CPSC 536N: Randomized Algorithms	2011-12 Term 2
Lecture 13	
Prof. Nick Harvey	*University of British Columbia*

1 Useful versions of the Ahlswede-Winter Inequality

Theorem 1 *Let Y be a random, symmetric, positive semi-definite $d \times d$ matrix such that $\mathrm{E}[Y] = I$. Suppose $\|Y\| \leq R$ for some fixed scalar $R \geq 1$. Let $Y_1, \ldots, Y_k$ be independent copies of Y (i.e., independently sampled matrices with the same distribution as Y). For any $\epsilon \in (0,1)$, we have*

$$\Pr\left[(1-\epsilon)I \ \preceq\ \frac{1}{k}\sum_{i=1}^{k} Y_i \ \preceq\ (1+\epsilon)I\right] \ \geq\ 1 - 2d \cdot \exp(-\epsilon^2 k/4R).$$

This event is equivalent to the sample average $\frac{1}{k}\sum_{i=1}^{k} Y_i$ having minimum eigenvalue at least $1-\epsilon$ and maximum eigenvalue at most $1+\epsilon$.

Proof: We apply the Ahlswede-Winter inequality with $X_i = \big(Y_i - \mathrm{E}[Y_i]\big)/R$. Note that $\mathrm{E}[X_i] = 0$, $\|X_i\| \leq 1$, and

$$\begin{aligned} \mathrm{E}[X_i^2] &= \frac{1}{R^2}\mathrm{E}\big[\big(Y_i - \mathrm{E}[Y_i]\big)^2\big] \\ &= \frac{1}{R^2}\Big(\mathrm{E}[Y_i^2] - 2\mathrm{E}[Y_i]^2 + \mathrm{E}[Y_i]^2\Big) \\ &\preceq \frac{1}{R^2}\mathrm{E}[Y_i^2] \qquad (\text{since } \mathrm{E}[Y_i]^2 \succeq 0) \\ &\preceq \frac{1}{R^2}\mathrm{E}[\|Y_i\| \cdot Y_i] \\ &\preceq \frac{R}{R^2}\mathrm{E}[Y_i] \end{aligned}$$

Finally, since $0 \preceq \mathrm{E}[Y_i] \preceq I$, we get

$$\lambda_{\max}\big(\mathrm{E}[X_i^2]\big) \ \leq\ 1/R. \tag{1}$$

Now we use Claim 15 from the Notes on Symmetric Matrices, together with the inequalities

$$\begin{aligned} 1+x &\leq e^x \quad \forall x \in \mathbb{R} \\ e^x &\leq 1 + x + x^2 \quad \forall x \in [-1,1]. \end{aligned}$$

Since $\|X_i\| \leq 1$, for any $\lambda \in [0,1]$, we have $e^{\lambda X_i} \preceq I + \lambda X_i + \lambda^2 X_i^2$, and so

$$\mathrm{E}[e^{\lambda X_i}] \ \preceq\ \mathrm{E}[I + \lambda X_i + \lambda^2 X_i^2] \ \preceq\ I + \lambda^2 \mathrm{E}[X_i^2] \ \preceq\ e^{\lambda^2 \mathrm{E}[X_i^2]}.$$

Thus by (1) we have

$$\|\mathrm{E}[e^{\lambda X_i}]\| \ \leq\ \|e^{\lambda^2 \mathrm{E}[X_i^2]}\| \ \leq\ e^{\lambda^2/R}.$$

The same analysis also shows that $\|\mathrm{E}[e^{-\lambda X_i}]\| \leq e^{\lambda^2/R}$. Substituting these two bounds into the basic Ahlswede-Winter inequality from the previous lecture, we obtain

$$\Pr\left[\ \Big\|\sum_{i=1}^{k}\frac{1}{R}\big(Y_i - \mathrm{E}[Y_i]\big)\Big\| > t\ \right] \ \leq\ 2d \cdot e^{-\lambda t}\prod_{i=1}^{k} e^{\lambda^2/R} \ =\ 2d \cdot \exp(-\lambda t + k\lambda^2/R).$$

Substituting $t = k\epsilon/R$ and $\lambda = \epsilon/2$ we get

$$\Pr\left[\ \left\| \frac{1}{R}\sum_{i=1}^{k} Y_i - \frac{k}{R}\mathrm{E}[Y_i] \right\| > \frac{k\epsilon}{R}\ \right] \ \leq\ 2d \cdot \exp(-k\epsilon^2/4R).$$

Multiplying by R/k and using the fact that $\mathrm{E}[Y_i] = I$, we have bounded the probability that any eigenvalue of the sample average matrix $\sum_{i=1}^{k} Y_i/k$ is less than $1-\epsilon$ or greater than $1+\epsilon$. □

Corollary 2 *Let Z be a random, symmetric, positive semi-definite $d \times d$ matrix. Define $U := \mathrm{E}[Z]$ and suppose $Z \preceq R \cdot U$ for some scalar $R \geq 1$. Let $Z_1, \ldots, Z_k$ be independent copies of Z. For any $\epsilon \in (0,1)$, we have*

$$\Pr\left[(1-\epsilon)U \ \preceq\ \frac{1}{k}\sum_{i=1}^{k} Z_i \ \preceq\ (1+\epsilon)U\right] \ \geq\ 1 - 2d \cdot \exp(-\epsilon^2 k/4R).$$

Proof: Let $U^{+/2} := (U^+)^{1/2}$ denote the square root of the pseudoinverse of U. Let $I_{\mathrm{im}\ U}$ denote the orthogonal projection on the image of U. Define the random, positive semi-definite matrices

$$Y \ :=\ U^{+/2} \cdot Z \cdot U^{+/2} \qquad \text{and} \qquad Y_i \ :=\ U^{+/2} \cdot Z_i \cdot U^{+/2}.$$

Because $Z_i \succeq 0$ and $U = \mathrm{E}[\sum_i Z_i]$, we have $\mathrm{im}(Z_i) \subseteq \mathrm{im}(U)$. So Claim 16 in Notes on Symmetric Matrices implies

$$(1-\epsilon)U \ \preceq\ \frac{1}{k}\sum_{i=1}^{k} Z_i \ \preceq\ (1+\epsilon)U \qquad \Longleftrightarrow \qquad (1-\epsilon)I_{\mathrm{im}\ U} \ \preceq\ \frac{1}{k}\sum_{i=1}^{k} Y_i \ \preceq\ (1+\epsilon)I_{\mathrm{im}\ U}.$$

We would like to use Theorem 1 to obtain our desired bound. We just need to check that the hypotheses of the theorem are satisfied. By Fact 6 from the Notes on Symmetric Matrices, we have

$$Y \ =\ U^{+/2} \cdot Z \cdot U^{+/2} \ \preceq\ U^{+/2} \cdot (R \cdot U) \cdot U^{+/2} \ =\ R \cdot I_{\mathrm{im}\ U},$$

showing that $\|Y\| \leq R$. Next,

$$\mathrm{E}[Y] \ =\ U^{+/2} \cdot \mathrm{E}[Z] \cdot U^{+/2} \ =\ U^{+/2} \cdot U \cdot U^{+/2} \ =\ I_{\mathrm{im}\ U}.$$

So the hypotheses of Theorem 1 are almost satisfied, with the small issue that $\mathrm{E}[Y]$ is not actually the identity, but merely the identity on the image of U. But, one may check that the proof of Theorem 1 still goes through as long as every eigenvalue of $\mathrm{E}[Y]$ is either 0 or 1, i.e., $\mathrm{E}[Y]$ is an orthogonal projection matrix. The details are left as an exercise. □