CPSC 536N: Randomized Algorithms 2011-12 Term 2

Lecture 14

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1 Spectral Sparsifiers

1.1 Graph Laplacians

Let $G = (V, E)$ be an unweighted graph. For notational simplicity, we will think of the vertex set as $V = \{1, \ldots, n\}$. Let $e_i \in \mathbb{R}^n$ be the ith standard basis vector, meaning that e_i has a 1 in the ith coordinate and 0s in all other coordinates. For an edge $uv \in E$, define the vector x_{uv} and the matrix X_{uv} as follows:

$$
\begin{aligned}
x_{uv} &:= e_u - u_v \\
X_{uv} &:= x_{uv} x_{uv}^{\mathsf{T}}
\end{aligned}
$$

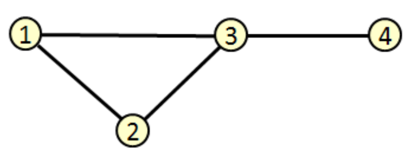
In the definition of x_{uv} it does not matter which vertex gets the $+1$ and which gets the -1 because the matrix X_{uv} is the same either way.

Definition 1 *The **Laplacian matrix** of G is the matrix*

$$L_G := \sum_{uv \in E} X_{uv}$$

Let us consider an example.

G



X_{13} =

	1	2	3	4
1	1	0	-1	0
2	0	0	0	0
3	-1	0	1	0
4	0	0	0	0

L_G =

	1	2	3	4
1	2	-1	-1	0
2	-1	2	-1	0
3	-1	-1	3	-1
4	0	0	-1	1

-1 indicates an edge from 1 to 3

degree of node 3 = # edges that hit 3

Note that each matrix X_{uv} has only four non-zero entries: we have $X_{uu} = X_{vv} = 1$ and $X_{uv} = X_{vu} = -1$. Consequently, the uth diagonal entry of L_G is simply the degree of vertex u. Moreover, we have the following fact.

Fact 2 *Let D be the diagonal matrix with $D_{u,u}$ equal to the degree of vertex u. Let A be the adjacency matrix of G. Then $L_G = D - A$.*

If G had weights $w : E \to \mathbb{R}$ on the edges we could define the weighted Laplacian as follows:

$$L_G \;=\; \sum_{uv \in E} w_{uv} \cdot X_{uv}.$$

Claim 3 *Let $G = (V, E)$ be a graph with non-negative weights $w : E \to \mathbb{R}$. Then the weighted Laplacian L_G is positive semi-definite.*

PROOF: Since $X_{uv} = x_{uv} x_{uv}^{\mathsf{T}}$, it is positive semi-definite. So L_G is a weighted sum of positive semi-definite matrices with non-negative coefficients. Fact 5 in the Notes on Symmetric Matrices implies L_G is positive semi-definite. □

The Laplacian can tell us many interesting things about the graph. For example:

Claim 4 *Let $G = (V, E)$ be a graph with Laplacian L_G. For any $U \subseteq V$, let $\chi(U) \in \mathbb{R}^n$ be the characteristic vector of U, i.e., the vector with $\chi(U)_v$ equal to 1 if $v \in U$ and equal to 0 otherwise. Then $\chi(U)^{\mathsf{T}} L_G \, \chi(U) = |\delta(U)|$.*

PROOF: For any edge uv we have $\chi(U)^{\mathsf{T}} X_{uv} \, \chi(U) = (\chi(U)^{\mathsf{T}} x_{uv})^2$. But $|\chi(U)^{\mathsf{T}} x_{uv}|$ is 1 if exactly one of u or v is in U, and otherwise it is 0. So $\chi(U)^{\mathsf{T}} X_{uv} \, \chi(U) = 1$ if $uv \in \delta(U)$, and otherwise it is 0. Summing over all edges proves the claim. □

Similarly, if $G = (V, E)$ is a graph with edge weights $w : E \to \mathbb{R}$ and L_G is the weighted Laplacian, then then $\chi(U)^{\mathsf{T}} L_G \, \chi(U) = w(\delta(U))$.

Fact 5 *If G is connected then $\mathrm{image}(L_G) \;=\; \{\, x \;:\; \sum_i x_i = 0 \,\}$, which is an $(n-1)$-dimensional subspace.*

1.2 Main Theorem

Theorem 6 *Let $G = (V, E)$ be a graph with $n = |V|$. There is a randomized algorithm to compute weights $w : E \to \mathbb{R}$ such that:*

- *only $O(n \log n / \epsilon^2)$ of the weights are non-zero, and*
- *with probability at least $1 - 2/n$,*

$$(1-\epsilon) \cdot L_G \;\preceq\; L_w \;\preceq\; (1+\epsilon) \cdot L_G,$$

where L_w denotes the weighted Laplacian of G with weights w. By Fact 4 in Notes on Symmetric Matrices, this is equivalent to

$$(1-\epsilon) x^{\mathsf{T}} L_G x \;\le\; x^{\mathsf{T}} L_w x \;\le\; (1+\epsilon) x^{\mathsf{T}} L_G x \qquad \forall x \in \mathbb{R}^n. \tag{1}$$

By (1) and Claim 4, the resulting weights are a graph sparsifier of G:

$$(1-\epsilon)\cdot|\delta(U)| \;\leq\; w(\delta(U)) \;\leq\; (1+\epsilon)\cdot|\delta(U)| \qquad \forall U \subseteq V.$$

The algorithm that proves Theorem 6 is as follows.

- Initially $w = 0$.
- Set $k = 8n\log(n)/\epsilon^2$.
- For every edge $e \in E$ compute $r_e = \operatorname{tr}(X_e L_G^+)$.
- For $i = 1, \ldots, k$
- Let e be a random edge chosen with probability $r_e/(n-1)$.
- Increase w_e by $\frac{n-1}{r_e\, k}$.

Claim 7 *The values $\{r_e/(n-1) \,:\, e \in E\}$ indeed form a probability distribution.*

Proof:(of Theorem 6). How does the matrix L_w change during the ith iteration? The edge e is chosen with probability $\frac{r_e}{n-1}$ and then L_w increases by $\frac{n-1}{r_e\cdot k}X_e$. Let Z_i be this random change in L_w during the ith iteration. So Z_i equals $\frac{n-1}{r_e\cdot k}X_e$ with probability $\frac{r_e}{n-1}$. The random matrices $Z_1, \ldots, Z_k$ are mutually independent and they all have this same distribution. Note that

$$\mathrm{E}[Z_i] \;=\; \sum_{e\in E}\frac{r_e}{n-1}\cdot\frac{n-1}{r_e\cdot k}X_e \;=\; \frac{1}{k}\sum_e X_e \;=\; \frac{L_G}{k}. \tag{2}$$

The final matrix L_w is simply $\sum_{i=1}^k Z_i$. To analyze this final matrix, we will use the Ahlswede-Winter inequality. All that we require is the following claim, which we prove later.

Claim 8 $Z_i \preceq (n-1)\cdot \mathrm{E}[Z_i]$.

We apply Corollary 2 from the previous lecture with $R = n-1$, obtaining

$$\begin{aligned}\Pr\big[(1-\epsilon)L_G \;\preceq\; L_w \;\preceq\; (1+\epsilon)L_G\big] \;&=\; \Pr\left[(1-\epsilon)\frac{L_G}{k} \;\preceq\; \frac{1}{k}\sum_{i=1}^k Z_i \;\preceq\; (1+\epsilon)\frac{L_G}{k}\right]\\ &\leq\; 2n\cdot\exp\big(-\epsilon^2 k/4(n-1)\big)\\ &\leq\; 2n\cdot\exp\big(-2\ln n\big) \;<\; 2/n.\end{aligned}$$

□

2 Appendix: Additional Proofs

Proof:(of Claim 7) First we check that the r_e values are non-negative. By the cyclic property of trace

$$\operatorname{tr}(X_e L_G^+) \;=\; \operatorname{tr}(x_e^{\mathsf{T}} L_G^+ x_e) \;=\; x_e^{\mathsf{T}} L_G^+ x_e,$$

This is non-negative since $L_G^+ \succeq 0$ because $L_G \succeq 0$. Thus $r_e \geq 0$.

Next, note that

$$\sum_e \mathrm{tr}(X_e L_G^+) \;=\; \mathrm{tr}(\sum_e X_e L_G^+) \;=\; \mathrm{tr}(L_G L_G^+) \;=\; \mathrm{tr}(I_{\mathrm{im}\ L_G}),$$

where $I_{\mathrm{im}\ L_G}$ is the orthogonal projection onto the image of L_G. The image has dimension $n-1$ by Fact 5, and so

$$\sum_e r_e \;=\; \frac{1}{n-1}\sum_e \mathrm{tr}(X_e L_G^+) \;=\; \frac{1}{n-1}\mathrm{tr}(I_{\mathrm{im}\ L_G}) \;=\; 1.$$

□

Proof:(of Claim 8). The maximum eigenvalue of a positive semi-definite matrix never exceeds its trace, so

$$\lambda_{\max}(L_G^{+/2}\cdot X_e \cdot L_G^{+/2}) \;\le\; \mathrm{tr}(L_G^{+/2}\cdot X_e \cdot L_G^{+/2}) \;=\; r_e.$$

By Fact 8 in the Notes on Symmetric Matrices,

$$L_G^{+/2}\cdot X_e \cdot L_G^{+/2} \;\preceq\; r_e \cdot I.$$

So, by Fact 4 in the Notes on Symmetric Matrices, for every vector v,

$$v^{\mathsf{T}} \frac{L_G^{+/2}\cdot X_e \cdot L_G^{+/2}}{r_e} v \;\le\; v^{\mathsf{T}} v. \tag{3}$$

Now let us write $v = v_1 + v_2$ where $v_1 = I_{\mathrm{im}\ L_G}\, v$ is the projection onto the image of L_G and $v_2 = I_{\ker\ L_G}\, v$ is the projection onto the kernel of L_G. Then $L_G^{+/2}\, v_2 = 0$. So

$$\begin{aligned}
v^{\mathsf{T}} \frac{L_G^{+/2}\cdot X_e \cdot L_G^{+/2}}{r_e} v \;&=\; (v_1+v_2)^{\mathsf{T}} \frac{L_G^{+/2}\cdot X_e \cdot L_G^{+/2}}{r_e} (v_1+v_2) \\
&=\; v_1^{\mathsf{T}} \frac{L_G^{+/2}\cdot X_e \cdot L_G^{+/2}}{r_e} v_1 \;+\; \underbrace{2v_1^{\mathsf{T}} \frac{L_G^{+/2}\cdot X_e \cdot L_G^{+/2}}{r_e} v_2 \;+\; v_2^{\mathsf{T}} \frac{L_G^{+/2}\cdot X_e \cdot L_G^{+/2}}{r_e} v_2}_{=0} \\
&=\; v_1^{\mathsf{T}} \frac{L_G^{+/2}\cdot X_e \cdot L_G^{+/2}}{r_e} v_1 \\
&\le\; v_1^{\mathsf{T}} v_1 \;=\; v^{\mathsf{T}} I_{\mathrm{im}\ L_G} v.
\end{aligned}$$

Here, the second equality is by the distributive law and the inequality is by (3). Since this holds for every vector v, Fact 4 in the Notes on Symmetric Matrices again implies

$$\frac{L_G^{+/2}\cdot X_e \cdot L_G^{+/2}}{r_e} \;\preceq\; I_{\mathrm{im}\ L_G}.$$

Since $\mathrm{im}\ X_e \subseteq \mathrm{im}\ L_G$, Claim 16 in the Notes on Symmetric Matrices shows this is equivalent to

$$\frac{n-1}{r_e \cdot k} X_e \;\preceq\; \frac{n-1}{k} L_G.$$

Since (2) shows that $\mathrm{E}[Z_i] = L_G/k$, this completes the proof of the claim. □