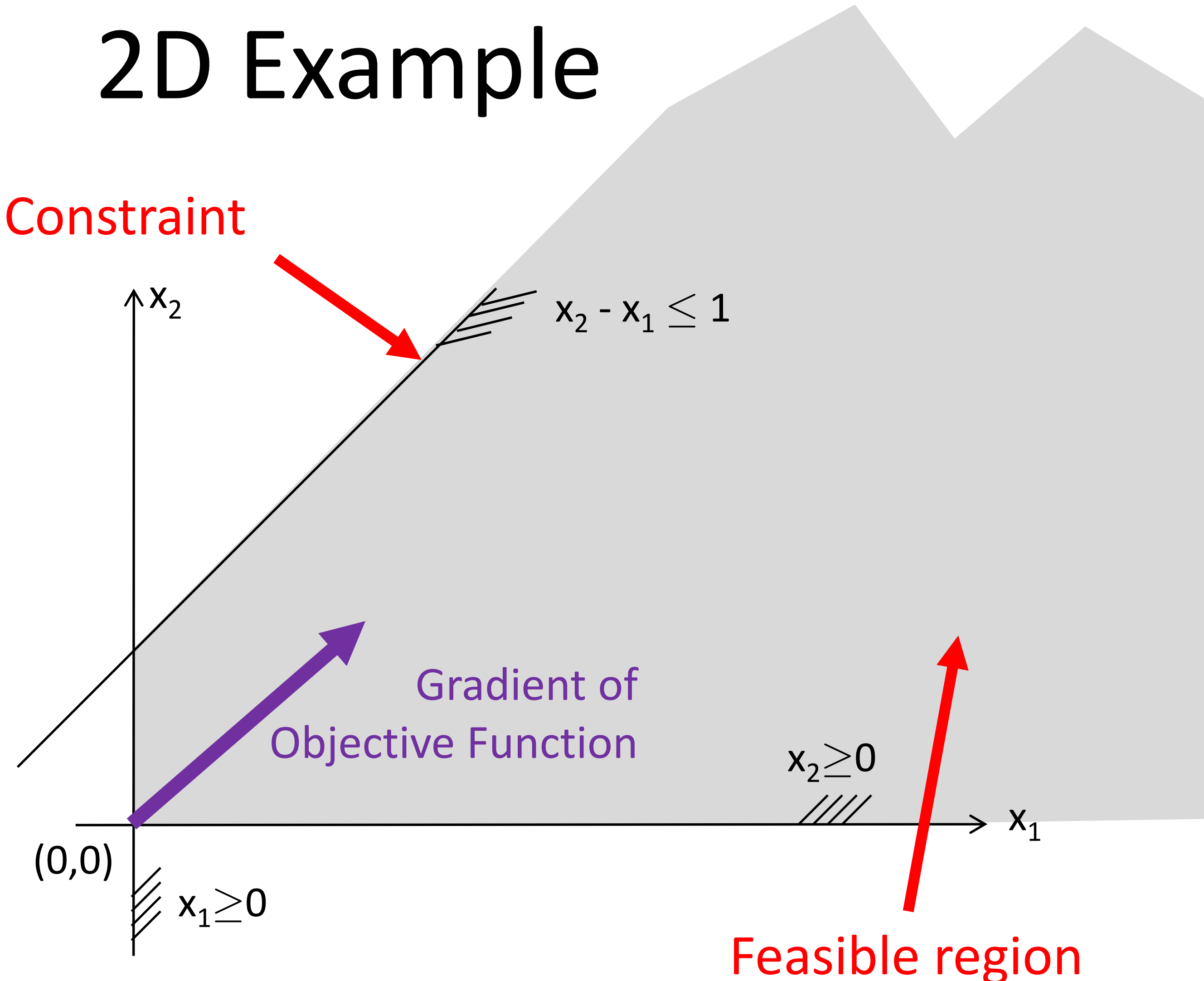
Infeasible

No feasible solutions

(so certainly no optimal solutions either)

2D Example

$$\begin{array}{llll} \max & x_1 + x_2 & & \\ \text{s.t.} & x_2 - x_1 & \leq 1 \\ & x_1 & \geq 0 \\ & x_2 & \geq 0 \end{array}$$



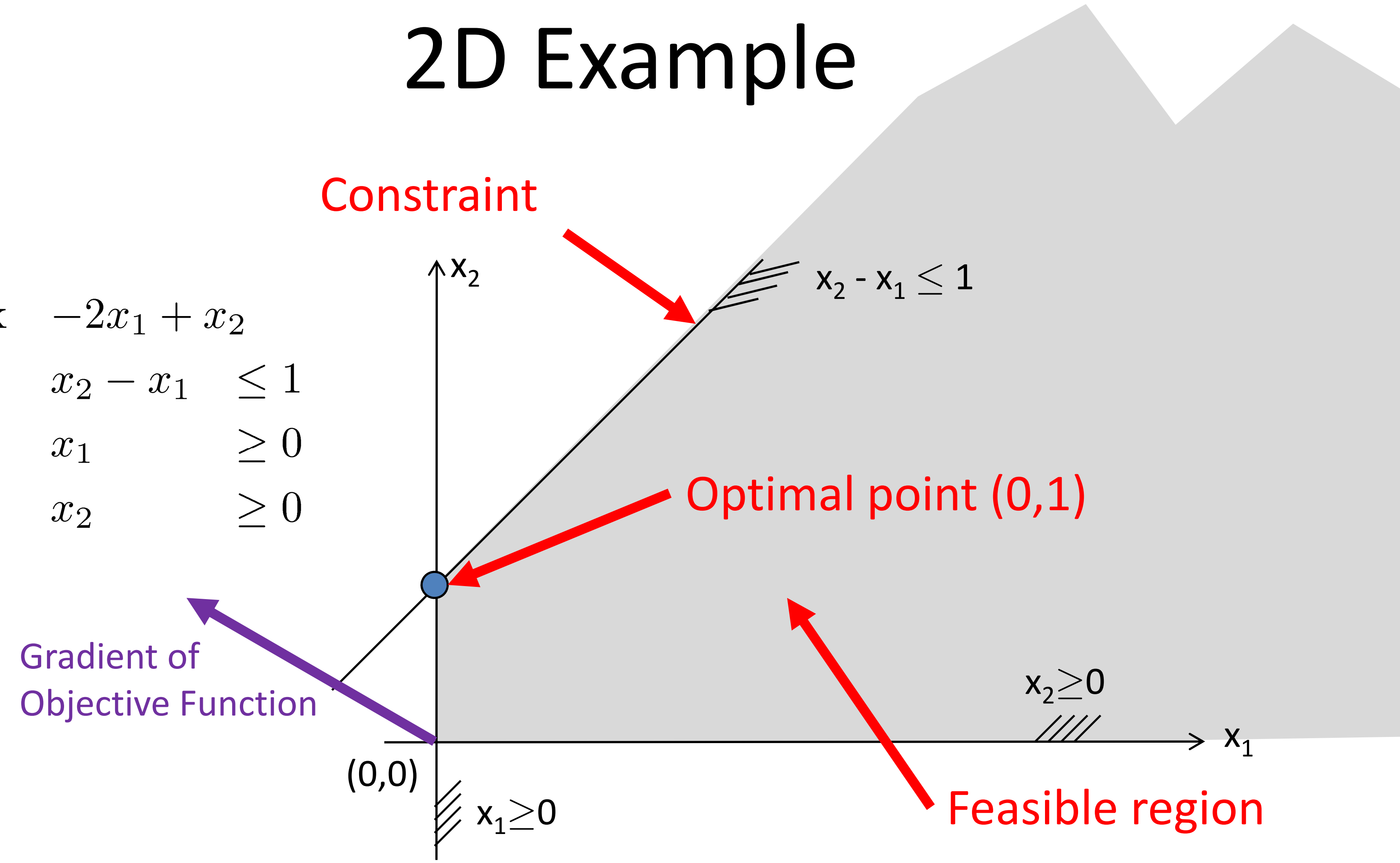
Unbounded

Feasible solutions, but no optimal solution

(Informally, “optimal value = ∞”)

2D Example

$$
\begin{array}{lll}
\max & -2x_1 + x_2 & \\
\text{s.t.} & x_2 - x_1 & \leq 1 \\
& x_1 & \geq 0 \\
& x_2 & \geq 0
\end{array}
$$



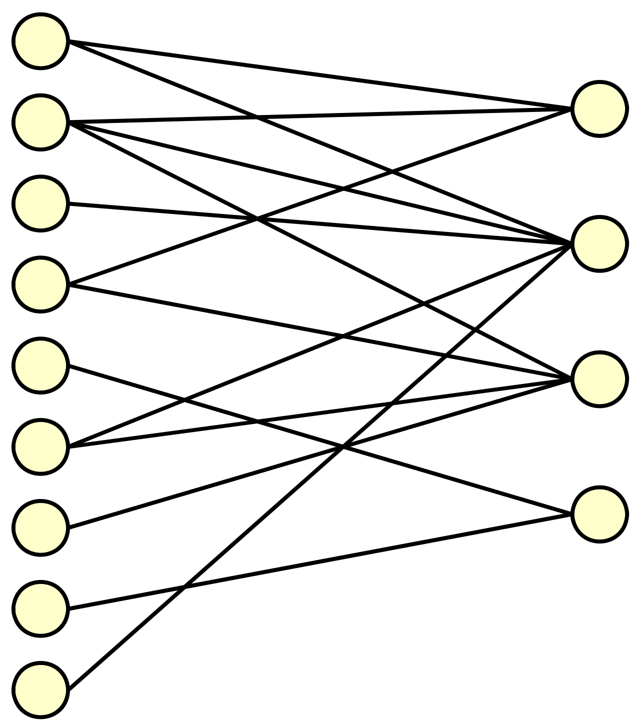
Important Point: This LP is **NOT** unbounded.
The feasible region is unbounded,
but optimal value is 1

“Fundamental Theorem” of LP

- **Theorem**: For any LP, the outcome is either:
  - Optimal solution (unique or infinitely many)
  - Infeasible
  - Unbounded
    (optimal value is ∞ for maximization problem, or $-\infty$ for minimization problem)
- The main point is: if the LP is feasible and not unbounded, then the supremum is achieved.

Example: Bipartite Matching

- Given bipartite graph G=(V, E)
- Find a maximum size matching
  - A set M $\subseteq$ E s.t. every vertex has at most one incident edge in M



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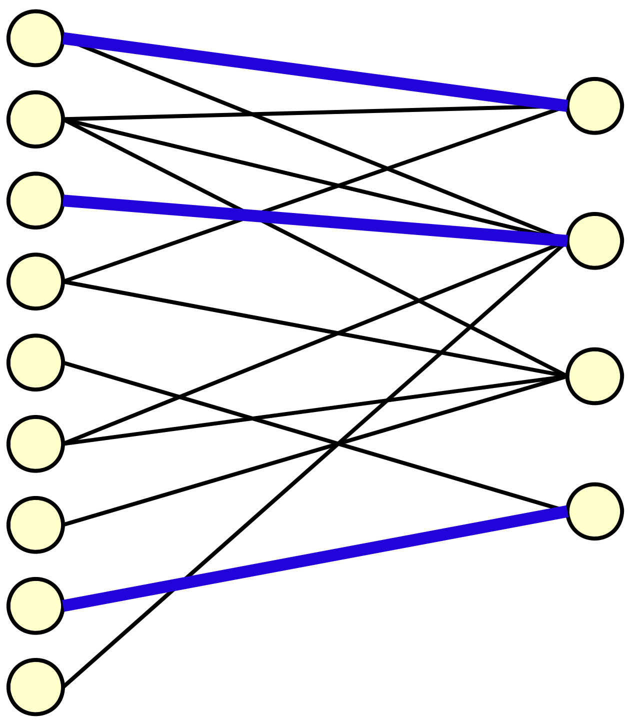


The blue edges are a matching M

Example: Bipartite Matching

- Given bipartite graph G=(V, E)
- Find a maximum size matching
  - A set M $\subseteq$ E s.t. every vertex has at most one incident edge in M
- The natural integer program

(IP)
$$
\begin{array}{llll}
\max & \sum_{e \in E} x_e & & \\
\text{s.t.} & \sum_{e \text{ incident to } v} x_e & \leq 1 & \forall v \in V \\
& x_e & \in \{0,1\} & \forall e \in E
\end{array}
$$

- Solving IPs is very hard. Try an LP instead.

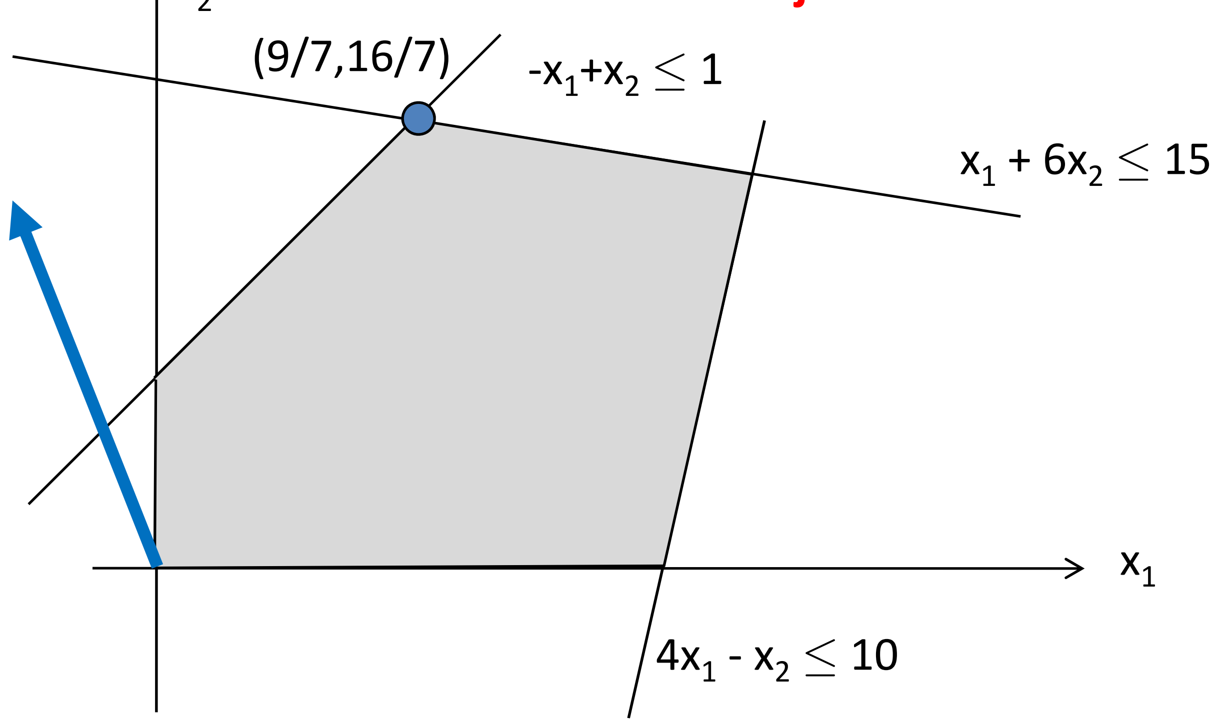
(LP)
$$
\begin{array}{llll}
\max & \sum_{e \in E} x_e & & \\
\text{s.t.} & \sum_{e \text{ incident to } v} x_e & \leq 1 & \forall v \in V \\
& x_e & \geq 0 & \forall e \in E
\end{array}
$$

- **Theorem**: (IP) and (LP) have the same solution!
- **Proof**: Later in the course!
- **Corollary**: Bipartite matching can be solved by LP algorithms.

Duality: Proving optimality

- **Question:** What is optimal point in direction c = (-7,14)?
- **Solution:** Optimal point is x=(9/7,16/7), optimal value is 23.
- How can I be sure?
  - **Every** feasible point satisfies $x_1+6x_2 \leq 15$
  - **Every** feasible point satisfies $-x_1+x_2 \leq 1 \;\Rightarrow\; -8x_1+8x_2 \leq 8$
  - **Every** feasible point satisfies their sum: $\underbrace{-7x_1+14x_2}_{\text{This is the objective function!}} \leq 23$

$$
\begin{array}{lll}
\max & -7x_1 + 14x_2 & \\
\text{s.t.} & -x_1 + x_2 & \leq 1 \\
 & x_1 + 6x_2 & \leq 15 \\
 & 4x_1 - x_2 & \leq 10 \\
 & x & \geq 0
\end{array}
$$



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  - **Every** feasible point satisfies their sum: $\underbrace{-7x_1+14x_2}_{\text{This is the objective function!}} \leq 23$

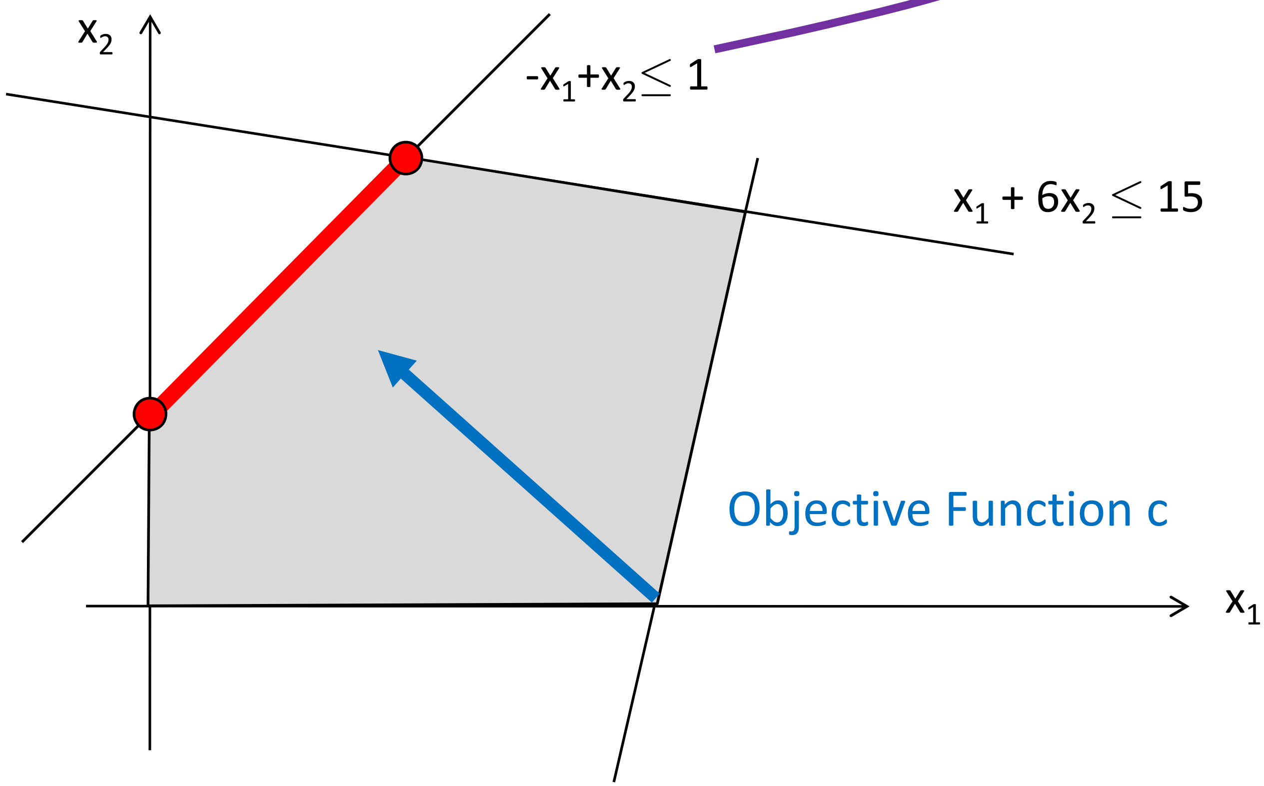
- **Certificates**
  - To convince you that optimal value is $\geq$ k, I can find x such that $c^T x \geq k$.
  - To convince you that optimal value is $\leq$ k, I can find a linear combination of the constraints which proves that $c^T x \leq k$.
- **"Strong Duality Theorem":** Such certificates always exists.

Duality: Geometric View

- Suppose c=[-1,1]
- Then **every** feasible x satisfies $c^T x = -x_1+x_2 \leq 1$
- If this constraint is tight at x $\Rightarrow$ x is optimal

i.e. $-x_1+x_2=1$ (because equality holds here)

i.e. x lies on the red line

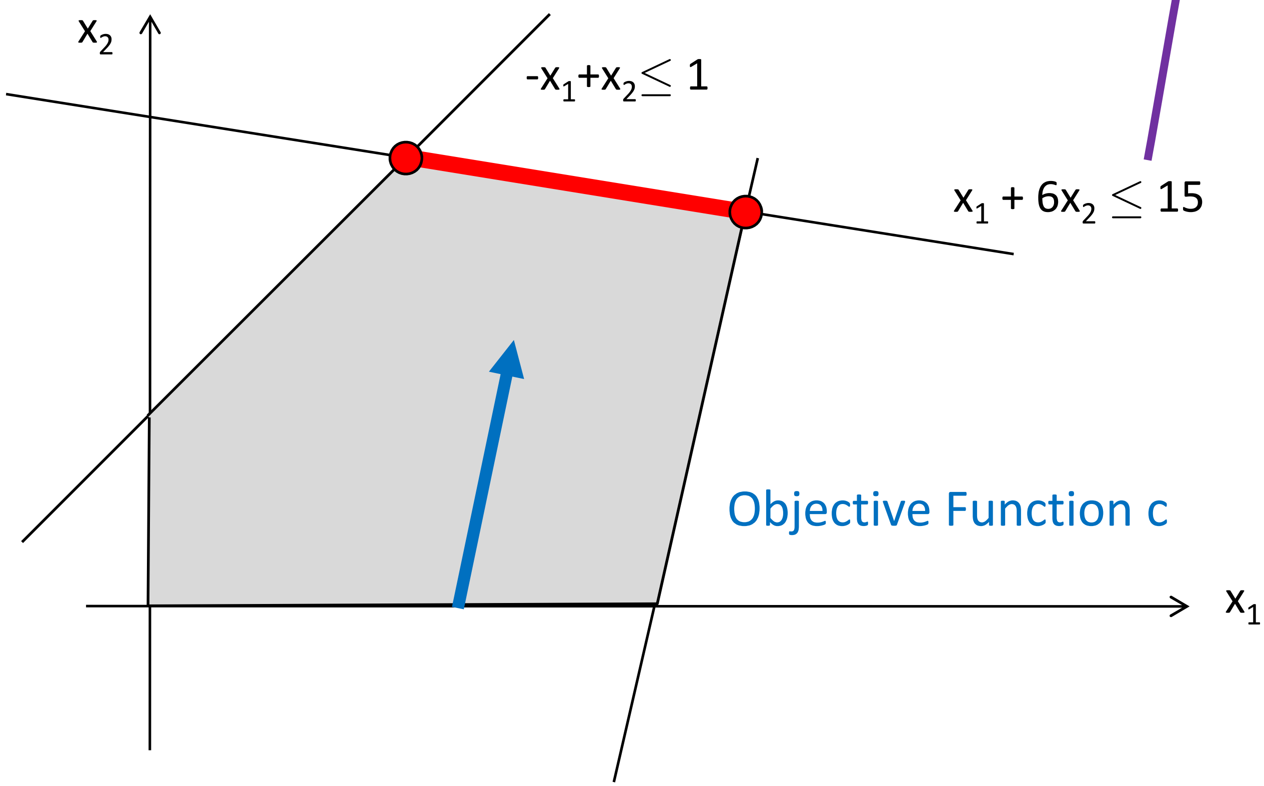


Duality: Geometric View

- Suppose c=[1,6]
- Then **every** feasible x satisfies $c^T x = x_1 + 6x_2 \leq 15$
- If this constraint is tight at x $\Rightarrow$ x is optimal

i.e. $x_1+6x_2=15$ (because equality holds here)

i.e. x lies on the red line



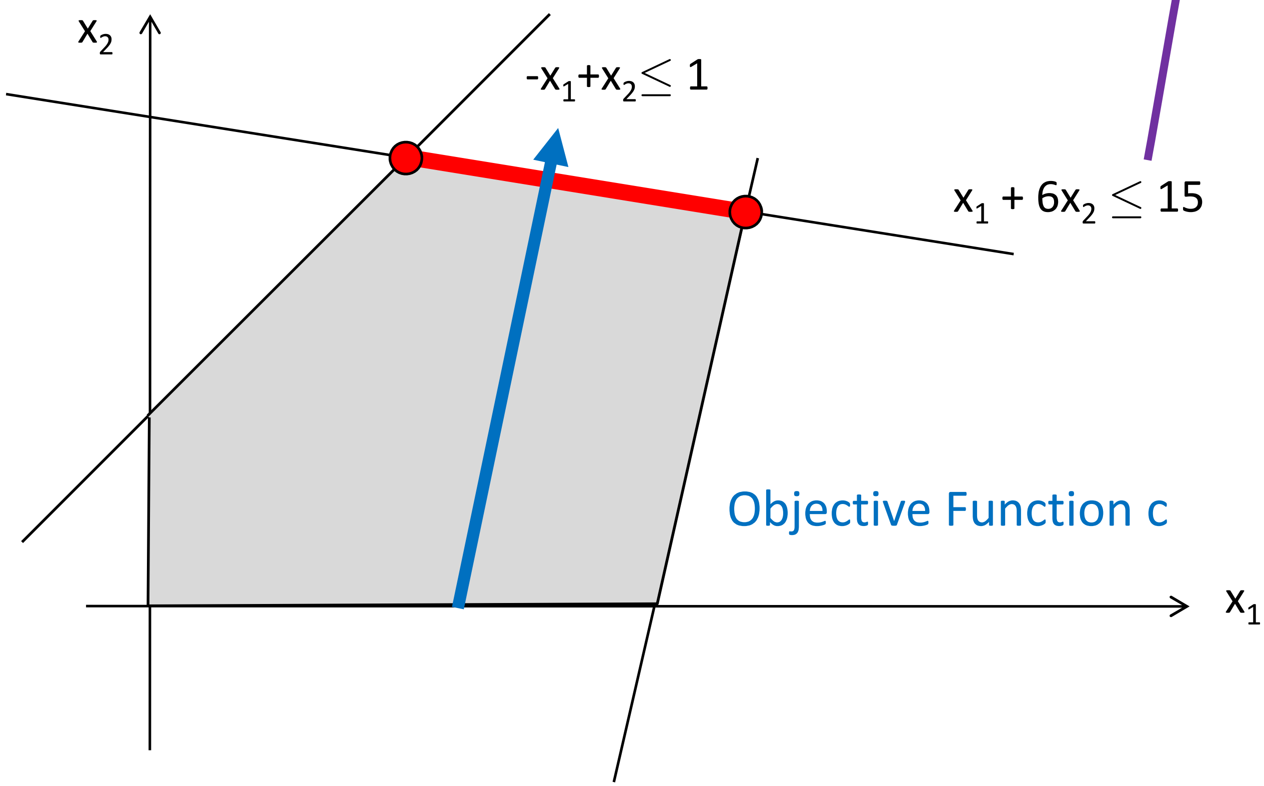
Duality: Geometric View

- Suppose c=$\alpha\cdot$[1,6], where $\alpha \geq 0$
- Then **every** feasible x satisfies $c^T x = \alpha\cdot(x_1+6x_2) \leq 15\alpha$
- If this constraint is tight at x $\Rightarrow$ x is optimal

i.e. $x_1+6x_2=15$

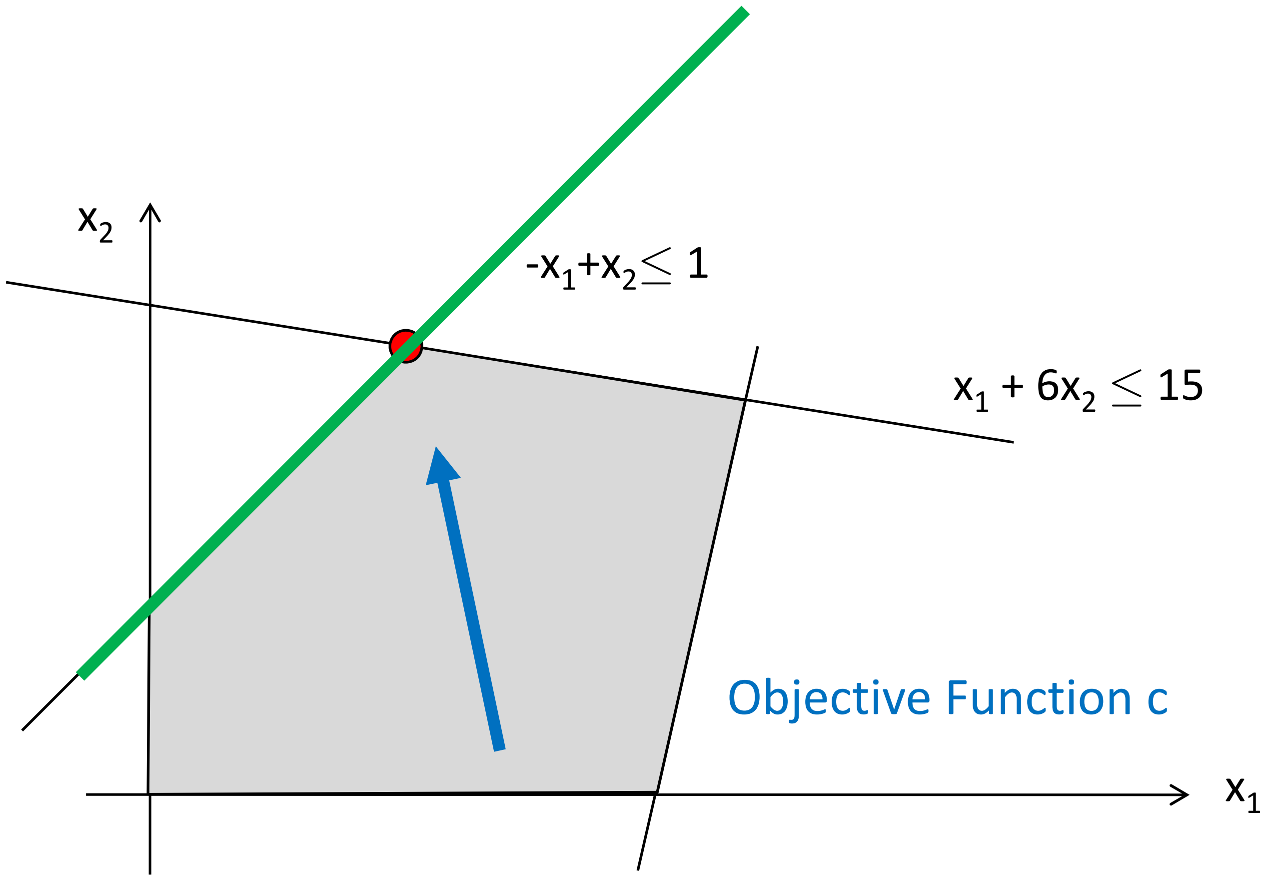
i.e. x lies on the red line

(because equality holds here)



Duality: Geometric View

- What if c does not align with any constraint?
- Can we "generate" a new constraint aligned with c?

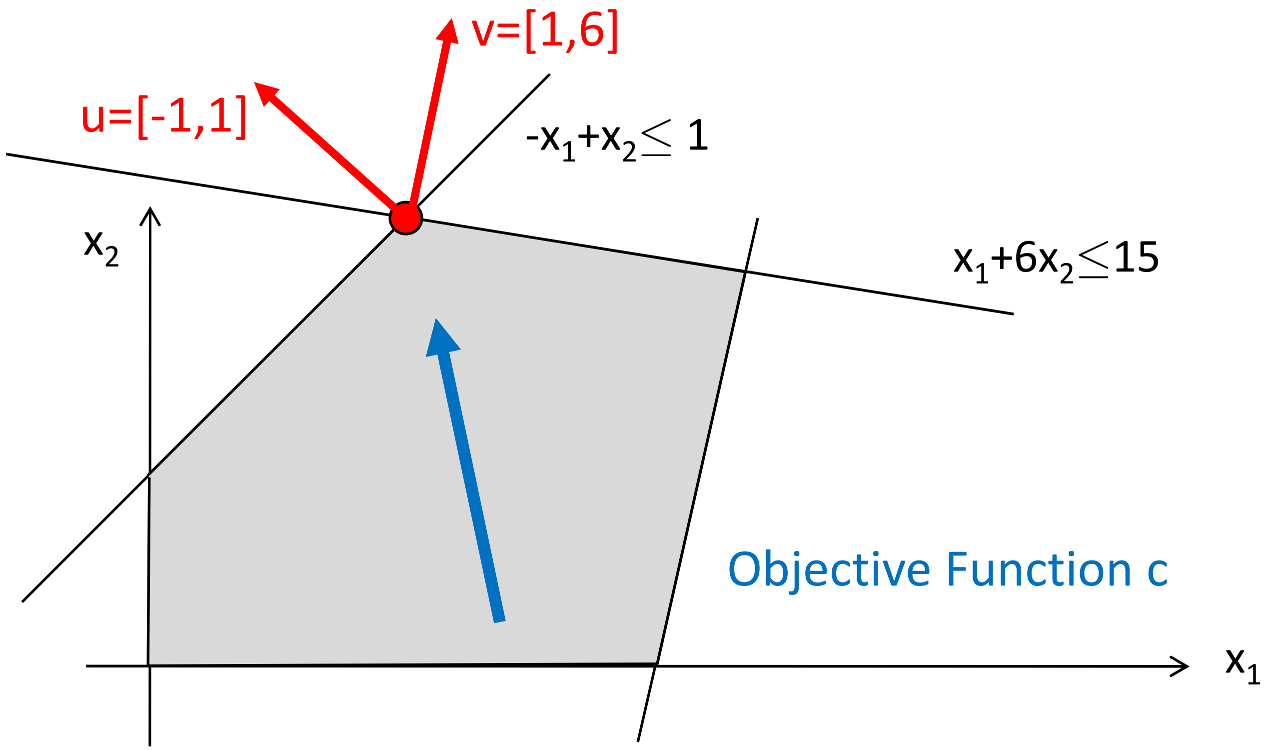


Duality: Geometric View

- Can we "generate" a new constraint aligned with c?
- One way is to "average" the tight constraints
- **Example:** Suppose c = u+v.
- Then every feasible x satisfies

$$c^T x = (u+v)^T x = (-x_1+x_2) + (x_1+6x_2) \leq 1 + 15 = 16$$

- x is feasible and **both** constraints tight $\Rightarrow$ x is optimal

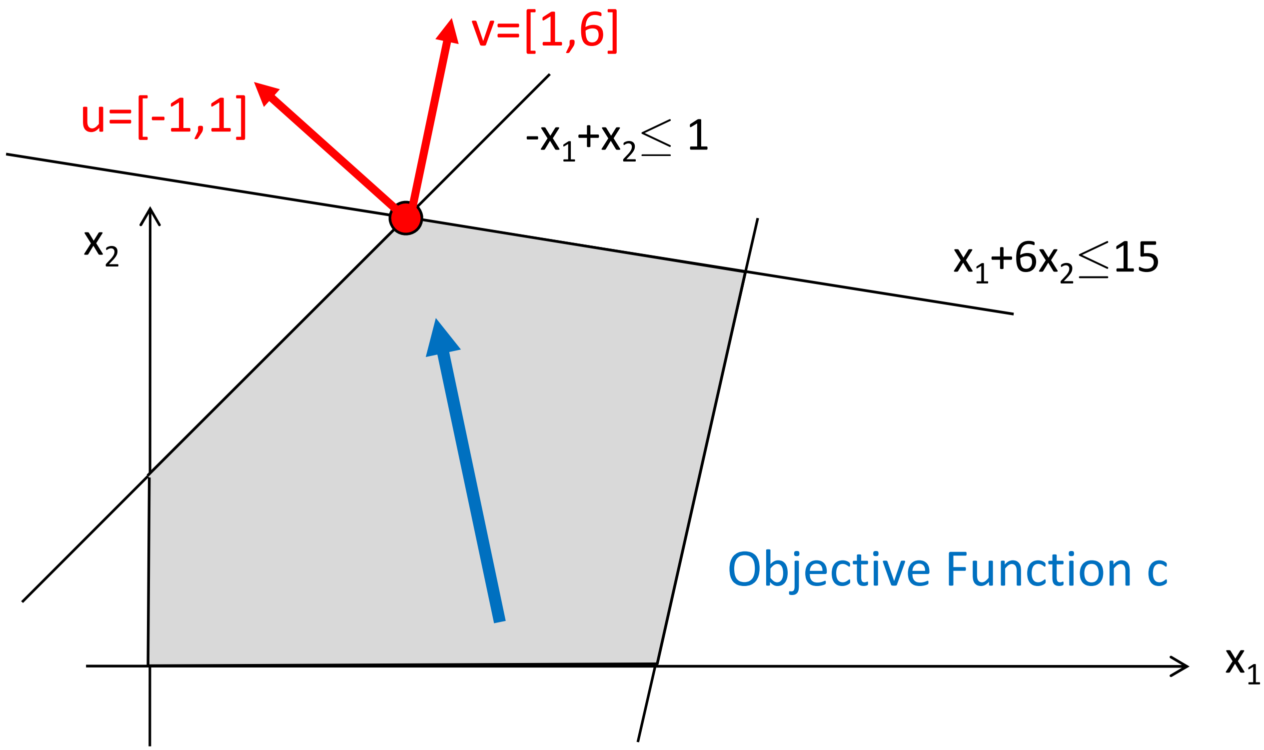


Duality: Geometric View

- Can we "generate" a new constraint aligned with c?
- One way is to "average" the tight constraints
- **More generally:** Suppose $c = \alpha u + \beta v$ for $\alpha, \beta \geq 0$
- Then every feasible x satisfies

$$c^T x = (\alpha u + \beta v)^T x = \alpha(-x_1 + x_2) + \beta(x_1 + 6x_2) \leq \alpha + 15\beta$$

- x is feasible and **both** constraints tight $\Rightarrow$ x is optimal



Duality: Algebraic View

$$\begin{aligned} \max \quad & c^{\mathsf{T}} x \\ \text{s.t.} \quad & a_i^{\mathsf{T}} x \le b_i \qquad \forall i = 1, \ldots, m \end{aligned}$$

Definition: A new constraint $a^T x \le b$ is **valid** if it is satisfied by all feasible points

x feasible $\Rightarrow$ $a_1^T x \le b_1$ and $a_2^T x \le b_2$

$\Rightarrow$ $(a_1+a_2)^T x \le b_1+b_2$ (new valid constraint)

More generally, for any $\lambda_1,\ldots,\lambda_m \ge 0$

x feasible $\Rightarrow$ $(\Sigma_i \lambda_i a_i)^T x \le \Sigma_i \lambda_i b_i$ (new valid constraint)

"Any **non-negative** linear combination of the constraints gives a new **valid constraint**"

To get upper bound on objective function $c^T x$, need $(\Sigma_i \lambda_i a_i) = c$

(because then our new valid constraint shows $c^T x \le \Sigma_i \lambda_i b_i$)

Want best upper bound $\Rightarrow$ want to minimize $\Sigma_i \lambda_i b_i$

Duality: Algebraic View

$$\begin{array}{lll} \max & c^{\mathsf{T}} x & \\ \text{s.t.} & a_i^{\mathsf{T}} x \le b_i & \forall i = 1, \ldots, m \end{array}$$

Primal LP

To get upper bound on objective function $c^T x$, need $(\Sigma_i \lambda_i a_i) = c$

Want best upper bound $\Rightarrow$ want to minimize $\Sigma_i \lambda_i b_i$

We can write this as an LP too!

$$\begin{array}{ll} \min & \sum_i \lambda_i b_i \\ \text{s.t.} & \sum_i \lambda_i a_i = c \\ & \lambda \ge 0 \end{array} \equiv \boxed{\begin{array}{ll} \min & b^{\mathsf{T}} \lambda \\ \text{s.t.} & A^{\mathsf{T}} \lambda = c \\ & \lambda \ge 0 \end{array}}$$

Dual LP

Theorem: **"Weak Duality Theorem"**

If x feasible for Primal and λ feasible for Dual then $c^T x \le b^T \lambda$.

Proof: $c^T x = (A^T \lambda)^T x = \lambda^T A x \le \lambda^T b.$ ■

Since $\lambda \ge 0$ and $Ax \le b$

Duality: Algebraic View

$$\boxed{\begin{array}{lll} \max & c^T x & \\ \text{s.t.} & a_i^T x \le b_i & \forall i = 1, ..., m \end{array}}$$

Primal LP

To get upper bound on objective function $c^T x$, need $(\Sigma_i \lambda_i a_i) = c$

Want best upper bound $\Rightarrow$ want to minimize $\Sigma_i \lambda_i b_i$

We can write this as an LP too!

$$\begin{array}{ll} \min & \sum_i \lambda_i b_i \\ \text{s.t.} & \sum_i \lambda_i a_i = c \\ & \lambda \ge 0 \end{array} \equiv \boxed{\begin{array}{ll} \min & b^T \lambda \\ \text{s.t.} & A^T \lambda = c \\ & \lambda \ge 0 \end{array}}$$

Dual LP

Theorem: "Weak Duality Theorem"

If x feasible for Primal and λ feasible for Dual then $c^T x \le b^T \lambda$.

Corollary: If x feasible for Primal and λ feasible for Dual and $c^T x = b^T \lambda$ then x **optimal** for Primal and λ **optimal** for Dual.

A has size m x n
$c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$

Dual of Dual

Dual LP

$$\begin{aligned} \min \quad & b^\top \lambda \\ \text{s.t.} \quad & A^\top \lambda = c \\ & \lambda \geq 0 \end{aligned}$$

$\equiv$

Inequality Form

$$\begin{aligned} \max \quad & -b^\top \lambda \\ \text{s.t.} \quad & \begin{pmatrix} A^\top \\ -A^\top \\ -I \end{pmatrix} \lambda \leq \begin{pmatrix} c \\ -c \\ 0 \end{pmatrix} \end{aligned}$$

transpose

transpose

transpose

Let x = v-u

$$\begin{aligned} \min \quad & -c^\top x \\ \text{s.t.} \quad & -Ax - w = -b \\ & w \geq 0 \end{aligned}$$

$\equiv$

Dual of Dual

$$\begin{aligned} \min \quad & (c^\top, \ -c^\top, \ 0) \begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ \text{s.t.} \quad & (A, \ -A, \ -I) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = -b \\ & u, v, w \geq 0 \end{aligned}$$

new variables
(non-negative)
$u \in \mathbb{R}^n$
$v \in \mathbb{R}^n$
$w \in \mathbb{R}^m$

$\equiv$

Primal:

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

Conclusion: "Dual of Dual is Primal!"

Primal vs Dual

Fundamental Theorem of LP: For any LP, the outcome is either: Infeasible, Unbounded, Optimum Point Exists.

Weak Duality Theorem:
If x feasible for Primal and λ feasible for Dual then $c^T x \leq b^T \lambda$.

Exercise!

Primal
(**max**imization)

Dual
(**min**imization)

	Infeasible	Unbounded	Opt. Exists
Infeasible	Possible	Possible	Impossible
Unbounded	Possible	Impossible	Impossible
Opt. Exists	Impossible	Impossible	Possible

Strong Duality Theorem:
If Primal has an opt. solution x, then Dual has an opt. solution λ.
Furthermore, optimal values are same: $c^T x = b^T \lambda$.

Strong Duality

Primal LP: $\begin{array}{ll} \max & c^\mathsf{T} x \\ \text{s.t.} & Ax \leq b \end{array}$

Dual LP: $\begin{array}{ll} \min & b^\mathsf{T} y \\ \text{s.t.} & A^\mathsf{T} y = c \\ & y \geq 0 \end{array}$

Strong Duality Theorem:

Primal has an opt. solution x $\Leftrightarrow$ Dual has an opt. solution y.
Furthermore, optimal values are same: $c^T x = b^T y$.

- Weak Duality implies $c^T x \leq b^T y$. So strong duality says $c^T x \geq b^T y$.

  (for any **feasible** x,y) (for **optimal** x,y)

- **Restatement of Theorem:**

  Primal has an optimal solution

  $\Leftrightarrow$ Dual has an optimal solution

  $\Leftrightarrow$ the following system is solvable:

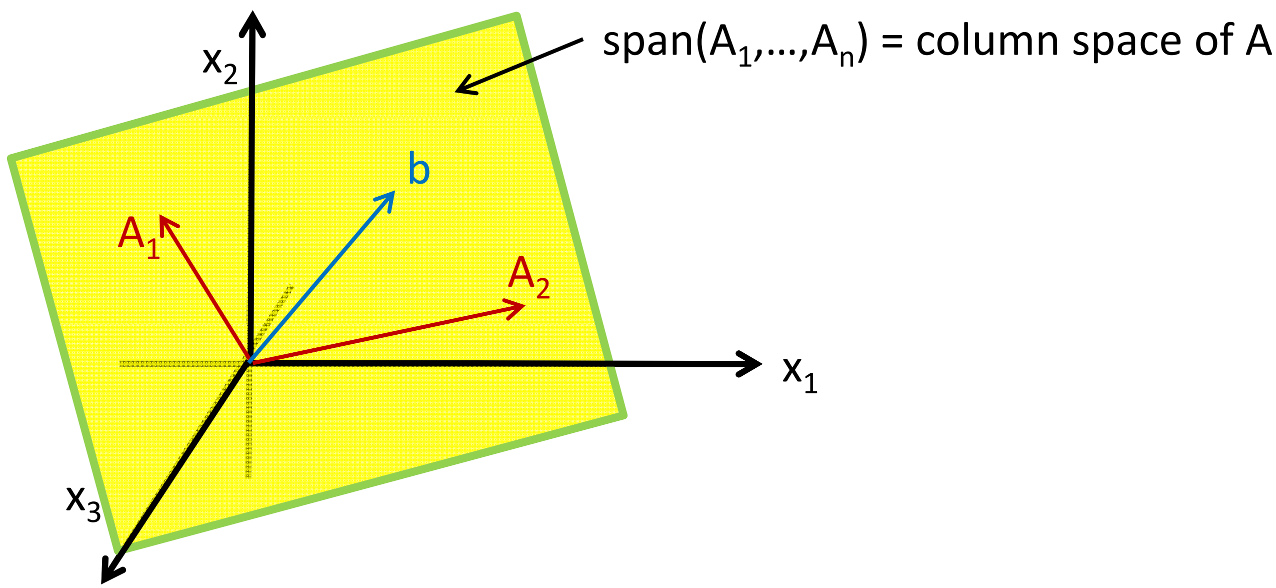
$$Ax \leq b \qquad A^\mathsf{T} y = c \qquad y \geq 0 \qquad c^\mathsf{T} x \geq b^\mathsf{T} y$$

Punchline: Finding optimal primal & dual LP solutions is equivalent to solving this system of inequalities.

- Can we characterize when systems of inequalities are solvable?

Systems of **Eq**ualities

- **Lemma:** Exactly one of the following holds:
  - There exists x satisfying Ax=b (b is in column space of A)
  - There exists y satisfying $y^TA=0$ and $y^Tb>0$
- **Geometrically…**

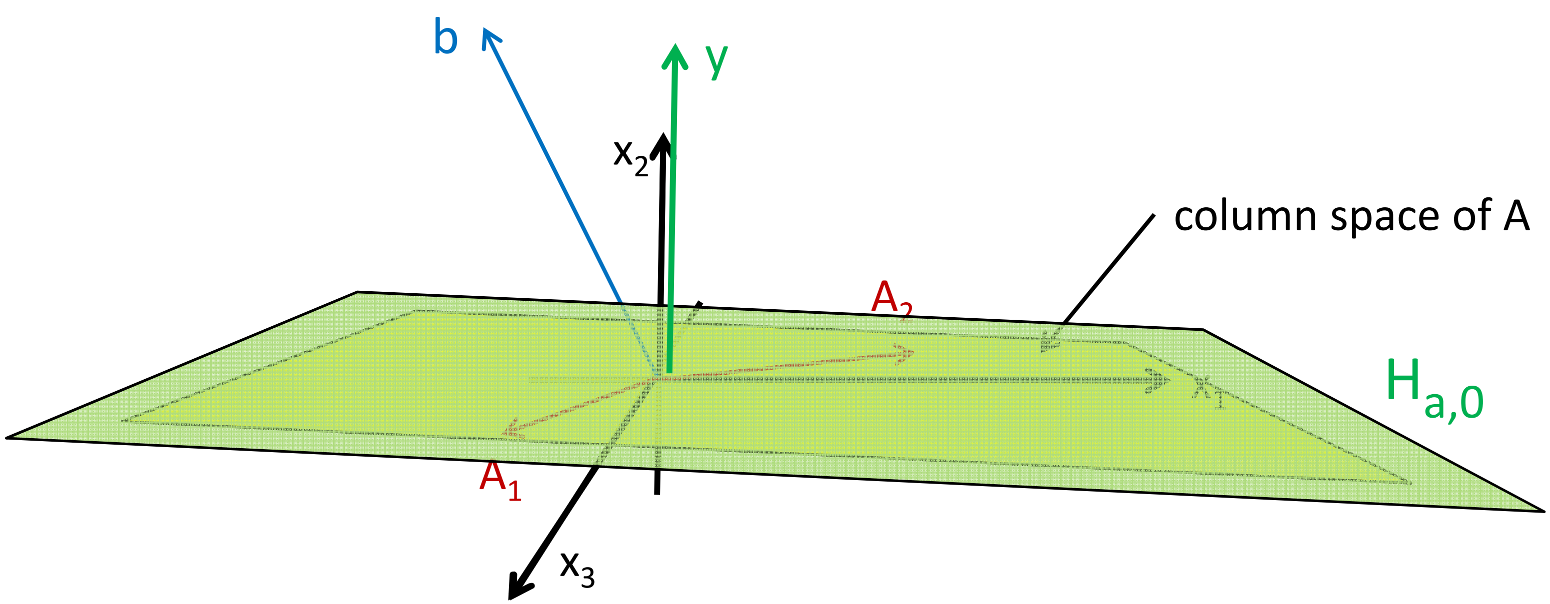


Systems of **Eq**ualities

- **Lemma:** Exactly one of the following holds:
  - There exists x satisfying Ax=b (b is in column space of A)
  - There exists y satisfying $y^T A=0$ and $y^T b>0$ (or it is not)
- **Geometrically…** col-space(A)$\subseteq H_{y,0}$ but $b \in H^{++}_{y,0}$

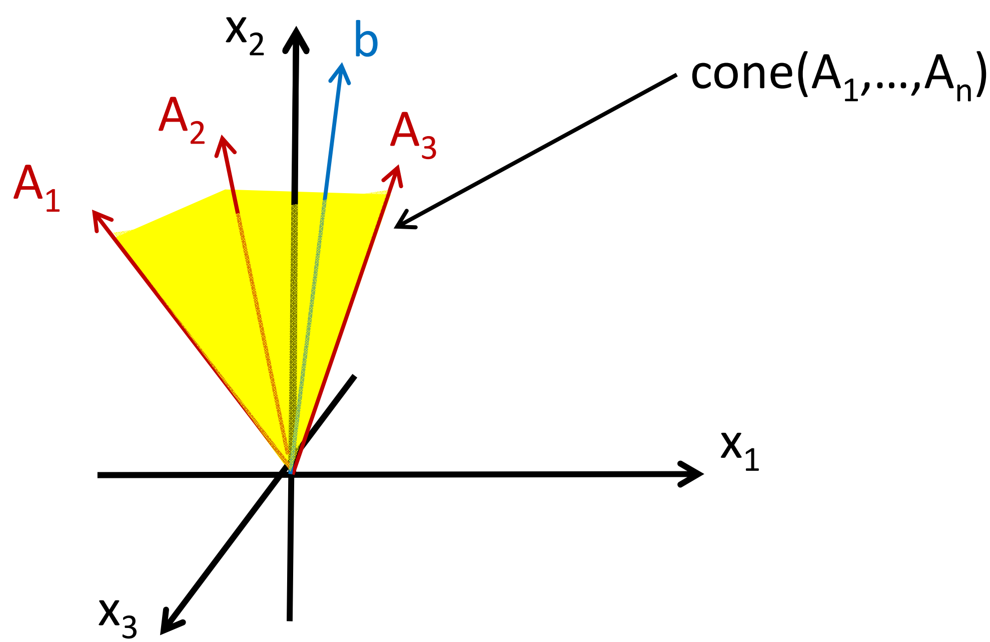
Hyperplane $H_{a,b} = \{ x \in \mathbb{R}^n : a^\mathsf{T} x = b \}$

Positive open halfspace $H^{++}_{a,b} = \{ x \in \mathbb{R}^n : a^\mathsf{T} x > b \}$



Systems of **Inequalities**

- **Lemma:** Exactly one of the following holds:
  - There exists $x \geq 0$ satisfying $Ax=b$ (b is in cone($A_1,\ldots,A_n$))
  - There exists y satisfying $y^T A \geq 0$ and $y^T b < 0$
- **Geometrically…**
  Let cone($A_1,\ldots,A_n$) = $\{ \Sigma_i \; x_i A_i : x \geq 0 \}$ "cone generated by $A_1,\ldots,A_n$"
  (Here A_i is the i^{th} column of A)

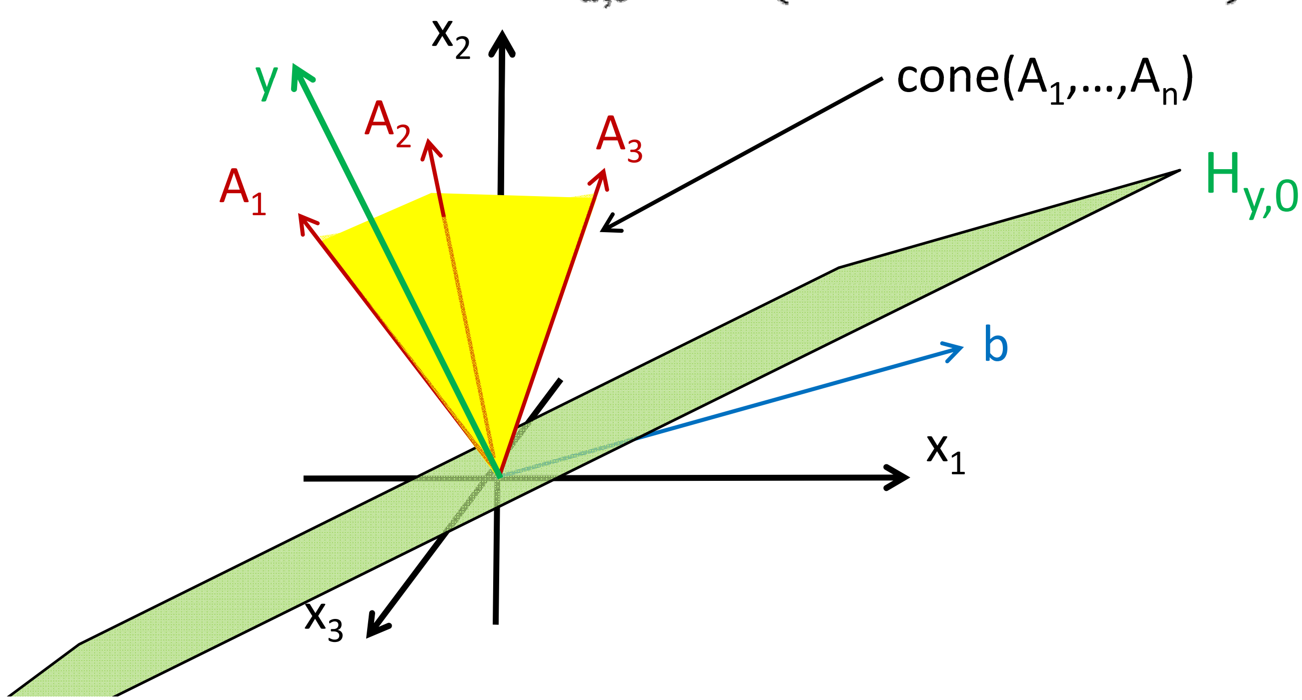


Systems of **Inequalities**

- **Lemma:** Exactly one of the following holds:
  - –There exists $x \geq 0$ satisfying $Ax=b$ (b is in $\text{cone}(A_1,\ldots,A_n)$)
  - –There exists y satisfying $y^T A \geq 0$ and $y^T b < 0$ (y gives a **separating hyperplane**)
- **Geometrically…** $\text{cone}(A_1,\ldots,A_n) \in H^{+}_{y,0}$ but $b \in H^{--}_{y,0}$

Positive closed halfspace $H^{+}_{a,b} = \{ x \in \mathbb{R}^n : a^T x \geq b \}$

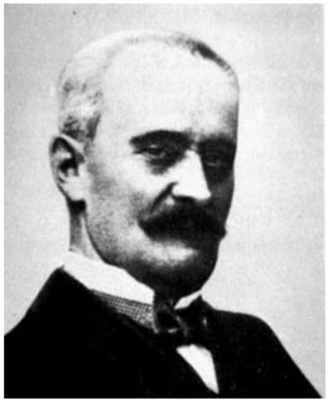
Negative open halfspace $H^{--}_{a,b} = \{ x \in \mathbb{R}^n : a^T x < b \}$



Systems of **Inequalities**

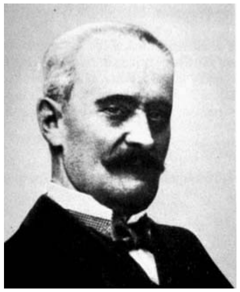
- **Lemma:** Exactly one of the following holds:
  - There exists $x \geq 0$ satisfying $Ax=b$ (b is in cone($A_1,\ldots,A_n$))
  - There exists y satisfying $y^T A \geq 0$ and $y^T b < 0$ (y gives a "separating hyperplane")

- This is called **"Farkas' Lemma"**
  - It has many interesting proofs.
  - It is "equivalent" to strong duality for LP.
  - There are several "equivalent" versions of it.



Gyula Farkas

Variants of Farkas' Lemma



Gyula Farkas

The System	$Ax \leq b$	$Ax = b$
has **no** solution $x \geq \mathbf{0}$ iff	$\exists y \geq 0$, $A^T y \geq \mathbf{0}$, $b^T y < 0$	$\exists y \in \mathbb{R}^n$, $A^T y \geq \mathbf{0}$, $b^T y < 0$
has **no** solution $x \in \mathbb{R}^n$ iff	$\exists y \geq 0$, $A^T y = \mathbf{0}$, $b^T y < 0$	$\exists y \in \mathbb{R}^n$, $A^T y = \mathbf{0}$, $b^T y < 0$

These are all "equivalent"

(each can be proved using another)

This is the simple lemma on systems of **eq**ualities