CPSC 320: Intermediate Algorithm Design	2016 Summer Term 1

Tutorial 10

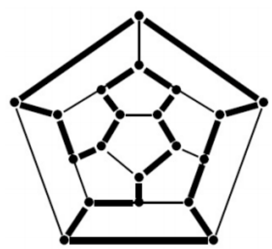
Prof. Nick Harvey — *University of British Columbia*

Reductions between problems, showing that some problems are algorithmically hard:
the theory of NP-completeness

Note: Tutorial A students are at a slight disadvantage as they have to do this tutorial before attending Lecture 14. Alireza will tailor his presentation appropriately.

1. **(NP Completeness of Bounded-Degree Spanning Trees)**
   The Hamiltonian Path Problem is defined as follows. In a graph $G = (V, E)$, a path that contains every vertex exactly once is called a Hamiltonian Path. The decision problem is "Given a graph G, does G contain a Hamiltonian Path?"

   The Hamiltonian Path Problem originates with a puzzle called the icosian game created by Sir William Rowan Hamilton.

   A generalization called the Traveling Salesman Problem has many interesting applications, which you might like to read about on this webpage.

   The **Bounded-Degree Spanning Tree Problem** is defined as follows. In a graph $G = (V, E)$, a spanning tree T has maximum degree at most d if every vertex in V has at most d incident edges that belong to T. The decision problem is "Given a graph G and an integer d, does G contain a spanning tree of maximum degree d?"

   The motivation for this problem is quite natural. If we wanted to form connections between many networked computers, a natural scheme is to form a spanning tree, because this uses the minimum number of connections to ensure that any computer can reach any other using a path of connections. However, if some computer had too many connections in this tree, those connections might overload its resources. So we may be interested to find a spanning tree where each computer has at most d connections.

   (a) Prove that the Bounded-Degree Spanning Tree Problem is in NP. More specifically, prove that, given a certificate corresponding to a subset of vertices in G, it is possible to verify, in polynomial time, that this subset is a spanning tree of G with degree at most d.

   (b) The Hamiltonian Path problem is known to be NP complete. (The closely related Hamiltonian Cycle problem is proven to be NP complete in Kleinberg-Tardos Section 8.5 and Erickson Section 30.11.)

   Prove that the Bounded-Degree Spanning Tree Problem is NP complete. More specifically, prove that you may reduce a known NP complete problem to the Bounded-Degree Spanning Tree problem in polynomial time.

2. **(NP Completeness of Clique)**
   The **independent set problem** is defined as follows. (See Kleinberg-Tardos Section 8.1 or Erickson Section 30.7). In a graph $G = (V, E)$, a set of nodes S is independent if no two nodes in S are joined by an edge. The decision problem is "Given a graph G and a number k, does G contain an independent set of size at least k?"

   The **clique problem** is as follows. In a graph $G = (V, E)$, a set of nodes S is called a clique if every pair of nodes in S are joined by an edge. The decision problem is "Given a graph G and a number k, does G contain a clique of size at least k?"

   (a) Prove that the clique problem is in NP. More specifically, prove that, given a certificate corresponding to a subset of nodes in G, it is possible to verify, in polynomial time, that this subset is a clique of G with size k.

   (b) The independent set problem is known to be NP complete. (See Kleinberg-Tardos Sections 8.2 and 8.4 or Erickson Section 30.7.) Prove that the clique problem is NP complete. More specifically, prove that you may reduce a known NP complete problem to the clique problem in polynomial time.