We continue our theorem from last time on random partitions of metric spaces.

1 Review of Previous Lecture

Define the partial Harmonic sum $H(a, b) = \sum_{i=a+1}^{b} \frac{1}{i}$. Let $B(x, r) = \{ y \in X : d(x, y) \leq r \}$ be the ball of radius $r$ around $x$.

**Theorem 1** Let $(X, d)$ be a metric with $|X| = n$. For every $\Delta > 0$, there is $\Delta$-bounded random partition $P$ of $X$ with

$$\Pr[B(x, r) \not\subseteq P(x)] \leq \frac{8r}{\Delta} \cdot H\left( |B(x, \Delta/4 - r)|, |B(x, \Delta/2 + r)| \right) \quad \forall x \in X, \forall r > 0. \quad (1)$$

The algorithm to construct $P$ is as follows.

- Pick $\alpha \in (1/4, 1/2]$ uniformly at random.
- Pick a bijection (i.e., ordering) $\pi : \{1, \ldots, n\} \to X$ uniformly at random.
- For $i = 1, \ldots, n$
  - Set $P_i = B(\pi(i), \alpha \Delta) \setminus \cup_{j=1}^{i-1} P_j$.
- Output the random partition $P = \{P_1, \ldots, P_n\}$.

We have already proven that this outputs a $\Delta$-bounded partition. So it remains to prove (1).

2 The Proof

Fix any point $x \in X$ and radius $r > 0$. For brevity let $B = B(x, r)$. Let us order all points of $X$ as $\{y_1, \ldots, y_n\}$ where $d(x, y_1) \leq \cdots \leq d(x, y_n)$. The proof involves two important definitions.

- **Sees**: A point $y$ sees $B$ if $d(x, y) \leq \alpha \Delta + r$.
- **Cuts**: A point $y$ cuts $B$ if $\alpha \Delta - r \leq d(x, y) \leq \alpha \Delta + r$.

Obviously “cuts” implies “sees”. To help visualize these definitions, the following claim interprets their meaning in Euclidean space. (In a finite metric, the ball $B$ is not a continuous object, so it doesn’t really have a “boundary”.)

**Claim 2** Consider the metric $(X, d)$ where $X = \mathbb{R}^n$ and $d$ is the Euclidean metric. Then
• $y$ sees $B$ if and only if $B = B(x, r)$ intersects $B(y, \alpha \Delta)$.
• $y$ cuts $B$ if and only if $B = B(x, r)$ intersects the boundary of $B(y, \alpha \Delta)$.

The following claim is in the same spirit, but holds for any metric.

**Claim 3** Let $(X, d)$ be an arbitrary metric. Then

• If $y$ does not see $B$ then $B \cap B(y, \alpha \Delta) = \emptyset$.
• If $y$ sees $B$ but does not cut $B$ then $B \subseteq B(y, \alpha \Delta)$.

To illustrate the definitions of “sees” and “cuts”, consider the following example. The blue ball around $x$ is $B$. The points $y_1$ and $y_2$ both see $B$; $y_3$ does not. The point $y_2$ cuts $B$; $y_1$ and $y_3$ do not. This example illustrates Claim 3: $y_1$ sees $B$ but does not cut $B$, and we have $B \subseteq B(y, \alpha \Delta)$.

The most important point for us to consider is the first point under the ordering $\pi$ that sees $B$. We call this point $y_{\pi(k)}$.

The first $k - 1$ iterations of the algorithm did not assign any point in $B$ to any $P_i$. To see this, note that $y_{\pi(1)}, \ldots, y_{\pi(k-1)}$ do not see $B$, by choice of $k$. So Claim 3 implies that $B \cap B(y_{\pi(i)}, \alpha \Delta) = \emptyset \ \forall i < k$. Consequently

$$B \cap P_i = \emptyset \ \forall i < k. \quad (2)$$

The point $y_{\pi(k)}$ sees $B$ by definition, but it may or may not cut $B$. If it does not cut $B$ then Claim 3 shows that $B \subseteq B(y_{\pi(k)}, \alpha \Delta)$. Thus

$$B \cap P_k = \left( B \cap B(y_{\pi(k)}, \alpha \Delta) \right) \setminus \bigcup_{i=1}^{k-1} B \cap P_i = B,$$
i.e., $B \subseteq P_k$. Since $\mathcal{P}(x) = P_k$, we have shown that

$$y \text{ does not cut } B \implies B \subseteq \mathcal{P}(x).$$

Taking the contrapositive of this statement, we obtain

$$\Pr[B \not\subseteq \mathcal{P}(x)] \leq \Pr[y_{\pi(k)} \text{ cuts } B] = \sum_{i=1}^{n} \Pr[y_{\pi(k)} = y_i \land y_i \text{ cuts } B].$$

Let us now simplify that sum by eliminating terms that are equal to 0.

**Claim 4** If $y \not\in B(x, \Delta/2 + r)$ then $y$ does not see $B$.

**Claim 5** If $y \in B(x, \Delta/4 - r)$ then $y$ sees $B$ but does not cut $B$.

So define $a = |B(x, \Delta/4 - r)|$ and $b = |B(x, \Delta/2 + r)|$. Then we have shown that

$$\Pr[B \not\subseteq \mathcal{P}(x)] \leq \sum_{i=a+1}^{b} \Pr[y_{\pi(k)} = y_i \land y_i \text{ cuts } B].$$

The remainder of the proof is quite interesting. The main point is that these two events are “nearly independent”, since $\alpha$ and $\pi$ are independent, “$y_i$ cuts $B$” depends only on $\alpha$, and “$y_{\pi(k)} = y_i$” depends primarily on $\pi$. Formally, we write

$$\Pr[B \not\subseteq \mathcal{P}(x)] \leq \sum_{i=a+1}^{b} \Pr[y_i \text{ cuts } B] \cdot \Pr[y_{\pi(k)} = y_i \mid y_i \text{ cuts } B]$$

and separately upper bound these two probabilities.

The first probability is easy to bound:

$$\Pr[y_i \text{ cuts } B] = \Pr[\alpha \Delta \in [d(x, y) - r, d(x, y) + r]] \leq \frac{2r}{\Delta/4},$$

because $2r$ is the length of the interval $[d(x, y) - r, d(x, y) + r]$, and $\Delta/4$ is the length of the interval from which $\alpha \Delta$ is randomly chosen.

Next we bound the second probability. Recall that $y_{\pi(k)}$ is defined to be the first element in the ordering $\pi$ that sees $B$. Since $y_i$ cuts $B$, we know that $d(x, y_i) \leq \alpha/2 + r$. Every $y_j$ coming earlier in the ordering has $d(x, y_j) \leq d(x, y_i) \leq \alpha/2 + r$, so $y_j$ also sees $B$. This shows that there are at least $i$ elements that see $B$. So the probability that $y_i$ is the first element in the random ordering to see $B$ is at most $1/i$.

Combining these bounds on the two probabilities we get

$$\Pr[B \not\subseteq \mathcal{P}(x)] \leq \sum_{i=a+1}^{b} \frac{8r}{\Delta} \cdot \frac{1}{i} = \frac{8r}{\Delta} \cdot H(a, b),$$

as required.
3 Optimality of these partitions

Theorem 1 from the previous lecture shows that there is a universal constant $L = O(1)$ such that every metric has a $\log(n)/10$-bounded, $L$-Lipschitz random partition. We now show that this is optimal.

**Theorem 6** There exist graphs $G$ whose shortest path metric $(X, d)$ has the property that any $\log(n)/10$-bounded, $L$-Lipschitz random partition must have $L = \Omega(1)$.

The graphs we need are expander graphs. In Lecture 20 we defined bipartite expanders. Today we need non-bipartite expanders. We say that $G = (V, E)$ is a non-bipartite expander if, for some constants $c > 0$ and $d \geq 3$:

- $G$ is $d$-regular, and
- $|\delta(S)| \geq c|S|$ for all $|S| \leq |V|/2$.

It is known that expanders exist for all $n = |V|$, $d = 3$ and $c \geq 1/1000$. (The constant $c$ can of course be improved.)

**Proof:** Suppose $(X, d)$ has a $\log(n)/10$-bounded, $L$-Lipschitz random partition. Then there exists a particular partition $P$ that is $\log(n)/10$-bounded and cuts at most an $L$-fraction of the edges. Every part $P_i$ in the partition has diameter at most $\log(n)/10$. Since the graph is 3-regular, the number of vertices in $P_i$ is at most $3^{\log(n)/10} < n/2$. So every part $P_i$ has size less than $n/2$. By the expansion condition, the number of edges cut is at least

$$\frac{1}{2} \sum_i c \cdot |P_i| = cn/2 = \Omega(|E|).$$

So $L = \Omega(1)$. \Halmos

4 Appendix: Proofs of Claims

**Proof:** (of Claim 3) Suppose $y$ does not see $B$. Then $d(x, y) > \alpha \Delta + r$. Every point $z \in B$ has $d(x, z) \leq r$, so $d(y, z) \geq d(y, x) - d(x, z) > \alpha \Delta + r - r$, implying that $z \not\in B(y, \alpha \Delta)$.

Suppose $y$ sees $B$ but does not cut $B$. Then $d(x, y) < \alpha \Delta - r$. Every point $z \in B$ has $d(x, z) \leq r$. So $d(y, z) \leq d(y, x) + d(x, z) < \alpha \Delta - r + r$, implying that $z \in B(y, \alpha \Delta)$. \Halmos

**Proof:** (of Claim 4) The hypothesis of the claim is that $d(x, y) > \Delta/2 + r$, which is at least $\alpha \Delta + r$. So $d(x, y) \geq \alpha \Delta + r$, implying that $y$ does not see $B$. \Halmos

**Proof:** (of Claim 5) The hypothesis of the claim is that $d(x, y) \leq \Delta/4 - r$, which is strictly less than $\alpha \Delta - r$. So $d(x, y) < \alpha \Delta - r$, which implies that $y$ sees $B$ but does not cut $B$. \Halmos