CPSC 536N Randomized Algorithms (Term 2, 2012)
Assignment 2

Due: Thursday, March 1st, in class.

Question 1: In class we discussed the bipartite matching problem. We defined a matrix A containing variables such that $\det A$ is identically zero if and only if G has no perfect matching. Using this fact we gave a randomized algorithm, based on polynomial identity testing, to decide if G has a perfect matching. The algorithm needs to compute the determinant of an $n \times n$ numeric matrix, which can be done in $O(n^{2.38})$ time.

However, that algorithm only says "yes" or "no"; it does not actually produce a perfect matching! Give a randomized algorithm with running time $O(n^{4.38})$ which can construct a perfect matching if one exists.

Question 2: Consider adapting the randomized minimum cut algorithm to the problem of finding a minimum s-t cut in an undirected graph. In this problem, we are given an undirected graph G together with two distinguished vertices s and t. An s-t cut is a set of edges whose removal from G disconnects s and t; we seek an s-t cut of minimum cardinality. As the algorithm proceeds, the vertex s may get amalgamated into a new vertex as a result of an edge being contracted; we call this vertex the s-vertex. (Initially the s-vertex is s itself.) Similarly, we have a t-vertex. As we run the contraction algorithm, we ensure that we never contract an edge between the s-vertex and the t-vertex.

Show that there are graphs (or multigraphs) in which the probability that this algorithm finds a minimum s-t cut is exponentially small.

Question 3: Let $c \geq 2$ be arbitrary. Let $\mathcal{E}_1, ..., \mathcal{E}_m$ be events. Let $k_1, ..., k_m$ be positive integers such that $\Pr[\,\mathcal{E}_i\,] \leq c^{-3k_i}$ for every i, and $|\{\, i : k_i \leq x \,\}| \leq c^x$ for all real numbers $x \geq 1$. Prove that $\Pr[\,\mathcal{E}_1 \vee \cdots \vee \mathcal{E}_m\,] \leq 1/2$.

Question 4: When you buy a box of cereal it comes with a randomly chosen toy. Suppose there are m different types of toys, and the toy in each cereal box is independently and uniformly chosen. In vague terms, the question we'd like to answer is: how many boxes of cereal must you buy in order to collect at least one toy of each type?

More formally, let $X_1, X_2, ...$ be independent random variables, each uniformly distributed in $\{1, ..., m\}$. We would like to prove that, for some constants c_1 and c_2, if $t \geq c_1\, m \log m$ then the random set $\{X_1, ..., X_t\}$ contains all numbers $\{1, ..., m\}$ with probability at least $1 - m^{-c_2}$.

Prove this using the **Ahlswede-Winter** inequality.