Settling the Sample Complexity for Learning Mixtures of Gaussians

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Abstract

We prove that $\tilde{\Theta}(kd^2/\varepsilon^2)$ samples are necessary and sufficient for learning a mixture of $k$ Gaussians in $\mathbb{R}^d$, up to error $\varepsilon$ in total variation distance. This improves both the known upper bound and lower bound for this problem. For mixtures of axis-aligned Gaussians, we show that $\tilde{O}(kd/\varepsilon^2)$ samples suffice, matching a known lower bound. Moreover, these results hold in an agnostic learning setting as well.

The upper bound is based on a novel technique for distribution learning based on a notion of sample compression. Any class of distributions that allows such a sample compression scheme can also be learned with few samples. Moreover, if a class of distributions has such a compression scheme, then so do the classes of products and mixtures of those distributions. The core of our main result is showing that the class of Gaussians in $\mathbb{R}^d$ has an efficient sample compression.

1 Introduction

Estimating distributions from observed data is a fundamental task in statistics that has been studied for over a century. This task frequently arises in applied machine learning and it is very common to assume that the distribution can be modeled using a mixture of Gaussians. Popular software packages have implemented heuristics, such as the EM algorithm, for learning a mixture of Gaussians. The theoretical machine learning community also has a rich literature on distribution learning. For example, the recent survey

This paper develops a general technique for distribution learning, then employs these techniques in the important setting of mixtures of Gaussians. The theoretical model that we adopt is density estimation: given i.i.d. samples from an unknown target distribution, find a distribution that is close to the target distribution in total variation (TV) distance. Our focus is on sample complexity bounds: using as few samples as possible to obtain a good estimate of the target distribution. For background on this model see, e.g., Devroye and Lugosi (2001, Chapter 5) and Diakonikolas (2016).

Our new technique for upper bounds on the sample complexity involves a form of sample compression. If it is possible to “encode” members of a class of distributions using a carefully chosen subset of the samples, then this yields an upper bound on the sample complexity of distribution learning for that class. In particular, by constructing compression schemes for mixtures of axis-aligned Gaussians and general Gaussians, we obtain new upper bounds on the sample complexity of learning with respect to these classes, which are optimal up to logarithmic factors.

The compression framework can incorporate a notion of robustness, which leads to sample complexity bounds for agnostic learning. Namely, if the target distribution is close to a mixture of Gaussians (in TV distance), our method uses few samples to find a mixture of Gaussians that is close to the target distribution (in TV distance).

1.1 Main results

In this section, all learning results refer to the problem of producing a distribution within total variation distance $\varepsilon$ from the target distribution.

Our first main result is an upper bound for learning mixtures of multivariate Gaussians. This bound is tight up to logarithmic factors.

**Theorem 1.1.** The class of $k$-mixtures of $d$-dimensional Gaussians can be learned using $\tilde{O}(kd^2/\varepsilon^2)$ samples. This result generalizes to the agnostic setting.

Previously, the best upper bounds on the sample complexity of this problem were $\tilde{O}(kd^2/\varepsilon^4)$, due to Ashtiani, Ben-David, and Mehrabian (2017), and $O(k^4 d^4/\varepsilon^2)$, based on a VC-dimension bound discussed later. For the case of a single Gaussian (i.e., $k=1$), a sample complexity bound of $O(d^2/\varepsilon^2)$ is well known (see, e.g., Ashtiani et al. (2017, Theorem 13)).

Our second main result is a lower bound matching Theorem 1.1 up to logarithmic factors.
Theorem 1.2. Any method for learning the class of $k$-mixtures of $d$-dimensional Gaussians has sample complexity $\tilde{\Omega}(kd^2/\varepsilon^2)$.

Previously, the best lower bound on the sample complexity was $\tilde{\Omega}(kd/\varepsilon^2)$ (Suresh, Orlitsky, Acharva, and Jafarpour, 2014). Even for a single Gaussian (i.e., $k=1$), an $\tilde{\Omega}(d^2/\varepsilon^2)$ lower bound was not known prior to this work.

Our third main result is an upper bound for learning mixtures of axis-aligned Gaussians, i.e., Gaussians with diagonal covariance matrix. This bound is tight up to logarithmic factors.

Theorem 1.3. The class of $k$-mixtures of axis-aligned $d$-dimensional Gaussians can be learned using $\tilde{O}(kd/\varepsilon^2)$ samples. This result generalizes to the agnostic setting.

A matching lower bound of $\tilde{\Omega}(kd/\varepsilon^2)$ samples was shown by Suresh et al. (2014). Previously, the best known upper bounds were $\tilde{O}(kd/\varepsilon^4)$, due to Ashtiani et al. (2017), and $O((k^4d^2 + k^3d^3)/\varepsilon^2)$, based on a VC-dimension discussed later.

Computational efficiency. Although our approach for proving sample complexity upper bounds is algorithmic, our focus is not on computational efficiency. The resulting algorithms are efficient in terms of sample complexity, but their runtime is exponential in the dimension $d$ and the number of mixture components $k$. The existence of a polynomial time algorithm for density estimation is unknown even for the class of mixtures of axis-aligned Gaussians (Diakonikolas, Kane, and Stewart, 2017a, Question 1.1).

Even for the case of a single Gaussian, the published proofs of the $O(d^2/\varepsilon^2)$ bound are not algorithmically efficient. Using ideas from our proof of Theorem 1.1, we show in Appendix B that an algorithmically efficient proof for the single Gaussian case can be obtained simply by computing the empirical mean and covariance matrix of $O(d^2/\varepsilon^2)$ samples.

1.2 Related work

Distribution learning is a vast topic and many approaches have been considered in the literature. We briefly review approaches that are most relevant to our problem.

For parametric families of distributions, a common approach is to use the samples to estimate the parameters of the distribution, possibly in a maximum likelihood sense, or possibly aiming to approximate the true parameters. For the specific case of mixtures of Gaussians, there is a substantial theoretical literature on algorithms that approximate the mixing weights, means and covariances. Kalai et al. (2012) gave a recent survey of this literature. The strictness of this objective cuts both ways. On the one hand, a
successful learner uncovers substantial structure of the target distribution. On the other hand, this objective is clearly impossible when the means and covariances are extremely close. Thus, algorithms for parameter estimation of mixtures necessarily require some assumptions on the target parameters. Also, the basic definition of parameter estimation does not immediately extend to an agnostic setting, although there is literature on agnostic parameter estimation, e.g., \cite{Lai2016}.

Density estimation has a long history in the statistics literature, where the focus is on the sample complexity question; see, e.g., \cite{Devroye1987,Devroye2001} for general background. It was first studied in the computational learning theory community under the name \textit{PAC learning of distributions} by \cite{Kearns1994}, whose focus is on the computational complexity of the learning algorithm.

For density estimation there are various possible measures of distance between distributions, the most popular ones being the TV distance and the Kullback-Leibler (KL) divergence. Here we focus on the TV distance since it has several appealing properties, such as being a metric and having a natural probabilistic interpretation. In contrast, KL divergence is not even symmetric and can be unbounded even for intuitively close distributions. For a detailed discussion on why TV is a natural choice, see \cite{Devroye2001}, Chapter 5).

A popular method for distribution learning in practice is kernel density estimation (see, e.g., \cite{Devroye2001}, Chapter 9)). The few rigorously proven sample complexity bounds for this method require certain smoothness assumptions on the class of densities (e.g., \cite{Devroye2001}, Theorem 9.5)). The class of Gaussians is not universally Lipschitz and does not satisfy these assumptions, so those results do not apply to the problems we consider.

Another elementary method for density estimation is using histogram estimators. Straightforward calculations show that histogram estimators for mixtures of Gaussians would result in a sample complexity that is exponential in the dimension. The same is true for estimators based on piecewise polynomials.

The \textit{minimum distance estimate} (\cite{Devroye2001}, Section 6.8) is another approach for deriving sample complexity upper bounds for distribution learning. This approach is based on uniform convergence theory. In particular, an upper bound for any class of distributions can be achieved by bounding the VC-dimension of an associated set system, called the Yatracos class (see \cite{Devroye2001}, page 58 for the definition). For example, \cite{Diakonikolas2017b} used this approach to bound the sample complexity of learning high-dimensional log-concave distributions. However, for mixtures of Gaussians and axis-aligned Gaussians in $\mathbb{R}^d$, the best known VC-dimension bound (\cite{Anthony1999}, Theorem 8.14) results in loose upper bounds of $O(k^4d^4/\varepsilon^2)$ and $O((k^4d^2 + k^3d^3)/\varepsilon^2)$ respectively.
Another approach is to first approximate the mixture class using a more manageable class such as piecewise polynomials, and then study the associated Yatracos class, see, e.g., Chan, Diakonikolas, Servedio, and Sun (2014). However, piecewise polynomials do a poor job in approximating $d$-dimensional Gaussians, resulting in an exponential dependence on $d$.

For density estimation of mixtures of Gaussians, the current best sample complexity upper bounds (in terms of $k$ and $d$) are $\tilde{O}(kd^2/\varepsilon^4)$ for general Gaussians and $\tilde{O}(kd/\varepsilon^4)$ for axis-aligned Gaussians, both due to Ashtiani et al. (2017). For the general Gaussian case, their method takes an i.i.d. sample of size $\tilde{O}(kd^2/\varepsilon^2)$ and partitions this sample in every possible way into $k$ subsets. Based on those partitions, $k^{\tilde{O}(kd^2/\varepsilon^2)}$ “candidate distributions” are generated. The problem is then reduced to learning with respect to that finite class of candidates. Their sample complexity has a suboptimal factor of $1/\varepsilon^4$, of which $1/\varepsilon^2$ arises in their approach for choosing the best candidate, and another factor $1/\varepsilon^2$ is due to the exponent in the number of candidates.

Our approach via compression schemes also ultimately reduces the problem to learning with respect to finite classes. However, our compression technique leads to a more refined bound. In the case of mixtures of Gaussians, one factor of $1/\varepsilon^2$ is again incurred due to learning with respect to finite classes. The key is that the number of compressed samples has no additional factor of $1/\varepsilon^2$, so the overall sample complexity bound has only a $\tilde{O}(1/\varepsilon^2)$ dependence on $\varepsilon$.

As for lower bounds on the sample complexity, much fewer results are known for learning mixtures of Gaussians. The only lower bound of which we are aware is due to Suresh et al. (2014), who show a bound of $\Omega(kd/\varepsilon^2)$ for learning mixtures of axis-aligned Gaussians (and hence for general Gaussians as well). This bound is tight for the axis-aligned case, as we show in Theorem 1.3, but loose in the general case, as we show in Theorem 1.2.

### 1.3 Our techniques

We introduce a novel method for learning distributions via a form of sample compression. Given a class of distributions, suppose there is a method for “compressing” the samples generated by any distribution in the class. Further, suppose there exists a fixed decoder for the class, such that given the compressed set of instances and a sequence of bits, it approximately recovers the original distribution. In this case, if the size of the compressed set and the number of bits is guaranteed to be small, we show that the sample complexity of learning that class is small as well.

More precisely, say a class of distributions admits $(\tau, t, m)$ compression if there exists a decoder function such that upon generating $m$ i.i.d. samples from any distribution in
the class, we are guaranteed, with reasonable probability, to have a subset of size at most \( \tau \) of that sample, and a sequence of at most \( t \) bits, on which the decoder outputs an approximation to the original distribution. Note that \( \tau, t, \) and \( m \) can be functions of \( \varepsilon \), the accuracy parameter.

This definition is generalized to a stronger notion of \textit{robust compression}, where the target distribution is to be encoded using samples that are not necessarily generated from the target itself, but are generated from a distribution that is close to the target. We prove that robust compression implies agnostic learning. In particular, if a class admits \((\tau, t, m)\) robust compression, then the sample complexity of agnostic learning with respect to this class is bounded by \( \tilde{O}(m + (\tau + t)/\varepsilon^2) \) (Theorem 3.5).

An attractive property of robust compression is that it enjoys two closure properties. Specifically, if a base class admits robust compression, then the class of \( k \)-mixtures of that base class, as well as the class of products of the base class, are robustly compressible (Lemmas 3.6 and 3.7).

Consequently, it suffices to provide a robust compression scheme for the class of single Gaussian distributions in order to obtain a compression scheme for classes of mixtures of Gaussians (and therefore, to be able to bound their sample complexity). We prove that the class of \( d \)-dimensional Gaussian distributions admits \((\tilde{O}(d), \tilde{O}(d^2), \tilde{O}(d))\) robust compression (Lemma 4.2). The high level idea is that by generating \( \tilde{O}(d) \) samples from a Gaussian, one can get some rough sketch of the geometry of the Gaussian. In particular, the convex hull of the points drawn from a Gaussian enclose an ellipsoid centered at the mean and whose principal axes are the eigenvectors of the covariance matrix. Using ideas from convex geometry and random matrix theory, we show one can in fact encode the center of the ellipsoid \textit{and} the principal axes using a convex combination of these samples. Then we discretize the coefficients and obtain an approximate encoding.

The above results together imply tight (up to logarithmic factors) upper bounds of \( \tilde{O}(kd^2/\varepsilon^2) \) for mixtures of \( k \) Gaussians, and \( \tilde{O}(kd/\varepsilon^2) \) for mixtures of \( k \) axis-aligned Gaussians over \( \mathbb{R}^d \). The \textit{robust compression} framework we introduce is quite flexible, and can be used to prove sample complexity upper bounds for other distribution classes as well.

\textbf{Lower bound.} For proving our lower bound for mixtures of Gaussians, we first prove a lower bound of \( \tilde{\Omega}(d^2/\varepsilon^2) \) for learning a single Gaussian. Although the approach is quite intuitive, the details are intricate and much care is required to make a formal proof. The main step is to construct a large family (of size \( 2^{\tilde{\Omega}(d^2)} \)) of covariance matrices such that the associated Gaussian distributions are well-separated in terms of their total variation distance while simultaneously ensuring that their Kullback-Leibler divergences are small. Once this is established, we can then apply a generalized version of Fano’s inequality to complete the proof.
To construct this family of covariance matrices, we sample $2^{\Omega(d^2)}$ matrices from the following probabilistic process: start with an identity covariance matrix. Then choose a random subspace of dimension $d/9$ and slightly increase the eigenvalues corresponding to this eigenspace from 1 to roughly $1 + \varepsilon/\sqrt{d}$. It is easy to bound the KL divergence between the constructed Gaussians. To lower bound the total variation, we show that for every pair of these distributions, there is some subspace for which a vector drawn from one Gaussian will have slightly larger projection than a vector drawn from the other Gaussian. Quantifying this gap will then give us the desired lower bound on the total variation distance.

1.4 Paper outline

We set up our formal framework and notations in Section 2. In Section 3, we define compression schemes for distributions, prove their closure properties, and show their connection with density estimation. Theorem 1.1 and Theorem 1.3 are proved in Section 4. Theorem 1.2 is proved in Section 5. All omitted proofs can be found in the appendix.

2 Preliminaries

A distribution learning method or density estimation method is an algorithm that takes as input a sequence of i.i.d. samples generated from a distribution $g$, and outputs (a description of) a distribution $\hat{g}$ as an estimation for $g$. We work with continuous distributions in this paper, and so we identify a probability distribution by its probability density function. Let $f_1$ and $f_2$ be two probability distributions defined over the Borel $\sigma$-algebra $\mathcal{B}$. The total variation (TV) distance between $f_1$ and $f_2$ is defined by

$$\text{TV}(f_1, f_2) := \sup_{B \in \mathcal{B}} \int_B (f_1(x) - f_2(x))dx = \frac{1}{2} \|f_1 - f_2\|_1,$$

where $\|f\|_1 := \int_{\mathbb{R}^d} |f(x)|dx$ is the $L_1$ norm of $f$. The Kullback-Leibler (KL) divergence between $f_1$ and $f_2$ is defined by

$$\text{KL}(f_1 \parallel f_2) := \int_{\mathbb{R}^d} f_1(x) \log \frac{f_1(x)}{f_2(x)}dx.$$

In the following definitions, $\mathcal{F}$ is a class of probability distributions, and $g$ is a distribution (not necessarily in $\mathcal{F}$).

**Definition 2.1** ($\varepsilon$-approximation, $(\varepsilon, C)$-approximation). A distribution $\hat{g}$ is an $\varepsilon$-approximation for $g$ if $\|\hat{g} - g\|_1 \leq \varepsilon$. A distribution $\hat{g}$ is an $(\varepsilon, C)$-approximation for $g$ with respect to $\mathcal{F}$ if

$$\|\hat{g} - g\|_1 \leq C \cdot \inf_{f \in \mathcal{F}} \|f - g\|_1 + \varepsilon.$$
Definition 2.2 (PAC-learning distributions, realizable setting). A distribution learning method is called a (realizable) PAC-learner for $\mathcal{F}$ with sample complexity $m_{\mathcal{F}}(\varepsilon, \delta)$, if for all distribution $g \in \mathcal{F}$ and all $\varepsilon, \delta \in (0, 1)$, given $\varepsilon, \delta$, and a sample of size $m_{\mathcal{F}}(\varepsilon, \delta)$ generated i.i.d. by that $g$, with probability at least $1 - \delta$ (over the samples) the method outputs an $\varepsilon$-approximation of $g$.

Definition 2.3 (PAC-learning distributions, agnostic setting). For $C > 0$, a distribution learning method is called a $C$-agnostic PAC-learner for $\mathcal{F}$ with sample complexity $m^C_{\mathcal{F}}(\varepsilon, \delta)$, if for all distributions $g$ and all $\varepsilon, \delta \in (0, 1)$, given $\varepsilon, \delta$, and a sample of size $m^C_{\mathcal{F}}(\varepsilon, \delta)$ generated i.i.d. from $g$, with probability at least $1 - \delta$ the method outputs an $(\varepsilon, C)$-approximation of $g$ w.r.t. $\mathcal{F}$.

We sometimes say a class can be “$C$-learned in the agnostic setting” to indicate the existence of a $C$-agnostic PAC-learner for the class. The case $C > 1$ is sometimes called semi-agnostic learning. Let $\Delta_n := \{ (w_1, \ldots, w_n) : w_i \geq 0, \sum w_i = 1 \}$ denote the $n$-dimensional simplex.

Definition 2.4 ($k$-mix($\mathcal{F}$)). Let $\mathcal{F}$ be a class of probability distributions. Then the class of $k$-mixtures of $\mathcal{F}$, written $k$-mix($\mathcal{F}$), is defined as

$$ k\text{-mix}(\mathcal{F}) := \{ \sum_{i=1}^k w_i f_i : (w_1, \ldots, w_k) \in \Delta_k, f_1, \ldots, f_k \in \mathcal{F} \} $$

Let $d$ denote the dimension. A Gaussian distribution with mean $\mu \in \mathbb{R}^d$ and covariance matrix $\Sigma \in \mathbb{R}^{d \times d}$ is denoted by $\mathcal{N}(\mu, \Sigma)$. If $\Sigma$ is a diagonal matrix, then $\mathcal{N}(\mu, \Sigma)$ is called an axis-aligned Gaussian. For a distribution $g$, we write $X \sim g$ to mean $X$ is a random variable with distribution $g$, and we write $S \sim g^n$ to mean that $S$ is an i.i.d. sample of size $m$ generated from $g$.

Definition 2.5. A random variable $X$ is said to be $\sigma$-subgaussian if $\Pr[|X| \geq t] \leq 2 \exp(-t^2/\sigma^2)$ for any $t > 0$.

Note that if $X \sim \mathcal{N}(0, 1)$ then $X$ is $\sqrt{2}$-subgaussian, see, e.g., Abramowitz and Stegun (1984, formula (7.1.13)).

Definition 2.6. Let $A, B$ be symmetric, positive definite matrices of the same size. The log-det divergence of $A$ and $B$ is defined as $\text{LD}(A, B) := \text{tr}(B^{-1}A - I) - \log \det(B^{-1}A)$.

We will use $\|v\|$ or $\|v\|_2$ to denote the Euclidean norm of a vector $v$, $\|A\|$ or $\|A\|_2$ to denote the operator norm of a matrix $A$, and $\|A\|_F := \sqrt{\text{tr}(A^T A)}$ to denote the Frobenius norm of a matrix $A$. For $x \in \mathbb{R}$, we will write $(x)_+ := \max\{0, x\}$.
3 Compression schemes and their connection with learning

Let \( \mathcal{F} \) be a class of distributions over a domain \( Z \).

**Definition 3.1** (distribution decoder). A distribution decoder for \( \mathcal{F} \) is a deterministic function \( J : \bigcup_{n=0}^{\infty} Z^n \times \bigcup_{n=0}^{\infty} \{0,1\}^n \to \mathcal{F} \), which takes a finite sequence of elements of \( Z \) and a finite sequence of bits, and outputs a member of \( \mathcal{F} \).

**Definition 3.2** (robust distribution compression schemes). Let \( \tau, t, m : (0,1) \to \mathbb{Z}_{\geq 0} \) be functions, and let \( r \geq 0 \). We say \( \mathcal{F} \) admits \( (\tau, t, m) \)-robust compression if there exists a decoder \( J \) for \( \mathcal{F} \) such that for any distribution \( g \in \mathcal{F} \), and for any distribution \( q \) on \( Z \) with \( \|g - q\|_1 \leq r \), the following holds:

For any \( \varepsilon \in (0,1) \), if the sample \( S \) is drawn from \( q^{m(\varepsilon)} \), then with probability at least \( 2/3 \), there exists a sequence \( L \) of at most \( \tau(\varepsilon) \) elements of \( S \), and a sequence \( B \) of at most \( t(\varepsilon) \) bits, such that \( \|J(L, B) - g\|_1 \leq \varepsilon \).

Essentially, the definition asserts that with high probability, there should be a (small) subset of \( S \) and some (small number of) additional bits, from which \( g \) can be approximately reconstructed. We say that the distribution \( g \) is “encoded” with \( L \) and \( B \), and in general we would like to have a compression scheme of a small size. This compression scheme is called “robust” since it requires \( g \) to be approximately reconstructed from a sample generated from \( q \) rather than \( g \) itself.

**Remark 3.3.** In the definition above we required the probability of existence of \( L \) and \( B \) to be at least \( 2/3 \), but one can boost this probability to \( 1 - \delta \) by generating a sample of size \( m(\varepsilon) \log(1/\delta) \).

Next we show that if a class of distributions can be compressed, then it can be learned; thus we build the connection between robust compression and agnostic learning. We will need the following useful result about PAC-learning of finite classes of distributions, which immediately follows from Devroye and Lugosi (2001, Theorem 6.3) and a standard Chernoff bound. It states that a finite class of size \( M \) can be 3-learned in the agnostic setting using \( O(\log(M/\delta)/\varepsilon^2) \) samples. Denote by \([M]\) the set \( \{1, 2, ..., M\} \). Throughout the paper, \( a/bc \) always means \( a/(bc) \).

**Theorem 3.4** (Devroye and Lugosi (2001)). There exists a deterministic algorithm that, given candidate distributions \( f_1, \ldots, f_M \), a parameter \( \varepsilon > 0 \), and \( \log(3M^2/\delta)/2\varepsilon^2 \) i.i.d. samples from an unknown distribution \( g \), outputs an index \( j \in [M] \) such that

\[
\|f_j - g\|_1 \leq 3 \min_{i \in [M]} \|f_i - g\|_1 + 4\varepsilon,
\]

with probability at least \( 1 - \delta/3 \).
The proof of the following theorem appears in Appendix C.1.

**Theorem 3.5** (compressibility implies learnability). Suppose \( \mathcal{F} \) admits \( (\tau, t, m) \) \( r \)-robust compression. Let \( \tau'(\varepsilon) := \tau(\varepsilon/6) + t(\varepsilon/6) \). Then \( \mathcal{F} \) can be max\( \{3, 2/r\} \)-learned in the agnostic setting using

\[
O \left( m \left( \frac{\varepsilon}{6} \right) \log \left( \frac{1}{\delta} \right) + \frac{\tau'(\varepsilon) \log(m(\varepsilon/6) \log(1/\delta)) + \log(1/\delta)}{\varepsilon^2} \right) = \tilde{O} \left( m \left( \frac{\varepsilon}{6} \right) + \frac{\tau'(\varepsilon)}{\varepsilon^2} \right) \text{ samples.}
\]

If \( \mathcal{F} \) admits \( (\tau, t, m) 0 \)-robust compression, then \( \mathcal{F} \) can be learned in the realizable setting using the same number of samples.

We next prove two closure properties of compression schemes. First, Lemma 3.6 below implies that if a class \( \mathcal{F} \) of distributions can be compressed, then the class of distributions that are formed by taking products of members of \( \mathcal{F} \) can also be compressed. If \( p_1, \ldots, p_d \) are distributions over domains \( Z_1, \ldots, Z_d \), then \( \prod_{i=1}^d p_i \) denotes the standard product distribution over \( \prod_{i=1}^d Z_i \). For a class \( \mathcal{F} \) of distributions, define \( \mathcal{F}^d := \left\{ \prod_{i=1}^d p_i : p_1, \ldots, p_d \in \mathcal{F} \right\} \). The following lemma is proved in Appendix C.2.

**Lemma 3.6** (compressing product distributions). If \( \mathcal{F} \) admits \( (\tau(\varepsilon), t(\varepsilon), m(\varepsilon)) \) \( r \)-robust compression, then \( \mathcal{F}^d \) admits \( (d\tau(\varepsilon/d), dt(\varepsilon/d) + \log(3d), m(\varepsilon/d) \log(3d)) \) \( r \)-robust compression.

Our next lemma implies that if a class \( \mathcal{F} \) of distributions can be compressed, then the class of distributions that are formed by taking mixtures of members of \( \mathcal{F} \) can also be compressed. The proof appears in Appendix C.3.

**Lemma 3.7** (compressing mixtures). If \( \mathcal{F} \) admits \( (\tau(\varepsilon), t(\varepsilon), m(\varepsilon)) \) \( r \)-robust compression, then \( k\text{-mix}(\mathcal{F}) \) admits \( (k\tau(\varepsilon/3), kt(\varepsilon/3) + k \log_2(4k/\varepsilon)), 48m(\varepsilon/3)k \log(6k)/\varepsilon \) \( r \)-robust compression.

4 Upper bound: learning mixtures of Gaussians by compression schemes

4.1 Warm-up: learning mixtures of axis-aligned Gaussians by compression schemes

In this section, we give a simple application of our compression framework to prove an upper bound of \( \tilde{O}(kd/\varepsilon^2) \) for the sample complexity of learning mixtures of \( k \) axis-aligned Gaussians in the realizable setting. In the following section, we generalize these arguments to work for general Gaussians in the agnostic setting.
Lemma 4.1. The class of single-dimensional Gaussians admits a $(3, O(\log(1/\varepsilon)), 3)$ 0-robust compression scheme.

Proof. Let $c < 1 < C$ be such that $\Pr_{X \sim \mathcal{N}(0,1)}[c < |X| < C] \geq 0.99$. Let $\mathcal{N}(\mu, \sigma^2)$ be the target distribution. We first show how to encode $\sigma$. Let $g_1, g_2 \sim \mathcal{N}(\mu, \sigma^2)$. Then $g = \frac{1}{\sqrt{2}}(g_1 - g_2) \sim \mathcal{N}(0, \sigma^2)$. So with probability at least 0.99, we have $\sigma C < |g| < \sigma C$. Conditioned on this event, this implies that there is a $\lambda \in [-1/c, 1/c]$ such that $\lambda g = \sigma$. We now choose $\hat{\lambda} \in \{0, \pm \varepsilon/2, \pm \varepsilon/2, \pm 3\varepsilon/2, \ldots, \pm 1/c\}$ satisfying $|\hat{\lambda} - \lambda| \leq \varepsilon/(4C^2)$, and encode the standard deviation by $(g_1, g_2, \hat{\lambda})$. The decoder then estimates $\hat{\sigma} := \hat{\lambda}(g_1 - g_2)/\sqrt{2}$. Note that $|\hat{\sigma} - \sigma| \leq |\hat{\lambda} - \lambda||g| \leq \sigma \varepsilon/(4C)$ and that the encoding requires two sample points and $O(\log(C^2/\varepsilon)) = O(\log(1/\varepsilon))$ bits (for encoding $\hat{\lambda}$).

Now we turn to encoding $\mu$. Let $g_3 \sim \mathcal{N}(\mu, \sigma^2)$. Then $|g_3 - \mu| \leq C\sigma$ with probability at least 0.99. We will condition on this event, which implies existence of some $\eta \in [-C, C]$ such that $g_3 + \sigma \eta = \mu$. We choose $\hat{\eta} \in \{0, \pm \varepsilon/2, \pm 3\varepsilon/2, \ldots, \pm C\}$ such that $|\hat{\eta} - \eta| \leq \varepsilon/4$, and encode the mean by $(g_3, \hat{\eta})$. The decoder estimates $\hat{\mu} := g_3 + \hat{\sigma} \hat{\eta}$. Again, note that $|\hat{\mu} - \mu| = |\sigma \eta - \hat{\sigma} \hat{\eta}| \leq |\sigma \eta - \eta\hat{\eta}| + |\sigma \hat{\eta} - \hat{\sigma} \hat{\eta}| \leq \sigma \varepsilon/2$. Moreover, encoding the mean requires one sample point and $O(\log(1/\varepsilon))$ bits.

To summarize, the decoder has $|\hat{\mu} - \mu| \leq \sigma \varepsilon/2$ and $|\hat{\sigma} - \sigma| \leq \sigma \varepsilon/2$. Plugging these bounds into Lemma A.4 gives $\|\mathcal{N}(\mu, \sigma^2) - \mathcal{N}(\hat{\mu}, \hat{\sigma}^2)\|_1 \leq \varepsilon$, as required. \qed

To complete the proof of Theorem 1.3 in the realizable setting, we note that Lemma 4.1 combined with Lemma 3.6 implies that the class of axis-aligned Gaussians in $\mathbb{R}^d$ admits a $(O(d), O(d \log(d/\varepsilon)), O(\log(3d)))$ 0-robust compression scheme. Then, by Lemma 3.7 the class of mixtures of $k$ axis-aligned Gaussians admit a $(O(kd), O(kd \log(d/\varepsilon) + k \log(k/\varepsilon)), O(k \log(k) \log(3d/\varepsilon)))$ 0-robust compression scheme. Applying Theorem 3.5 implies that the class of $k$-mixtures of axis-aligned Gaussians in $\mathbb{R}^d$ can be learned using $O(kd/\varepsilon^2)$ many samples in the realizable setting.

4.2 Agnostic learning mixtures of Gaussians by compression schemes

In this section we prove an upper bound of $O(kd^2/\varepsilon^2)$ for the sample complexity of learning mixtures of $k$ Gaussians in $d$ dimensions, and an upper bound of $O(kd^2/\varepsilon^2)$ for the sample complexity of learning mixtures of $k$ axis-aligned Gaussians, both in the agnostic sense. The heart of the proof is to show that Gaussians have robust compression schemes in any dimension.

Lemma 4.2. For any positive integer $d$, the class of $d$-dimensional Gaussians admits an $(O(d \log(2d)), O(d^2 \log(2d) \log(d/\varepsilon)), O(d \log(2d)))$ $2/3$-robust compression scheme.
Remark 4.3. This lemma can be boosted to give an \( r \)-robust compression schemes for any \( r < 1 \) at the expense of worse constants hidden in the big Oh, but this will not yield any improvement in the final results.

Remark 4.4. In the special case \( d = 1 \), there also exists a \((4, 1, O(1/\varepsilon)) \) (i.e., constant size) 0.773-robust compression scheme using completely different ideas. The proof appears in Appendix D.4. Remarkably, this compression scheme has constant size, as the value of \( \tau + t \) is independent of \( \varepsilon \) (unlike Lemma 4.2). This scheme could be used instead of Lemma 4.2 in the proof of Theorem 1.3, although it would not improve the sample complexity bound asymptotically.

Proof of Theorem 1.1. Combining Lemma 4.2 and Lemma 3.7 implies that the class of \( k \)-mixtures of \( d \)-dimensional Gaussians admits a

\[
\left( O(kd \log(2d)), O(kd^2 \log(d/\varepsilon) + k \log(k/\varepsilon)), O(dk \log d \log(2d)/\varepsilon) \right)
\]

2/3-robust compression scheme. Applying Theorem 3.5 with \( m(\varepsilon) = \tilde{O}(dk/\varepsilon) \) and \( \tau'(\varepsilon) = \tilde{O}(d^2k) \) shows that the sample complexity of learning this class is \( \tilde{O}(kd^2/\varepsilon^2) \). This proves Theorem 1.1. ■

Proof of Theorem 1.3. Let \( \mathcal{G} \) denote the class of 1-dimensional Gaussian distributions. By Lemma 4.2, \( \mathcal{G} \) admits an \((O(1), O(\log(1/\varepsilon)), O(1)) \) 2/3-robust compression scheme. By Lemma 3.6, the class \( \mathcal{G}^d \) admits a \((O(d), O(d \log(d/\varepsilon)), O(\log(3d))) \) 2/3-robust compression scheme. Then, by Lemma 3.7, the class \( k\text{-mix} (\mathcal{G}^d) \) admits \((O(kd), O(kd \log(d/\varepsilon) + k \log(k/\varepsilon)), O(k \log(k \log(3d)/\varepsilon))) \) 2/3-robust compression. Applying Theorem 3.5 implies that the class of \( k \)-mixtures of axis-aligned Gaussians in \( \mathbb{R}^d \) can be 3-agnostically learned using \( \tilde{O}(kd/\varepsilon^2) \) many samples. ■

4.3 Proof of Lemma 4.2

Let \( Q \) denote the target distribution, which satisfies \( \|Q - \mathcal{N}(\mu, \Sigma)\|_1 \leq 2/3 \) for some Gaussian \( \mathcal{N}(\mu, \Sigma) \) which we are to encode. Note that this implies \( \text{TV}(Q, \mathcal{N}(\mu, \Sigma)) \leq 1/3 \).

Remark 4.5. The case of rank-deficient \( \Sigma \) can easily be reduced to the case of full-rank \( \Sigma \). If the rank of \( \Sigma \) is \( k < d \), any \( X \sim \mathcal{N}(\mu, \Sigma) \) lies in some affine subspace \( S \) of dimension \( k \). Thus, any \( X \sim Q \) lies in \( S \) with probability at least 2/3. With high probability, after seeing 10d samples from \( Q \), at least \( k + 1 \) points from \( S \) will appear in the sample. We encode \( S \) using these samples, and for the rest of the process we work in this affine space, and discard outside points. Hence, we may assume \( \Sigma \) has full rank \( d \).

We first prove a lemma that is similar to known results in random matrix theory (see Litvak, Pajor, Rudelson, and Tomczak-Jaegermann, 2005, Corollary 4.1), but is tailored
for our purposes. Its proof appears in Appendix [D.1]. Let \( S^{d-1} := \{ y \in \mathbb{R}^d : \|y\| = 1 \} \) and \( B^d_2 := \{ y \in \mathbb{R}^d : \|y\| \leq 1 \} \).

**Lemma 4.6.** Let \( q_1, \ldots, q_m \) be i.i.d. samples from a distribution \( Q \) where \( \text{TV}(Q, \mathcal{N}(0, I_d)) \leq 2/3 \). Let

\[
T := \{ \pm q_i : \|q_i\| \leq 4\sqrt{d} \}.
\]

Then for a large enough constant \( C > 0 \), if \( m \geq Cd(1 + \log d) \) then

\[
\Pr \left[ \frac{1}{20} B^d_2 \not\subseteq \text{conv}(T) \right] \leq \frac{1}{6}.
\]

Suppose \( \Sigma = \sum_{i=1}^d v_i v_i^T \), where the \( v_i \) vectors are orthogonal. Let \( \Psi := \sum_{i=1}^d v_i v_i^T / \|v_i\| \). Note that both \( \Sigma \) and \( \Psi \) are positive definite, and that \( \Sigma = \Psi^2 \). Moreover, it is easy to see that \( \Sigma^{-1} = \sum_{i=1}^d v_i v_i^T / \|v_i\|^4 \) and \( \Psi^{-1} = \sum_{i=1}^d v_i v_i^T / \|v_i\|^3 \).

The following lemma is proved in Appendix [D.2].

**Lemma 4.7.** Let \( C > 0 \) be a sufficiently large constant. Given \( m = 2Cd(1 + \log d) \) samples \( S \) from \( Q \), where \( \text{TV}(Q, \mathcal{N}(\mu, \Sigma)) \leq 1/3 \), with probability at least 2/3, one can encode vectors \( \hat{v}_1, \ldots, \hat{v}_d, \hat{\mu} \in \mathbb{R}^d \) satisfying

\[
\|\Psi^{-1} (\hat{v}_j - v_j)\| \leq \frac{\varepsilon}{6d^2} \quad \forall j \in [d],
\]

and

\[
\|\Psi^{-1} (\hat{\mu} - \mu)\| \leq \varepsilon,
\]

using \( O(d^2 \log(2d) \log(d/\varepsilon)) \) bits and the points in \( S \).

Lemma 4.2 now follows immediately from the following lemma, which is proved in Appendix [D.3].

**Lemma 4.8.** Suppose \( \Sigma = \Psi^2 = \sum_{i \in [d]} v_i v_i^T \), where \( v_i \) are orthogonal and \( \Sigma \) is full rank, and that

\[
\|\Psi^{-1} (\hat{v}_j - v_j)\| \leq \rho \leq 1 \quad \forall j \in [d],
\]

and

\[
\|\Psi^{-1} (\hat{\mu} - \mu)\| \leq \zeta.
\]

Then

\[
\text{TV}(\mathcal{N}(\mu, \sum_{i \in [d]} v_i v_i^T), \mathcal{N}(\hat{\mu}, \sum_{i \in [d]} \hat{v}_i \hat{v}_i^T)) \leq \sqrt{9d^3 \rho^2 + \zeta^2}/2.
\]
5 The lower bound for Gaussians and their mixtures

In this section, we establish a lower bound of $\tilde{\Omega}(d^2/\varepsilon^2)$ for learning a single Gaussian, and then lift it to obtain a lower bound of $\tilde{\Omega}(kd^2/\varepsilon^2)$ for learning mixtures of $k$ Gaussians in $d$ dimensions. Both our lower bounds consider the realizable setting (so they also hold in the agnostic setting).

Our lower bound is based on the following lemma, which follows from Fano’s inequality in information theory (see Lemma E.1). Its proof appears in Appendix E.1.

**Lemma 5.1.** Let $\mathcal{F}$ be a class of distributions such that for all small enough $\varepsilon > 0$ there exist $N$ densities $f_1, \ldots, f_N \in \mathcal{F}$ with

$$\text{KL}(f_i \parallel f_j) \leq \kappa(\varepsilon)$$

and

$$\text{TV}(f_i, f_j) = \Omega(\varepsilon) \quad \forall i \neq j \in [N].$$

Then any algorithm that learns $\mathcal{F}$ to within total variation distance $\varepsilon$ with success probability at least $2/3$ has sample complexity $\Omega\left(\log N/(\kappa(\varepsilon) \log(1/\varepsilon))\right)$.

**Theorem 5.2.** Any algorithm that learns a general Gaussian in $\mathbb{R}^d$ in the realizable setting within total variation distance $\varepsilon$ and with success probability $2/3$ has sample complexity $\Omega\left(d^2/\varepsilon^2 \log(1/\varepsilon)\right)$.

**Proof.** Let $r = 9$ and $\lambda = \Theta(\varepsilon d^{-1/2} \log(1/\varepsilon))$. Guided by Lemma 5.1, we will build $2^{O(d^2)}$ Gaussian distributions of the form $f_a := \mathcal{N}(0, \Sigma_a)$ where $\Sigma_a = I_d + \lambda U_a U_a^T$, where each $U_a$ is a $d \times d/r$ matrix with orthonormal columns. To apply Lemma 5.1, we need to give an upper bound on the KL-divergence between any two $f_a$ and $f_b$, and a lower bound on their total variation distance. Upper bounding the KL divergence is easy: by Lemma A.1 and since $\|U_a^T U_b\|_F^2 \geq 0$,

$$2 \text{KL}(f_a \parallel f_b) = \text{Tr}(\Sigma_a^{-1} \Sigma_b - I) = \text{Tr}
\left((I - \frac{\lambda}{1 + \lambda} U_a U_a^T)(I + \lambda U_b U_b^T) - I\right)
= \text{Tr}(\lambda U_b U_b^T - \frac{\lambda}{1 + \lambda} U_a U_a^T - \frac{\lambda^2}{1 + \lambda} U_a U_a^T U_b U_b^T)
= \lambda (d/r) - \frac{\lambda}{1 + \lambda} (d/r) - \frac{\lambda^2}{1 + \lambda} \|U_a^T U_b\|_F^2
\leq \frac{\lambda^2 d}{r + r \lambda} \leq \lambda^2 d/(2r) = O(\varepsilon^2 \log^2(1/\varepsilon),
$$
as required.

Our next goal is to give a lower bound on the total variation distance between $f_a$ and $f_b$. For this, we would like the matrices $\{U_a\}$ to be “spread out,” in the sense that their columns should be nearly orthogonal. This is formalized in Lemma 5.3 below, where we show if we choose the $U_a$ randomly, we can achieve $\|U_a^T U_b\|_F^2 \leq \frac{d}{2r}$ for any $a \neq b$. Then,
if \( S_a \) is the subspace spanned by the columns of \( U_a \), then we expect that a Gaussian drawn from \( \mathcal{N}(0, \Sigma_a) \) should have a slightly larger projection onto \( S_a \) than a Gaussian drawn from \( \mathcal{N}(0, \Sigma_b) \). This will then allow us to give a lower bound on the total variation distance between \( \mathcal{N}(0, \Sigma_a) \) and \( \mathcal{N}(0, \Sigma_b) \). More precisely, in Lemma 5.4 we show that \( \|U_a^T U_b\|^2_F \leq \frac{d}{2r} \) implies \( \text{TV}(f_a, f_b) = \Omega \left( \frac{\lambda \sqrt{d/r}}{\log(r/\lambda \sqrt{d})} \right) = \Omega(\varepsilon) \), completing the proof.

We defer the proofs of the following lemmas to Appendix E.2 and Appendix E.3, respectively.

**Lemma 5.3.** Suppose \( d \geq r \geq 9 \). Then there exists \( 2^{\Omega(d^2/r)} \) orthonormal \( d \times d/r \) matrices \( \{U_a\} \) such that for any \( a \neq b \) we have \( \|U_a^T U_b\|^2_F \leq \frac{d}{2r} \).

**Lemma 5.4.** Suppose that \( \lambda \leq 1 \leq r \), and \( \lambda \sqrt{d/r} \in (0, 1/3) \). If \( \|U_a^T U_b\|^2_F \leq d/(2r) \), then \( \text{TV}(f_a, f_b) = \Omega \left( \frac{\lambda \sqrt{d/r}}{\log(r/\lambda \sqrt{d})} \right) \).

Finally, in Appendix E.4 we prove our lower bound for mixtures.

**Theorem 5.5.** Any algorithm that learns a mixture of \( k \) general Gaussians in \( \mathbb{R}^d \) in the realizable setting within total variation distance \( \varepsilon \) and with success probability at least \( 2/3 \) has sample complexity \( \Omega \left( \frac{kd^2}{\varepsilon^2 \log^3(1/\varepsilon)} \right) \).

### 6 Further discussion

A central open problem in distribution learning and density estimation is characterizing the sample complexity of learning a distribution class. An insight from supervised learning theory is that the sample complexity of learning a class (of concepts, functions, or distributions) may be proportional to some kind of intrinsic dimension of the class divided by \( \varepsilon^2 \), where \( \varepsilon \) is the error tolerance. For the case of agnostic binary classification, the intrinsic dimension is captured by the VC-dimension of the concept class (see Vapnik and Chervonenkis (1971); Blumer, Ehrenfeucht, Haussler, and Warmuth (1989)). For the case of distribution learning with respect to ‘natural’ parametric classes, we expect this dimension to be equal to the number of parameters. In this paper, we showed that this is indeed the case for the class of Gaussians, axis-aligned Gaussians, and their mixtures in any dimension.

In binary classification, the combinatorial notion of Littlestone-Warmuth compression has been shown to be sufficient (Littlestone and Warmuth, 1986) and necessary (Moran and Yehudayoff, 2016) for learning. In this work, we showed that the new but related notion of robust distribution compression is sufficient for distribution learning. Whether the existence of compression schemes is necessary for learning an arbitrary class of distributions remains an intriguing open problem.
We would like to mention that while it may first seem that the VC-dimension of the Yatracos set associated with a class of distributions can characterize its sample complexity, it is not hard to come up with examples where this VC-dimension is infinite while the class can be learned with finite samples. Covering numbers do not work, either; for instance the class of Gaussians do not have a bounded covering number in the TV metric, nevertheless it is learnable with finite samples.

A concept related to compression is that of core-sets. In a sense, core-sets can be viewed as a special case of compression, where the decoder is required to be the empirical error minimizer. See the work of (Lucic, Faulkner, Krause, and Feldman, 2017) for using core-sets in maximum likelihood estimation.

A Standard results

Lemma A.1 (Rasmussen and Williams (2006, Equation A.23)). For two full-rank Gaussians $\mathcal{N}(\mu, \Sigma)$ and $\mathcal{N}(\mu', \Sigma')$, their KL divergence is

$$KL(\mathcal{N}(\mu, \Sigma) \parallel \mathcal{N}(\mu', \Sigma')) = \frac{1}{2} \left( \text{Tr}(\Sigma^{-1} \Sigma' - I) + (\mu - \mu')^T \Sigma^{-1} (\mu - \mu') - \log \det(\Sigma' \Sigma^{-1}) \right).$$

Lemma A.2 (Pinsker’s Inequality (Tsybakov, 2009, Lemma 2.5)). For any two distributions $A$ and $B$, we have $2 \text{TV}(A, B)^2 \leq KL(A \parallel B)$.

Lemma A.3. For two full-rank Gaussians $\mathcal{N}(\mu, \Sigma)$ and $\mathcal{N}(\mu', \Sigma')$, their total variation distance is bounded by

$$2 \text{TV}(\mathcal{N}(\mu, \Sigma), \mathcal{N}(\mu', \Sigma'))^2 \leq KL(\mathcal{N}(\mu, \Sigma) \parallel \mathcal{N}(\mu', \Sigma'))$$

$$= \frac{1}{2} \left( \text{LD}(\Sigma, \Sigma') + (\mu - \mu')^T \Sigma^{-1} (\mu - \mu') \right).$$

Proof. Follows from Lemma A.1 and Lemma A.2.

Lemma A.4. For any $\mu, \sigma, \hat{\mu}, \hat{\sigma} \in \mathbb{R}$ with $|\hat{\mu} - \mu| \leq \varepsilon \sigma$ and $|\hat{\sigma} - \sigma| \leq \varepsilon \sigma$ and $\varepsilon \in [0, 2/3]$ we have

$$\|\mathcal{N}(\mu, \sigma^2) - \mathcal{N}(\hat{\mu}, \hat{\sigma}^2)\|_1 \leq 2\varepsilon.$$

Proof. By Lemma A.3.

$$4 \text{TV}(\mathcal{N}(\mu, \sigma^2), \mathcal{N}((\hat{\mu}, \hat{\sigma}^2))^2 \leq \frac{\hat{\sigma}^2}{\sigma^2} - 1 - \log \left( \frac{\hat{\sigma}^2}{\sigma^2} \right) + \frac{|\mu - \hat{\mu}|^2}{\sigma^2} \leq \left( \frac{\hat{\sigma}}{\sigma} \right)^2 - 1 - \log \left( \frac{(\hat{\sigma})^2}{\sigma^2} \right) + \varepsilon^2.$$

Since $z := \hat{\sigma}/\sigma \in [1 - \varepsilon, 1 + \varepsilon]$ and $\varepsilon \leq 2/3$, using the inequality $x^2 - 1 - \log(x^2) \leq 3(x - 1)^2$ valid for all $|x - 1| \leq 2/3$, we find

$$\text{TV}(\mathcal{N}(\mu, \sigma^2), \mathcal{N}(\hat{\mu}, \hat{\sigma}^2))^2 \leq \frac{1}{4} (3(z - 1)^2 + \varepsilon^2) \leq \frac{1}{4} (4\varepsilon^2) = \varepsilon^2.$$
And the lemma follows since the $L_1$ distance is twice the TV distance. \hfill \Box

**Fact A.5.** Let $X$ and $Y$ be arbitrary random variables on the same space. For any function $f$, we have

$$\text{TV}(f(X), f(Y)) \leq \text{TV}(X, Y).$$

**Proof.** This follows from the observation that

$$\Pr[f(X) \in A] - \Pr[f(Y) \in A] = \Pr[X \in f^{-1}(A)] - \Pr[Y \in f^{-1}(A)] \leq \text{TV}(X, Y),$$

so taking supremum of the left-hand side gives the result. \hfill \Box

**Lemma A.6** (Laurent and Massart (2000, Lemma 1)). Let $X$ have the chi-squared distribution with parameter $d$; that is, $X = \sum_{i=1}^{d} X_i^2$ where the $X_i$ are i.i.d. standard normal. Then,

$$\Pr[X - d \geq 2\sqrt{d} t + 2t] \leq \exp(-t) \text{ and } \Pr[d - X \geq 2\sqrt{d} t] \leq \exp(-t).$$

The first inequality above implies, in particular, that $\Pr[X \geq 16d] \leq \exp(-3)$ for any $d$.

**Lemma A.7.** Let $g_1, \ldots, g_m \in \mathbb{R}^d$ be independent samples from $\mathcal{N}(0, I)$. For $\varepsilon \in [0, 1]$,

$$\Pr\left[\left\| \frac{1}{m} \sum_{i=1}^{m} g_i \right\|^2 \geq (1 + \varepsilon)d/m \right] \leq \exp(-\varepsilon^2 d/9).$$

**Proof.** Note that $X = \left\| \frac{1}{\sqrt{m}} \sum_{i=1}^{m} g_i \right\|^2$ has the chi-squared distribution with parameter $d$. Applying Lemma A.6 with $t = \varepsilon^2 d/9$ shows that $\Pr[X \geq (1 + \varepsilon)d] \leq \exp(-\varepsilon^2 d/9)$. \hfill \Box

**Lemma A.8** (Theorem 3.1.1 in Vershynin (2018)). Let $g \sim \mathcal{N}(0, I_d)$. Then $(\|g\|_2 - \sqrt{d})$ is $O(1)$-subgaussian. Consequently, $(\|g\|_2 - \sqrt{d})_+$ is also $O(1)$-subgaussian.

**Lemma A.9** (Proposition 2.5.2 in Vershynin (2018)). A random variable $X$ is $\sigma$-subgaussian if and only if $\sup_{p \geq 1} p^{-1/2} (\mathbb{E}|X|^p)^{1/p} \leq C\sigma$ for some global constant $C > 0$.

**Lemma A.10** (Hoeffding’s Inequality, Proposition 2.6.1 in Vershynin (2018)). Let $X_1, \ldots, X_n$ be independent, mean-zero random variables and suppose $X_i$ is $\sigma_i$-subgaussian. Then, for some global constant $c > 0$ and any $t \geq 0$,

$$\Pr\left[ \left| \sum_{i} X_i \right| > t \right] \leq 2 \exp\left( \frac{-ct^2}{\sum_i \sigma_i^2} \right).$$
Lemma A.11 (Bernstein’s Inequality, Theorem 2.8.1 in Vershynin (2018)). Let \( g_1, \ldots, g_n \sim \mathcal{N}(0, 1) \) and \( a_1, \ldots, a_n > 0 \). Then, there is a global constant \( c > 0 \) such that for every \( t \geq 0 \),

\[
\Pr \left[ \left| \sum_{i=1}^{n} a_i g_i^2 - \sum_{i=1}^{n} a_i \right| \geq t \right] \leq 2 \exp \left( -c \min \left\{ \frac{t^2}{\sum_{i=1}^{n} a_i^2}, \frac{t}{\max_i a_i} \right\} \right).
\]

Theorem A.12 (Gordon’s Theorem, Theorem 5.32 in Vershynin (2012)). Let \( G \) be a \( m \times n \) matrix with entries independently drawn from \( \mathcal{N}(0, 1) \). Then

\[
\mathbb{E} \sigma_{\min}(G) \geq \sqrt{m - \sqrt{n}}.
\]

Lemma A.13 (Corollary 5.50 and Remark 5.51 in Vershynin (2012)). Let \( X_1, \ldots, X_m \sim \mathcal{N}(0, \Sigma) \), where \( \Sigma \) is \( d \times d \), and let \( 0 < \varepsilon < 1 < t \). If \( m \geq C (t/\varepsilon)^2 d \), then with probability \( 1 - 2 \exp(-t^2 d) \) we have

\[
\left\| \frac{1}{m} \sum_{i=1}^{m} X_i X_i^T - \Sigma \right\| \leq \varepsilon \| \Sigma \|.
\]

Lemma A.14 (Corollary 4.2.13 in Vershynin (2018)). For any \( \varepsilon \in (0, 1) \), there exists an \( \varepsilon \)-net for \( B_d^2 \) of size \((3/\varepsilon)^d\).

### B Efficient algorithm for learning a single Gaussian by empirical mean/covariance estimation

In this section we give a simple algorithm for learning a single high dimensional Gaussian, with sample complexity \( O(d^2/\varepsilon^2) \) and computational complexity \( O(d^4/\varepsilon^2) \).

Lemma B.1. Let \( v_1, \ldots, v_m \in \mathbb{R}^d \) be independent samples from \( \mathcal{N}(\mu, \Sigma) \). Let \( \bar{v} = \frac{1}{m} \sum_{i=1}^{m} v_i \). Then

\[
\Pr[(\bar{v} - \mu)^T \Sigma^{-1}(\bar{v} - \mu) \geq 2d/m] \leq \exp(-d/9).
\]

Proof. Let \( g_i = \Sigma^{-1/2}(v_i - \mu) \), so that \( g_1, \ldots, g_m \) are independent samples from \( \mathcal{N}(0, I) \). Then

\[
\Pr[(\bar{v} - \mu)^T \Sigma^{-1}(\bar{v} - \mu) \geq (1 + \varepsilon)d/m] = \Pr[\left\| \frac{1}{m} \sum_{i=1}^{m} g_i \right\|^2 \geq (1 + \varepsilon)d/m] \leq \exp(-\varepsilon^2 d/9),
\]

by Lemma A.7.

We write \( B \preceq A \) if \( A - B \) is a positive semidefinite matrix. Observe that, \( x - 1 - \log x \leq (x - 1)^2 \) for any \( x \in [1/2, \infty) \).
Lemma B.2. Let $A, B$ be symmetric, positive definite matrices, satisfying $(1 - \alpha) B \preceq A \preceq (1 + \alpha) B$ for some $\alpha \in [0, 1/2]$. Then $\text{LD}(A, B) \leq d\alpha^2$.

Proof. Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of $B^{-1} A$. By the hypothesis, each $\lambda_i \in [1 - \alpha, 1 + \alpha]$. So,

$$\text{LD}(A, B) = \text{tr}(B^{-1} A - I) - \log \det(B^{-1} A) = \sum_{i=1}^d (\lambda_i - 1) - \log \prod_{i=1}^d \lambda_i$$

$$= \sum_{i=1}^d (\lambda_i - 1 - \log(\lambda_i)) \leq \sum_{i=1}^d (\lambda_i - 1)^2 \leq d\alpha^2.$$

Our result immediately follows from the following theorem.

Theorem B.3. Let $m = Cd^2/\varepsilon^2$ for a large enough constant $C$. Let $v_1, \ldots, v_m$ be i.i.d. samples from $\mathcal{N}(\mu, \Sigma)$. Let $\tilde{\mu} = \frac{1}{m} \sum_{i=1}^m v_i$ and $\tilde{\Sigma} = \frac{1}{m} \sum_{i=1}^m v_i v_i^T$ respectively be the empirical mean and the empirical covariance matrix. Then $\text{TV}(\mathcal{N}(\tilde{\mu}, \tilde{\Sigma}), \mathcal{N}(\mu, \Sigma)) \leq \varepsilon$ with probability at least $1 - 3 \exp(-d/9)$.

Proof. We will show that, with probability at least $1 - 3 \exp(-d/9)$, $\text{KL}(\mathcal{N}(\tilde{\mu}, \tilde{\Sigma}) \parallel \mathcal{N}(\mu, \Sigma)) \leq \varepsilon^2$, and the theorem follows from Pinsker’s inequality (Lemma A.2). By standard concentration for Gaussian matrices (see Lemma A.13) we have $\|\tilde{\Sigma} - \Sigma\| \leq \varepsilon/\sqrt{d} =: \alpha$ with probability at least $1 - 2 \exp(-d)$. That is, $(1 - \alpha) \Sigma \preceq \tilde{\Sigma} \preceq (1 + \alpha) \Sigma$. Applying Lemma B.2 shows that $\text{LD}(\tilde{\Sigma}, \Sigma) \leq d\alpha^2 = \varepsilon^2$. Next, by Lemma B.1 we have $(\tilde{\mu} - \mu)^T \Sigma^{-1} (\tilde{\mu} - \mu) \leq 2d/m = \varepsilon^2/18d$ with probability at least $1 - \exp(-d/9)$. So

$$\text{KL}(\mathcal{N}(\tilde{\mu}, \tilde{\Sigma}) \parallel \mathcal{N}(\mu, \Sigma)) = \frac{1}{2} \left( \text{LD}(\tilde{\Sigma}, \Sigma) + (\tilde{\mu} - \mu)^T \Sigma^{-1} (\tilde{\mu} - \mu) \right) \leq \varepsilon^2,$$

with probability at least $1 - 3 \exp(-d/9)$.

It is easy to see that, by multiplying the sample size by $\log(1/\delta)$, one can boost the success probability to $1 - \delta$, for any $\delta \in (0, 1)$.

C Omitted proofs from Section 3

C.1 Proof of Theorem 3.5

We give the proof for the agnostic case. The proof for the realizable case is similar. Let $q$ be the target distribution that the samples are being generated from. Let $\alpha = \inf_{f \in \mathcal{F}} \|f - q\|_1$
be the approximation error of \( q \) with respect to \( \mathcal{F} \). The goal of the learner is to find a distribution \( \hat{h} \) such that \( \| \hat{h} - q \|_1 \leq \max \{ 3, 2/r \} \cdot \alpha + \varepsilon \).

First, consider the case \( \alpha \leq r \). In this case, we develop a learner that finds a distribution \( h \) such that \( \| h - q \|_1 \leq 3\alpha + \varepsilon \). Let \( g \in \mathcal{F} \) be a distribution such that \( \| g - q \|_1 \leq \alpha + \frac{\varepsilon}{12} \) (such a \( g \) exists by the definition of \( \alpha \)). By assumption, \( \mathcal{F} \) admits \((\tau, t, m)\) compression. Let \( \mathcal{J} \) denote the corresponding decoder. Given \( \varepsilon \), the learner first asks for an i.i.d. sample \( S \sim q^{m(\varepsilon/6)\log(2/\delta)} \). By the definition of robust compression, we know that with probability at least \( 1 - \delta/2 \), there exist \( L \in S^{\tau(\varepsilon/6)} \) and \( B \in \{0,1\}^{t(\varepsilon/6)} \) such that \( \| \mathcal{J}(L,B) - g \| \leq \varepsilon/6 \) (see Remark 3.3). Let \( h^* := \mathcal{J}(L,B) \).

The learner is of course unaware of \( L \) and \( B \). However, given the sample \( S \), it can try all of the possibilities for \( L \) and \( B \) and create a candidate set of distributions. More concretely, let \( H = \{ \mathcal{J}(L,B) : L \in S^{\tau(\varepsilon/6)}, B \in \{0,1\}^{t(\varepsilon/6)} \} \). Note that
\[
|H| \leq (m(\varepsilon/6) \log(2/\delta))^{\tau(\varepsilon/6)2^{t(\varepsilon/6)}} \leq (m(\varepsilon/6) \log(2/\delta))^{\tau(\varepsilon)}.
\]
Since \( H \) is finite, we can use the algorithm of Theorem 3.4 to find a good candidate \( \hat{h} \) from \( H \). In particular, we set the accuracy parameter in Theorem 3.4 to be \( \varepsilon/16 \) and the confidence parameter to be \( \delta/2 \). In this case, Theorem 3.4 requires
\[
\frac{\log(6|H|^2/\delta)}{2(\varepsilon/16)^2} = O\left( \frac{\tau(\varepsilon) \log(m(\varepsilon/6) \log(1/\delta)) + \log(1/\delta)}{\varepsilon^2} \right) = \widetilde{O}(\tau(\varepsilon)/\varepsilon^2)
\]
additional samples, and its output \( \hat{h} \) will be an \((\varepsilon,3)\)-approximation of \( q \):
\[
\| \hat{h} - q \|_1 \leq 3\| h^* - q \|_1 + 4\frac{\varepsilon}{16} \leq 3(\| h^* - g \|_1 + \| g - q \|_1) + \frac{\varepsilon}{4} \leq 3(\varepsilon/6 + (\alpha + \varepsilon/12)) + \frac{\varepsilon}{4} \leq 3\alpha + \varepsilon.
\]
Note that the above procedure uses \( \widetilde{O}(m(\varepsilon/6) + \tau(\varepsilon)/\varepsilon^2) \) samples, and the probability of failure is at most \( \delta \) (i.e., the probability of either \( H \) not containing a good \( h^* \), or the failure of Theorem 3.4 in choosing a good candidate among \( H \), is bounded by \( \delta/2 + \delta/2 = \delta \)).

The other case, \( \alpha > r \), is trivial: the learner outputs some distribution \( \hat{h} \). Since \( \hat{h} \) and \( q \) are density functions, we have \( \| \hat{h} - q \|_1 \leq 2 < \frac{2}{r} \cdot \alpha < \max \{ 3, 2/r \} \cdot \alpha + \varepsilon \).

### C.2 Proof of Lemma 3.6

The following proposition is standard.

**Proposition C.1** (Lemma 3.3.7 in [Reiss 1989]). For \( i \in [d] \), let \( p_i \) and \( q_i \) be probability distributions over the same domain \( Z \). Then \( \| \Pi_{i=1}^d p_i - \Pi_{i=1}^d q_i \|_1 \leq \sum_{i=1}^d \| p_i - q_i \|_1 \).

---

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Proof of Lemma 3.6  Let $G = \Pi_{i=1}^{d} g_i$ be an arbitrary element of $\mathcal{F}^d$. Let $Q$ be an arbitrary distribution over $Z^d$, subject to $\|G - Q\|_1 \leq r$. Let $q_1, \ldots, q_d$ be the marginal distributions of $Q$ on the $d$ components. First, observe that, since projection onto a coordinate cannot increase the total variation distance (see Fact A.5), we have $\|q_j - g_j\|_1 \leq r$ for each $j \in [d]$.

We know that $\mathcal{F}$ admits $(\tau, t, m)$ $r$-robust compression. Call the corresponding decoder $\mathcal{J}$, and let $m_0 := m(\varepsilon/d) \log(3d)$, and $S \sim Q^m$. The goal is then to encode an $\varepsilon$-approximation of $G$ using $d\tau(\varepsilon/d)$ elements of $S$ and $dt(\varepsilon/d)$ bits.

Note that each element of $S$ is a $d$-dimensional vector. For each $i \in [d]$, let $S_i \in Z^{m_0}$ be the set of the $i$th components of elements of $S$. By definition of $q_i$, we have $S_i \sim q_i^m$ for each $i$. Thus, for each $i \in [d]$, since $\|q_i - g_i\| \leq r$, with probability at least $1 - 1/3d$ there exists a sequence $L_i$ of at most $\tau(\varepsilon/d)$ elements of $S_i$, and a sequence $B_i$ of at most $t(\varepsilon/d)$ bits, such that $\|\mathcal{J}(L_i, B_i) - g_i\|_1 \leq \varepsilon/d$. By the union bound, this assertion holds for all $i \in [d]$, with probability at least $2/3$. We may encode these $L_1, \ldots, L_d, B_1, \ldots, B_d$ using $d\tau(\varepsilon/d)$ elements of $S$ and $dt(\varepsilon/d)$ bits. Our decoder for $\mathcal{F}^d$ then extracts $L_1, \ldots, L_d, B_1, \ldots, B_d$ from these elements and bits, and then outputs $\Pi_{i=1}^{d} \mathcal{J}(L_i, B_i) \in \mathcal{F}^d$. Finally, Proposition C.1 gives $\|\Pi_{i=1}^{d} \mathcal{J}(L_i, B_i) - G\|_1 \leq \sum_{i=1}^{d} \|\mathcal{J}(L_i, B_i) - g_i\|_1 \leq d \times \varepsilon/d \leq \varepsilon$, completing the proof. 

C.3 Proof of Lemma 3.7

We will need the following standard proposition.

Proposition C.2. Let $g$ and $g^*$ be probability densities with $\|g - g^*\|_1 = \rho$ and $g^* = \sum_{i \in [k]} w_i f_i$, with $(w_1, \ldots, w_k) \in \Delta_k$ and where each $f_i$ is a density. Then we may write $g = \sum_{i \in [k]} w_i G_i$, such that each $G_i$ is a density, and for each $i$ we have $\|f_i - G_i\|_1 \leq \rho$.

Proof. Write

$$g = g^* + h = \sum_{i=1}^{k} w_i f_i + h = \sum_{i=1}^{k} w_i (f_i + h)$$

(1)

with $\|h\|_1 = \rho$. Note that $f_i + h$ is not necessarily a probability density function. Let $\mathcal{D}$ denote the set of probability density functions, that is, the set of nonnegative functions with unit $L_1$ norm. Note that this is a convex set. Since projection is a linear operator, by projecting both sides of (1) onto $\mathcal{D}$ we find $g = \sum_{i=1}^{k} w_i G_i$, where $G_i$ is the $L_1$ projection of $f_i + h$ onto $\mathcal{D}$ (since $g \in \mathcal{D}$, the projection of $g$ onto $\mathcal{D}$ is itself). Also, since $f_i \in \mathcal{D}$ and projection onto a convex set does not increases distances, we have

$$\|f_i - G_i\|_1 \leq \|f_i - (f_i + h)\|_1 = \|h\|_1 = \rho,$$

as required.

\[\square\]
Proof of Lemma 3.7. Let $g$ be the distribution from which we have $48m(\varepsilon/3)k\log(6k)/\varepsilon$ samples, and suppose $g^* \in k\text{-mix}(F)$ is the distribution to be compressed, so $\|g-g^*\|_1 \leq r$. Thus we have $g^* = \sum_{i \in [k]} w_i f_i$, with each $f_i \in F$, and $(w_1, \ldots, w_k) \in \Delta_k$. By Proposition 6.2 we also have $g = \sum_{i \in [k]} w_i G_i$ for some distributions $G_1, \ldots, G_k$, such that for each $i$ we have $\|f_i - G_i\|_1 \leq r$. We view $g$ as a mixture of these $k$ distributions, so the samples from $g$ can be partitioned into $k$ parts, so that samples from the $i$th part have distribution $G_i$. We compress each of the parts individually.

Moreover, we compress the mixing weights $w_1, \ldots, w_k$ using bits, as follows. Consider an $(\varepsilon/3k)$-net in $\ell_\infty$ for $\Delta_k$, of size $(1 + 3k/\varepsilon)^k$. Such a net can be obtained from a mesh of grid-size $\varepsilon/3k$ for $[0,1]^k$, and projecting each of its points onto $\Delta_k$. Let $(\hat{w}_1, \ldots, \hat{w}_k)$ be an element in the net that has

$$\|(\hat{w}_1, \ldots, \hat{w}_k) - (w_1, \ldots, w_k)\|_\infty \leq \varepsilon/3k;$$

then, $w_i - \hat{w}_i \leq \varepsilon/3k$ for all $i$. Moreover, the particular element $(\hat{w}_1, \ldots, \hat{w}_k)$ of the net can be encoded using $\log_2(1+3k/\varepsilon)^k) \leq k\log_2(4k/\varepsilon)$ bits.

For any $i \in [k]$, we say component $i$ is negligible if $w_i \leq \varepsilon/(6k)$. Since the total number of samples is $48m(\varepsilon/3)k\log(6k)/\varepsilon$, by a standard Chernoff bound combined with a union bound over the $k$ components, with probability at least $5/6$, for each non-negligible component $i$, we have at least $m(\varepsilon/3)\log(6k)$ samples from $i$. Let $i$ be a non-negligible component. Since $F$ admits $(\tau, t, m)$ robust compression and $f_i \in F$ and $\|f_i - G_i\|_1 \leq r$, with probability at least $1 - 1/6k$ there exists $\tau(\varepsilon/3)$ samples from part $i$ and $t(\varepsilon/3)$ bits, from which the decoder can construct a distribution $\hat{f}_i$ with $\|f_i - \hat{f}_i\|_1 \leq \varepsilon/3$. Using a union bound over the $k$ components, this is true uniformly over all non-negligible components, with probability at least $5/6$. (Note that, for negligible components $i$, there is no guarantee about $\hat{f}_i$.) Hence, given the mixing weights $\hat{w}_1, \ldots, \hat{w}_k$, the decoder outputs $\sum \hat{w}_i \hat{f}_i$.

The total number of instances used to encode is $k\tau(\varepsilon/3)$. Similarly, the total number of used bits is not more than $kt(\varepsilon/3) + k\log_2(4k/\varepsilon)$. Thus to complete the proof of the lemma, we need only show that $\|\sum w_i f_i - \sum \hat{w}_i \hat{f}_i\|_1 \leq \varepsilon$. Let $L \subseteq [k]$ denote the set of negligible components. We have

$$\left\| \sum_{i \in [k]} (\hat{w}_i \hat{f}_i - w_i f_i) \right\|_1 \leq \left\| \sum_{i \in [k]} w_i (\hat{f}_i - f_i) \right\|_1 + \left\| \sum_{i \in [k]} (\hat{w}_i - w_i) \hat{f}_i \right\|_1$$

$$\leq \left\| \sum_{i \in L} w_i (\hat{f}_i - f_i) \right\|_1 + \left\| \sum_{i \in L} w_i (\hat{f}_i - f_i) \right\|_1 + \left\| \sum_{i \in [k]} (\hat{w}_i - w_i) f_i \right\|_1$$

$$\leq 2 \sum_{i \in L} w_i + \sum_{i \in L} w_i (\varepsilon/3) + \sum_{i \in [k]} \varepsilon/3k \times 1 \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

completing the proof of the lemma.
D Omitted proofs from Section 4

D.1 Proof of Lemma 4.6

First we show the following proposition implies \( \frac{1}{10} B^d_2 \subseteq \operatorname{conv}(T) \):

\[
\max_{q \in S} |\langle y, q \rangle| \geq \frac{1}{20} \quad \forall y \in S^{d-1}. \tag{2}
\]

For, let \( P := \operatorname{conv}(T) \). Its polar is \( P^o = \{ y \in \mathbb{R}^d : |\langle y, q \rangle| \leq 1 \ \forall q \in T \} \). So (2) implies \( P^o \subseteq 20B^d_2 \). As polarity reverses containment, we obtain \( P \supseteq (20B^d_2)^o = (\frac{1}{20}) B^d_2 \).

We now bound the probability that (2) fails. For this, let \( g \sim \mathcal{N}(0, I_d) \) and let \( X_y := \langle y, g \rangle \). Notice that \( X_y \sim \mathcal{N}(0, 1) \). Since the pdf of \( X_y \) is bounded above by 1, we have \( \Pr [ |X_y| \leq \frac{1}{10} ] \leq 1/5 \). Moreover, by Lemma 4.6, the probability that \( \|g\|_2 \geq 4\sqrt{d} \) is \( \leq \exp(-3) \). Hence

\[
\Pr \left[ |X_y| \leq \frac{1}{10} \lor \|g\|_2 \geq 4\sqrt{d} \right] \leq 1/5 + \exp(-3) < 0.25.
\]

Now let \( Y_{y,i} := \langle y, q_i \rangle \) and let \( E_{y,i} \) be the event \( \{ |Y_{y,i}| \leq \frac{1}{10} \lor \|q_i\| > 4\sqrt{d} \} \). As \( \operatorname{TV}(Q, \mathcal{N}(0, I_d)) \leq 2/3 \), we have \( \Pr [E_{y,i}] \leq 0.25 + 2/3 < 0.92 \). Thus

\[
\Pr \left[ \bigwedge_{i \in [m]} E_{y,i} \right] < (0.92)^m
\]

Let \( N \) be an \((1/80\sqrt{d})\)-net of \( S^{d-1} \) with \( |N| \leq (240\sqrt{d})^d \). By a union bound, since \( m \geq Cd(1+\log d) \) for \( C \) large enough, with probability at least \( 1-(240\sqrt{d})^d(0.92)^m \geq 5/6 \), for all \( y \in N \) there exists \( i \in [m] \) such that \( |Y_{y,i}| \geq \frac{1}{10} \) and \( \|q_i\| \leq 4\sqrt{d} \).

Suppose this event holds. Let \( y \in S^{d-1} \), and let \( y' \in N \) satisfy \( \|y - y'\|_2 \leq 1/80\sqrt{d} \). Let \( q_i \) be such that \( \|q_i\| \leq 4\sqrt{d} \) and \( |Y_{y',i}| \geq \frac{1}{10} \). These imply \( \pm q_i \in T \) and

\[
|Y_{y,i}| \geq |Y_{y',i}| - \|q_i\|/80\sqrt{d} \geq 1/10 - 1/20 = \frac{1}{20},
\]

as required.

D.2 Proof of Lemma 4.7

Let \( X_1, \ldots, X_{2m} \) be the samples, and let \( Y_i := \frac{1}{\sqrt{2}} \Psi^{-1}(X_{2i} - X_{2i-1}) \) for \( i \in [m] \). Observe that if \( X_{2i} \) and \( X_{2i-1} \) were \( \mathcal{N}(\mu, \Sigma) \), then \( \frac{1}{\sqrt{2}} \Psi^{-1}(X_{2i} - X_{2i-1}) \) would have been \( \mathcal{N}(0, I) \). Since \( X_{2i} \) and \( X_{2i-1} \) have TV distance at most 1/3 from \( \mathcal{N}(\mu, \Sigma) \), \( Y_i \) has TV distance at most 2/3 from \( \mathcal{N}(0, I) \) (this can be seen, e.g., by the coupling characterization of the TV
distance). Let \( I := \{ i \in [m] : \|Y_i\| \leq 4\sqrt{d} \} \). By Lemma 4.6 with probability \( \geq 5/6 \) we have

\[
\frac{1}{C} B_2^d \subseteq \text{conv}\{ \pm Y_i : i \in I \}
\]

with \( C = 20 \). We give the encoding for \( \hat{v}_j \) conditioned on this event.

Fix \( j \in [d] \). Observe that \( \Psi^{-1} v_j = v_j/\|v_j\| \) has unit norm, so we can write

\[
\Psi^{-1} v_j / C = \sum_{i \in [m]} \theta_{j,i} Y_i
\]

for some vector \( \theta_j \in [-1, 1]^m \) supported on \( I \). Applying \( \Psi \) to both sides, we obtain

\[
v_j = \frac{C}{\sqrt{2}} \sum_{i \in [m]} \theta_{j,i} (X_{2i} - X_{2i-1}).
\]

For encoding \( \hat{v}_j \), consider an \( (\varepsilon/24Cmd^3) \)-net for \([-1, 1]^m \) in \( \ell_\infty \) distance. The size of the net is \( (48Cmd^3/\varepsilon)^m \), so any element of the net can be described using \( O(m \log(d/\varepsilon)) \) bits. Let \( \hat{\theta}_j \) be an element in the net that is closest to \( \theta_j \) subject to \( \text{supp}(\hat{\theta}_j) \subseteq I \), and let \( \hat{v}_j := \frac{C}{\sqrt{2}} \sum_{i \in [m]} \hat{\theta}_{j,i} (X_{2i} - X_{2i-1}) \). We have,

\[
\|\Psi^{-1}(\hat{v}_j - v_j)\| = \frac{C}{\sqrt{2}} \| \sum_{i=1}^m (\theta_{j,i} - \hat{\theta}_{j,i}) \Psi^{-1}(X_{2i} - X_{2i-1}) \|
\leq \frac{C}{\sqrt{2}} m \max_i |\theta_{j,i} - \hat{\theta}_{j,i}| \| \Psi^{-1}(X_{2i} - X_{2i-1}) \|
\leq \frac{C}{\sqrt{2}} m (\varepsilon/24Cmd^3)(4\sqrt{2}\sqrt{d}) \leq \varepsilon/6d^2,
\]

as required. The total number of bits used to encode each \( \hat{v}_j \) is \( O(m \log(d/\varepsilon)) \), giving a total number of \( O(d^2 \log(2d) \log(d/\varepsilon)) \) bits for encoding them all.

We next describe the encoding of \( \hat{\mu} \). Let \( Z_i := \Psi^{-1}(X_i - \mu) \) and observe that \( Z_i \) has a distribution with TV distance at most 2/3 to \( \mathcal{N}(0, I) \). So, using Lemma A.6

\[
\Pr[\|Z_i\| \geq 4\sqrt{d}] \leq \exp(-3) + 1/3 < 1/\sqrt{1/6},
\]

which means, with probability at least 5/6, \( \min\{\|Z_1\|, \|Z_2\|\} \leq 4\sqrt{d} \). We give an encoding for \( \hat{\mu} \) provided this event occurs.

Without loss of generality, assume \( \|Z_1\| \leq 4\sqrt{d} \), and suppose \( Z_1 = \sum_{j \in [d]} \lambda_j e_j \), \( \{e_i\} \) being the standard basis. This implies \( \mu = X_1 - \sum_{j \in [d]} \lambda_j v_j \), with \( \sum \lambda_j^2 \leq 16d^2 \). Consider an \( (\varepsilon/3d) \)-net for \( 4\sqrt{d} B_2^d \) of size \( (36d^2/\varepsilon)^d \), and let \( \hat{\lambda} \) be the closest element to \( \lambda \) in this net. We encode \( \hat{\mu} = X_1 - \sum_{j \in [d]} \lambda_j \hat{v}_j \). Note that this requires \( O(d \log(d/\varepsilon)) \) more bits, which is dominated by the number of bits for encoding the \( \hat{v}_j \).
Finally, observe that,
\[
\| \Psi^{-1}(\hat{\mu} - \mu) \| = \| \sum_j (\hat{\lambda}_j - \lambda_j) \Psi^{-1}(v_j - \hat{v}_j) \| \\
\leq \sum_j \| \hat{\lambda}_j (\Psi^{-1}v_j - \Psi^{-1}\hat{v}_j) + (\lambda_j - \hat{\lambda}_j) \Psi^{-1}v_j \| \\
\leq d \max_j \left\{ |\hat{\lambda}_j| \| \Psi^{-1}v_j - \Psi^{-1}\hat{v}_j \| + |\lambda_j - \hat{\lambda}_j| \| \Psi^{-1}v_j \| \right\} \\
\leq d \cdot 4\sqrt{d} \cdot \frac{\varepsilon}{6d^2} + d(\varepsilon/3d) \leq \varepsilon,
\]
as required.

### D.3 Proof of Lemma 4.8

Let \( \hat{\Sigma} := \sum_i \hat{v}_i\hat{v}_i^T \). We will show that
\[
LD(\Sigma, \hat{\Sigma}) \leq 9d^3 \rho^2 \tag{3}
\]
If this is true, Lemma [A.3] gives
\[
TV(\mathcal{N}(\mu, \Sigma), \mathcal{N}(\hat{\mu}, \hat{\Sigma})) \leq \frac{1}{4} \left( LD(\Sigma, \hat{\Sigma}) + (\mu - \hat{\mu})^T \Sigma^{-1}(\mu - \hat{\mu}) \right) \leq \frac{1}{4} (9d^3 \rho^2 + \zeta^2),
\]
completing the proof. For showing (3), note that we have
\[
LD(\Sigma, \hat{\Sigma}) = LD(\Psi^{-1} \Sigma \Psi^{-1}, \Psi^{-1} \hat{\Sigma} \Psi^{-1}) \\
= LD\left( \sum_i \Psi^{-1}v_i v_i^T \Psi^{-1}, \sum_i \Psi^{-1}\hat{v}_i \hat{v}_i^T \Psi^{-1} \right) = LD(I, \sum_i \Psi^{-1}\hat{v}_i \hat{v}_i^T \Psi^{-1})
\]
For the first equality we have used the fact that, if \( A \) and \( B \) are positive definite and \( C \) is invertible, then \( LD(A, B) = LD(CAC, CBC) \). Indeed, as we proved in Lemma [B.2], \( LD(A, B) \) only depends on the spectrum of \( B^{-1}A \). Observe that \((v, \lambda)\) is an eigenvector/eigenvalue pair for \( B^{-1}A \) if and only if \((C^{-1}v, \lambda)\) is an eigenvector/eigenvalue pair for \((CBC)^{-1}CAC\); hence these two matrices have the same spectrum.

Let \( B := \sum_i \Psi^{-1}\hat{v}_i \hat{v}_i^T \Psi^{-1} \). We shall show \( \|B - I\| \leq 3d\rho \), which implies \(-3d\rho I \preceq B - I \preceq 3d\rho I\) and together with Lemma [B.2] implies \( LD(\Sigma, \hat{\Sigma}) = LD(I, B) \leq 9d^3 \rho^2 \). We have
\[
\|B - I\| = \| \sum_i (\Psi^{-1}\hat{v}_i \hat{v}_i^T \Psi^{-1} - \Psi^{-1}v_i v_i^T \Psi^{-1}) \| = \| \sum_i \Psi^{-1}\hat{v}_i \hat{v}_i^T \Psi^{-1} - \Psi^{-1}v_i v_i^T \Psi^{-1} \|
\]
with \( x_i := \Psi^{-1}\hat{v}_i \) and \( y_i := \Psi^{-1}v_i \) (here we have used the fact that \( \Psi^{-1} \) is symmetric). Note that \( \| y_i \| = \| \Psi^{-1}v_i \| = 1 \), and that, by the lemma hypothesis, \( \| x_i - y_i \| \leq \rho \). Applying Lemma [D.1] below concludes the proof.

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Lemma D.1. Suppose \( x, y \) satisfy \( \|y\| = 1 \) and \( \|x - y\| \leq \varepsilon \leq 1 \). Then we have \( \|xx^T - yy^T\| \leq 3\varepsilon \).

Proof. Suppose \( x = y + z \) with \( \|z\| \leq \varepsilon \). Then,

\[
\|xx^T - yy^T\| = \|yz^T + zy^T + zz^T\| \leq \|yz^T\| + \|zy^T\| + \|zz^T\| \leq \varepsilon + \varepsilon + \varepsilon^2 \leq 3\varepsilon.
\]

\( \blacksquare \)

D.4 Proof of Remark 4.4

Recall that \( \mathcal{N}(\mu, \sigma^2) \) denotes a 1-dimensional Gaussian distribution with mean \( \mu \) and standard deviation \( \sigma \). We will need a lemma bounding the \( L_1 \) distance of two Gaussians in terms of their parameters.

Any vector \( (p_1, \ldots, p_n) \in \Delta_n \) induces a discrete probability distribution over \([n]\) defined by \( \Pr(i) := p_i \). Let \( x \vee y := \max\{x, y\} \).

Lemma D.2. Let \( (p_1, \ldots, p_{2n+1}) \in \Delta_{2n+1} \) and \( (q_1, \ldots, q_{2n+1}) \in \Delta_{2n+1} \) be discrete probability distributions with \( \ell_1 \) distance between them \( \leq t \). Suppose we have \( 2n + 1 \) bins, numbered 1 to \( 2n + 1 \). We throw \( m \) balls into these bins, where each ball chooses a bin independently according to \( q_i \). We pair bin 1 with bin 2, bin 3 with bin 4, \ldots, and bin \( 2n - 1 \) with bin \( 2n \); so bin \( 2n + 1 \) is unpaired. The probability that, for all pairs of bins, at most one them gets a ball, is not more than

\[
2^n \left( \frac{t}{2} + \frac{p_{2n+1}}{2} + \sum_{i=1}^n \max\{p_{2i-1}, p_{2i}\} \right)^m
\]

Proof. Let \( P_1 = \{1, 2\}, P_2 = \{3, 4\}, \ldots, P_n = \{2n - 1, 2n\} \), and let \( \mathcal{A} := \{A \subset [2n] : |A \cap P_i| = 1 \forall i \in [n]\} \). Clearly \( |\mathcal{A}| = 2^n \). For any \( A \in \mathcal{A} \), let \( E_A \) be the event that, the first ball does not choose a bin in \( A \), and let \( F_A \) be the event that, none of the balls choose a bin in \( A \). Then,

\[
\Pr[E_A] = \sum_{i \in [2n+1] \setminus A} q_i \leq \text{TV}(p, q) + \sum_{i \in [2n+1] \setminus A} p_i \leq \frac{t}{2} + \sum_{i \notin A} p_i
\]

\[
\leq \frac{t}{2} + p_{2n+1} + \sum_{i=1}^n \max\{p_{2i-1}, p_{2i}\},
\]

and so \( \Pr[F_A] = \Pr[E_A]^m \leq \left( \frac{t}{2} + p_{2n+1} + \sum_{i=1}^n (p_{2i-1} \vee p_{2i}) \right)^m \). Finally, observe that, if for each pair of bins, at most one them gets a ball, then there exists at least one \( A \in \mathcal{A} \), such that none of the balls chooses a bin in \( A \). The lemma is thus proved by applying the union bound over all events \( \{F_A\}_{A \in \mathcal{A}} \).

\( \blacksquare \)
Theorem D.3. The class of all Gaussian distributions over the real line admits \((4, 1, O(1/\varepsilon))\) 0.773-robust compression.

Proof. Let \(q\) be any distribution (not necessarily a Gaussian) such that there exists a Gaussian \(g = \mathcal{N}(\mu, \sigma^2)\) with \(\|q - g\|_1 \leq r \leq 0.773\). Our goal is to encode \(q\) using samples generated from \(q\). Let \(m = C/\varepsilon\) for a large enough constant \(C\) to be determined, and let \(S \sim q^m\) be an i.i.d. sample. The goal is to approximately encode \(\mu\) and \(\sigma\) using only four elements of \(S\) and a single bit.

We start by defining the decoder \(\mathcal{J}\). Our proposed decoder takes as input four points \(x_1, x_2, y_1, y_2 \in \mathbb{R}\), and one bit \(b \in \{0, 1\}\). The decoder then outputs a Gaussian distribution based on the following rule:

\[
\mathcal{J}(x_1, x_2, y_1, y_2, b) =
\begin{cases} 
\mathcal{N}(\frac{x_1 + x_2}{2}, \frac{|y_1 - y_2|^2}{4} : & \text{if } b = 1 \\
\mathcal{N}(\frac{x_1 + x_2}{2}, |y_1 - y_2|^2) : & \text{if } b = 0
\end{cases}
\]

Our goal is thus to show that, with probability at least 2/3, there exists \(x_1, x_2, y_1, y_2 \in S\) and \(b \in \{0, 1\}\) such that \(\|\mathcal{J}(x_1, x_2, y_1, y_2, b) - g\|_1 \leq \varepsilon\).

Let \(M = 1/\varepsilon\) and partition the interval \([\mu - 2\sigma, \mu + 2\sigma]\) into \(4M\) subintervals of length \(\varepsilon\sigma\). Enumerate these intervals as \(I_1 \to I_{4M}\), i.e., \(I_i = [\mu - 2\sigma + (i - 1)(\varepsilon\sigma), \mu - 2\sigma + i(\varepsilon\sigma)]\). Also let \(I_{4M+1} = \mathbb{R} \setminus \bigcup_{i=1}^{4M} I_i\). We state two claims which will imply the theorem, and will be proved later.

Claim 1. With probability at least 5/6, there exist \(y_1, y_2 \in S\) such that at least one of the following two conditions holds: (a) \(y_1 \in I_i\) and \(y_2 \in I_{i+M}\) for some \(i \in \{M + 1, 2M + 2, ..., 2M\}\). In this case, we let \(b = 0\), and so \(\mathcal{J}(x_1, x_2, y_1, y_2, b)\) will have standard deviation \(|y_1 - y_2|\).

(b) \(y_1 \in I_i\) and \(y_2 \in I_{i+3M}\) for some \(i \in [M]\). In this case, we let \(b = 1\), and so \(\mathcal{J}(x_1, x_2, y_1, y_2, b)\) will have standard deviation \(|y_1 - y_2|/3\).

Also, if both of (a) and (b) happen, we will go with the first rule. Note that if Claim 1 holds, and \(\hat{\sigma}\) is the standard deviation of \(\mathcal{J}(x_1, x_2, y_1, y_2, b)\), then we will have \(|\hat{\sigma} - \sigma| \leq \varepsilon\).

Claim 2. With probability at least 5/6, there exist \(x_1, x_2 \in S\) such that \(x_1 \in I_i\) and \(x_2 \in I_{4M-i+1}\) for some \(i \in [2M]\). If so, \(\mathcal{J}(x_1, x_2, y_1, y_2, b)\) will have mean \(\frac{x_1 + x_2}{2} = \hat{\mu}\).

Also note that if Claim 2 holds, then \(|\hat{\mu} - \mu| \leq \varepsilon\). Therefore, if both claims hold, Lemma A.4 gives that \(\mathcal{J}(x_1, x_2, y_1, y_2, b) = \mathcal{N}(\hat{\mu}, \hat{\sigma}^2)\) is a \(2\varepsilon\)-approximation for \(\mathcal{N}(\mu, \sigma^2) = g\). In other words, \(g\) can be approximately reconstructed, up to error \(2\varepsilon\), using only four data points (i.e., \(\{x_1, x_2, y_1, y_2\}\)) from a sample \(S\) of size \(O(1/\varepsilon)\) and a single bit \(b\) (the definition of robust compression requires error \(\leq \varepsilon\). For getting this, one just needs to refine the partition by a constant factor, which multiplies \(M\) by a constant factor, and as we will see below, this will only multiply \(m\) by a constant factor). Note also that the
probability of existence of such four points is at least $1 - (1 - 5/6) - (1 - 5/6) \geq 2/3$. Therefore, it remains to prove Claim 1 and Claim 2.

We start with Claim 1. View the sets $I_1, \ldots, I_{4M}, I_{4M+1}$ as bins, and consider the i.i.d. samples as balls landing in these bins according to $q$. Let $p_i := \int_{I_i} g(x)dx$ and $q_i := \int_{I_i} q(x)dx$ for $i \in [4M + 1]$. Note that, by triangle’s inequality, the $\ell_1$ distance between $(p_1, \ldots, p_{4M+1})$ and $(q_1, \ldots, q_{4M+1})$ is not more than the $L_1$ distance between $g$ and $q$, which is at most $r$.

We pair the bins as follows: $I_i$ is paired with $I_{i+M}$ for $i \in \{M + 1, \ldots, 2M\}$, and $I_i$ is paired with $I_{i+3M}$ for $i \in \{M\}$. Therefore, by Lemma D.2, the probability that Claim 1 does not hold can be bounded by

\[
2^{2M} \left( \sum_{i=M+1}^{2M} (p_i \lor p_{i+M}) + \sum_{i=1}^{M} (p_i \lor p_{i+3M}) + p_{4M+1} + \frac{r}{2} \right)^m
\]

\[
= 2^{2M} \left( \sum_{i=\frac{M}{2}+1}^{\frac{5M}{2}} p_i + \sum_{i=M+1}^{\frac{5M}{2}} p_i + \sum_{i=3M+1}^{\frac{7M}{2}} p_i + p_{4M+1} + \frac{r}{2} \right)^m,
\]

where we have used the fact that $p_i$ are coming from a Gaussian, and thus $p_1 \leq \cdots \leq p_{2M} = p_{2M+1} \geq \cdots \geq p_{4M}$ (we have also assumed, for simplicity, that $M$ is even). Let $X \sim N(\mu, \sigma^2)$ and $\Phi(A) := \Pr[N(0,1) \in A]$. Then using known numerical bounds for $\Phi$, we obtain

\[
\sum_{i=\frac{M}{2}+1}^{\frac{5M}{2}} p_i + \sum_{i=M+1}^{\frac{5M}{2}} p_i + \sum_{i=3M+1}^{\frac{7M}{2}} p_i + p_{4M+1} + \frac{r}{2}
\]

\[
= \Pr[X \in [\mu - \sigma/2, \mu + \sigma/2]] + 2\Pr[X \in [\mu - 3\sigma/2, \mu - \sigma]] + \Pr[X \not\in [\mu - 2\sigma, \mu + 2\sigma]] + \frac{r}{2}
\]

\[
= \Phi([-0.5, 0.5]) + 2\Phi([-1.5, -1]) + 2\Phi((-\infty, -2]) + \frac{r}{2} < 0.383 + 0.184 + 0.046 + \frac{r}{2}
\]

\[
= 0.613 + \frac{r}{2} \leq 0.9995.
\]

Therefore since $M = \Theta(1/\varepsilon)$, by making $m = C/\varepsilon$ for a large enough $C$, we can make this probability arbitrarily small, completing the proof of Claim 1.

Via a similar argument, the probability that Claim 2 does not hold can be bounded by

\[
2^{2M} \left( \sum_{i=1}^{2M} \max\{p_i, p_{4M-i+1}\} + p_{4M+1} + r/2 \right)^m = 2^{2M} \left( \sum_{i=1}^{2M} p_i + p_{4M+1} + r/2 \right)^m
\]

\[
= 2^{2M} (\Phi([-1, 1]) + \Phi([2, \infty] + r/2)^m < 2^{2M} (0.5 + 0.023 + r/2)^m < 2^{2M} (0.91)^m < 1/6,
\]

for $m = C/\varepsilon$ with a large enough $C$.  \hfill \Box
Remark D.4. By using more bits and adding more scales, one can show that 1-dimensional Gaussians admit \((4, b(r), O(1/\varepsilon))\) \(r\)-robust compression for any fixed \(r < 1\) (the number of required bits and the implicit constant in the \(O\) will depend on the value of \(r\)).

E Omitted proofs from Section 5

E.1 Proof of Lemma 5.1

The proof of the following lemma, which is called the ‘generalized Fano’s inequality,’ uses Fano’s inequality in information theory (Cover and Thomas, 2006, Theorem 2.10.1). It was first proved in (Devroye, 1987, page 77). We write here a slightly stronger version, which appears in (Yu, 1997, Lemma 3).

Lemma E.1 (generalized Fano’s inequality). Suppose we have \(M > 1\) distributions \(f_1, \ldots, f_M\) with

\[
\text{KL}(f_i \parallel f_j) \leq \beta \quad \text{and} \quad \|f_i - f_j\|_1 > \alpha \quad \forall i \neq j \in [M].
\]

Consider any density estimation method that gets \(n\) i.i.d. samples from some \(f_i\), and outputs an estimate \(\hat{f}\) (the method does not know \(i\)). For each \(i\), define \(e_i\) as follows: assume the method receives samples from \(f_i\), and outputs \(\hat{f}\). Then \(e_i := \mathbf{E}\|f_i - \hat{f}\|_1\). Then, we have

\[
\max_i e_i \geq \alpha (\log M - n\beta + \log 2)/(2 \log M).
\]

To prove Lemma 5.1, consider a distribution learning method for learning \(\mathcal{F}\) with sample complexity \(m(\varepsilon)\), and consider \(M\) distributions \(f_1, \ldots, f_M\) satisfying the hypotheses. Suppose we are in the setup of generalized Fano’s inequality: we get samples from some unknown \(j \in [M]\), and we are to find which \(f_j\) are the samples coming from. When \(m(\varepsilon)\) samples are given to the method, with probability \(\geq 2/3\) it outputs some \(g\) within distance \(\varepsilon\) to \(f_j\). Suppose we repeat this procedure for \(k\) times, and the method outputs \(k\) distributions. If more than half of the times the method’s output was \(\varepsilon\)-close to some \(f_i\), then we output that \(f_i\) as the answer; otherwise we output \(f_1\). Our error would be 0 with probability \(\Pr[\text{Bin}(k, 2/3) > k/2]\), and at most 2 with the remaining probability. Thus, the expected error can be upper bounded by \(\exp(-\Omega(k))\) by the Chernoff bound. Thus, generalized Fano’s inequality gives

\[
\alpha (\log M - (km(\varepsilon))\kappa(\varepsilon) + \log 2)/(2 \log M) \leq \exp(-\Omega(k)).
\]

Choosing \(k = \Theta(\log(1/\varepsilon))\) and rearranging gives \(m(\varepsilon) = \Omega(\log M/\kappa(\varepsilon) \log(1/\varepsilon))\), as required.
E.2 Proof of Lemma 5.3

We use the probabilistic method. We let the $d/r$ columns of each $U_a$ to be the first $d/r$ columns of a uniformly random orthonormal basis of $\mathbb{R}^d$. To complete the proof, we need only show that for two such random matrices $U_a$ and $U_b$, with probability $1 - 2^{-\Omega(d^2/r)}$ we have $\|U_a^T U_b\|_F^2 \leq \frac{d}{2r}$.

In the following, $U \overset{d}{=} V$ means $U$ and $V$ have the same distribution. By rotation invariance, we may assume $U_a = [e_1, \ldots, e_{d/r}]$, so that $\|U_a^T U_b\|_F^2 \overset{d}{=} \|U_{d/r}\|_F^2$, where $U_{d/r}$ is the $d/r \times d/r$ principal submatrix of a uniformly random orthogonal matrix $U$ (alternatively, the columns of $U_{d/r}$ are the first $d/r$ coordinates of $d/r$ orthonormal vectors in $\mathbb{R}^d$ chosen uniformly at random). Hence, it suffices to show that $\|U_{d/r}\|_F^2 \leq d/(2r)$ with probability at least $1 - 2^{-\Omega(d^2/r)}$. We will do this indirectly by relating $U_{d/r}$ to a matrix with independent Gaussian entries.

To that end, let $G$ be a $d \times d/r$ matrix with i.i.d. $\mathcal{N}(0, 1/d)$ entries. Let $G = U_G \Sigma_G V_G^T$ be its SVD, where $U_G \in \mathbb{R}^{d \times d/r}$ and $\Sigma_G, V_G \in \mathbb{R}^{d/r \times d/r}$. Observe that, by rotation invariance of the Gaussian matrix $G$, the columns of $U_G$ are $d/r$ uniformly random orthonormal vectors and hence, the top $d/r$ rows of $U_G$ have the same distribution as $U_{d/r}$. Moreover, by rotation invariance again, $\Sigma_G$ is independent of $U_G$.

Now let $G_{d/r}$ be the first $d/r$ rows of $G$. Then,

$$\|G_{d/r}\| = (U_G)_{d/r} \Sigma_G V_G \overset{d}{=} U_{d/r} \Sigma_G V_G,$$

and, taking Frobenius norms of both sides,

$$\|G_{d/r}\|_F \overset{d}{=} \|U_{d/r} \Sigma_G V_G\|_F = \|U_{d/r} \Sigma G\|_F \geq \sigma_{\min}(\Sigma_G) \|U_{d/r}\|_F,$$  \hspace{1cm} (4)

with $\sigma_{\min}(\Sigma_G)$ being the smallest singular value of $\Sigma_G$. Since $\|G_{d/r}\|_F^2$ is a sum of i.i.d. random variables and concentrates sharply around its mean, and $\mathbb{E} \sigma_{\min}(\Sigma_G) \geq 1 - 1/\sqrt{r}$ by Gordon’s Theorem (Theorem A.12), this allows us to control $\|U_{d/r}\|_F$. In particular, for any $p \geq 1$, we can bound a suitably translated moment of $\|U_{d/r}\|_F$. Let $(x)_+ := \max\{0, x\}$. Then, from (4) we get

$$\mathbb{E}_G(\|G_{d/r}\|_F - \sqrt{d/r})_+^p \geq \mathbb{E}_{U_{d/r}, \Sigma_G}(\sigma_{\min}(\Sigma_G)) \|U_{d/r}\|_F - \sqrt{d/r})_+^p$$

$$= \mathbb{E}_{U_{d/r}} \mathbb{E}_{\Sigma_G}(\sigma_{\min}(\Sigma_G)) \|U_{d/r}\|_F - \sqrt{d/r})_+^p$$

$$\geq \mathbb{E}_{U_{d/r}} (\mathbb{E}_{\Sigma_G}(\sigma_{\min}(\Sigma_G)) \|U_{d/r}\|_F - \sqrt{d/r})_+^p$$

$$\geq \mathbb{E}_{U_{d/r}} ((1 - 1/\sqrt{r}) \|U_{d/r}\|_F - \sqrt{d/r})_+^p,$$

where the second inequality is Jensen’s inequality and the third inequality is Gordon’s Theorem. Lemma A.6 gives that $(\|G_{d/r}\|_F - \sqrt{d/r})$ is $O(1/\sqrt{d})$-subgaussian, and since the moments of $(1 - 1/\sqrt{r}) \|U_{d/r}\|_F - \sqrt{d/r})_+$ are bounded by the moments of this random
variable, by equivalence of having moment bounds and being subgaussian (Lemma A.9), we find that 
\[
(1 - \sqrt{1/r})\|U_{d/r}\|_F - \sqrt{d/r} \leq t
\]

is also \(O(1/\sqrt{d})\)-subgaussian. Hence, for any \(t > 0\), we have 
\[
\Pr \left[ (1 - \sqrt{1/r})\|U_{d/r}\|_F - \sqrt{d/r} \leq t \right] \geq 1 - 2^{-\Omega(t^2d)}.
\]

Choosing \(t = \sqrt{d/12r}\) and the assumption that \(r \geq 9\) gives \(\|U_{d/r}\|_F^2 \leq d/(2r)\) with probability at least \(1 - 2^{-\Omega(d^2/r)}\), completing the proof.

### E.3 Proof of Lemma 5.4

Assume that \(\|U_a^T U_b\|_F^2 \leq d/(2r)\). Our goal is to show that \(TV(f_a, f_b) = \Omega \left( \frac{\lambda \sqrt{d/r}}{\log(r/\lambda \sqrt{d})} \right)\).

Recall that \(f_a = \mathcal{N}(0, \Sigma_a)\) with \(\Sigma_a = I_d + \lambda U_a U_a^T\). Let \(g \sim \mathcal{N}(0, I_d)\). Then \(\sqrt{\Sigma_a g} \sim f_a\). Thus our goal is to lower bound \(TV(\sqrt{\Sigma_a g}, \sqrt{\Sigma_a g})\). Since total variation distance never increases under a mapping (Fact A.5), we need only show that

\[
TV(U_a U_a^T \sqrt{\Sigma_a g}, U_a U_a^T \sqrt{\Sigma_a g}) = \Omega \left( \frac{\lambda \sqrt{d/r}}{\log(r/\lambda \sqrt{d})} \right).
\]

Now let \(C := U_a U_a^T \sqrt{\Sigma_a g}\) and since \(C\) is symmetric, its SVD has form \(C = \sum_{i \in [d/r]} \sigma_i w_i w_i^T\) for orthonormal \(\{w_i\}\) (some \(\sigma_i\) may be zero). Let \(S\) denote the column space of \(U_a\); so \(w_1, \ldots, w_{d/r} \in S\), and we may assume \(w_1, \ldots, w_{d/r}\) is an orthonormal basis for \(S\). This means \(U_a U_a^T \sqrt{\Sigma_a g} = \sum_{i \in [d/r]} \sigma_i g_i w_i\), where the \(g_i\) are the components of \(g\). Note that

\[
\sum_{i \in [d/r]} \sigma_i^2 = \|C\|_F^2 = \text{Tr}(C C^T) = \text{Tr}(U_a U_a^T (1 + \lambda U_b U_b^T) U_a U_a^T) \]
\[
= \text{Tr}(U_a U_a^T) + \lambda \text{Tr}(U_a U_a^T U_b U_b^T U_a U_a^T) = \frac{d}{r} + \lambda \text{Tr}(U_a U_a^T U_b U_b^T U_a U_a^T) \]
\[
= \frac{d}{r} + \lambda \|U_a^T U_b\|_F^2 \leq \frac{d}{r} (1 + \lambda/2).
\]

On the other hand, suppose \(u_1, \ldots, u_{d/r}\) are the columns of \(U_a\). Then, \((U_a U_a^T \sqrt{\Sigma_a})^2 = (1+\lambda)U_a U_a^T \sqrt{\Sigma_a}\) and hence, \(U_a U_a^T \sqrt{\Sigma_a g} \leq \sqrt{1 + \lambda} \sum_{i \in [d/r]} g_i u_i\). That is, \(U_a U_a^T \sqrt{\Sigma_a g}\) is a spherical Gaussian in the subspace \(S\). So, by its rotation invariance, we have \(\sqrt{1 + \lambda} \sum_{i \in [d/r]} g_i u_i \leq \sqrt{1 + \lambda} \sum_{i \in [d/r]} g_i w_i\). Hence our goal is to show

\[
TV(\sqrt{1 + \lambda} \sum_{i = 1}^{d/r} g_i u_i, \sum_{i = 1}^{d/r} \sigma_i g_i w_i) = \Omega \left( \frac{\lambda \sqrt{d/r}}{\log(r/\lambda \sqrt{d})} \right),
\]

provided \(\sum_{i \in [d/r]} \sigma_i^2 \leq d(1 + \lambda/2)/r\).
By reordering the $w_i$, we may assume $\sigma_1^2 \leq \cdots \leq \sigma_{d/r}^2$. At most half of the $\sigma_i^2$ can be twice their average, which means $\sigma_1^2 \leq \cdots \leq \sigma_{d/2r}^2 \leq 2(1 + \lambda/2) \leq 3$. Now we may project the two random vectors onto the subspace generated by the $d/(2r)$ smallest eigenvectors, and this can only decrease the total variation distance. Taking the norms of the projected vectors and dividing by $\sqrt{d}$ can only decrease the total variation distance; hence our new goal is to show

$$TV((1 + \lambda)\sum_{i=1}^{d/2r} \frac{g_i^2}{\sqrt{d}}, \sum_{i=1}^{d/2r} \sigma_i^2 \frac{g_i^2}{\sqrt{d}}) = \Omega \left(\frac{\lambda \sqrt{d}/r}{\log(r/\lambda \sqrt{d})}\right),$$

provided $\sum_{i \in [d/2r]} \sigma_i^2 \leq d(1 + \lambda/2)/2r$ and $\max_i \sigma_i \leq 3$.

Observe that

$$\E \left[ (1 + \lambda)\sum_{i=1}^{d/2r} \frac{g_i^2}{\sqrt{d}} \right] - \E \left[ \sum_{i=1}^{d/2r} \sigma_i^2 \frac{g_i^2}{\sqrt{d}} \right] \geq (1 + \lambda)(\sqrt{d}/2r) - \sqrt{d}(1 + \lambda/2)/2r = \lambda \sqrt{d}/(4r).$$

Moreover, by Bernstein’s inequality (Lemma A.11), there exists a global constant $c > 0$ (independent of $\lambda$) such that for any $t > 0$,

$$\Pr \left[ \left| (1 + \lambda)\sum_{i=1}^{d/(2r)} \frac{g_i^2}{\sqrt{d}} - \E (1 + \lambda)\sum_{i=1}^{d/(2r)} \frac{g_i^2}{\sqrt{d}} \right| > t \right] \leq 2 \exp \left( -c \min\{t^2, t\sqrt{d}\} \right),$$

and, since $\sigma_i^2 \leq 3$ for all $i$,

$$\Pr \left[ \left| \sum_{i=1}^{d/2r} \sigma_i^2 \frac{g_i^2}{\sqrt{d}} - \E \sum_{i=1}^{d/2r} \sigma_i^2 \frac{g_i^2}{\sqrt{d}} \right| > t \right] \leq 2 \exp \left( -c \min\{t^2, t\sqrt{d}\} \right).$$

Applying Lemma E.2 gives (5) (note that $\zeta \leq O(\log(r/\lambda \sqrt{d}))$ in the lemma), as required.

**Lemma E.2.** Let $X, Y$ be continuous random variables such that $|EX - EY| \geq \Delta$. Suppose also there exist $c, C, \beta$ such that for any $t > 0$ we have

$$\max\{\Pr [|X - EX| \geq t] , \Pr [|Y - EY| \geq t]\} \leq C \exp(-c \min\{t^2, \beta t\}).$$

Let

$$\zeta := \max\{1, \Delta, \log(4C)/c\beta, \sqrt{\log(4C)/c}, \log(8C/c\beta \Delta)/c\beta, \sqrt{\log(4C/c\Delta)/c}\}.$$

Then, $TV(X, Y) \geq \Delta/8 \zeta$. 

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Proof. We shall show that, for any $h > 0$ we have
\[
\text{TV}(X, Y) \geq \frac{\Delta - C \exp(-c \min(h^2, \beta h))\Delta - C \exp(-ch^2)/ch - 2C \exp(-c\beta h)/c\beta}{h + \Delta},
\]
and the lemma will follow by choosing $h = \zeta$.

Without loss of generality, we may assume $\mathbb{E}Y \geq \mathbb{E}X$ and by translating, we may assume $\mathbb{E}X = -\Delta/2$ and $\mathbb{E}Y = \Delta/2$. Let $I = [\mathbb{E}X - h, \mathbb{E}Y + h]$, $I' = \mathbb{R} \setminus I$ and let $f(x), g(x)$ denote the densities of $X$ and $Y$.

Define random variable $Z$ to be $|X - \mathbb{E}X|$ if $|X - \mathbb{E}X| > h$, and 0 otherwise. Then, since $I' \subseteq (-\infty, \mathbb{E}X - h) \cup (\mathbb{E}X + h, \infty)$, we have
\[
\mathbb{E}Z = \int_{(-\infty, \mathbb{E}X-h)\cup(\mathbb{E}X+h,\infty)} |(x - \mathbb{E}X)f(x)| \, dx \\
\geq \int_{I'} (|x|f(x) - |\mathbb{E}X|f(x)) \, dx \\
= \int_{I'} |xf(x)| \, dx - (\Delta/2)\mathbb{P} \left[ |X - \mathbb{E}X| > h \right] \\
\geq \int_{I'} |xf(x)| \, dx - C \exp(-c \min(h^2, \beta h))\Delta/2
\]
On the other hand,
\[
\mathbb{E}Z = \int_0^\infty \mathbb{P} \left[ Z > t \right] \, dt = \int_h^\infty \mathbb{P} \left[ |X - \mathbb{E}X| > t \right] \, dt \\
\leq \int_h^\infty C \exp(-c \min\{t^2, \beta t\}) \, dt \leq \int_h^\infty C \exp(-ct^2) \, dt + \int_h^\infty C \exp(-c\beta t) \, dt \\
\leq C \exp(-ch^2)/2ch + C \exp(-c\beta h)/c\beta,
\]
where for the last inequality we have used tail bounds for the standard normal distribution (see [Abramowitz and Stegun, 1984], formula (7.1.13)). Thus we find
\[
\int_{I'} |xf(x)| \, dx \leq C \exp(-ch^2)/2ch + C \exp(-c\beta h)/c\beta + C \exp(-c \min(h^2, \beta h))\Delta/2.
\]
A similar calculation gives the same upper bound for $\int_{I'} |xg(x)| \, dx$. Finally, observe that
\[
\Delta = \int_I x(g(x) - f(x)) \, dx + \int_{I'} x(g(x) - f(x)) \, dx \\
\leq \int_I |x||g(x) - f(x)| \, dx + \int_{I'} |x|(|g(x)| + |f(x)|) \, dx \\
\leq \|f(x) - g(x)\|_1(h + \Delta)/2 + C \exp(-ch^2)/ch + 2C \exp(-c\beta h)/c\beta + C \exp(-c \min(h^2, \beta h))\Delta,
\]
since $|x| \leq (h + \Delta)/2$ for all $x \in I$. Re-arranging and noting total variation distance is half the $L_1$ distance gives (6).
E.4 Proof of Theorem 5.5

We begin with a combinatorial lemma. Let $d_H(\cdot, \cdot)$ denote the Hamming distance between two tuples.

**Lemma E.3.** Let $T \geq 2$ and $k \in \mathbb{N}$. There exists a set of tuples $\mathcal{X} \subseteq [T]^k$ such that $|\mathcal{X}| \geq 2^{\Omega(k \log(T))}$ and $d_H(x, y) \geq k/4$ for any pair of distinct $x, y \in \mathcal{X}$.

**Proof.** Let $x$ and $y$ be strings of length $k$ where each coordinate is drawn from $[T]$ independently and uniformly at random. Let $X_i$ be the indicator random variable which is 1 if $x_i = y_i$. Then $k - d_H(x, y) = \sum_{i=1}^{k} X_i$ and $E[k - d_H(x, y)] = k/T \leq k/2$. Observe that $X_i = 1$ with probability $1/T$ and 0 otherwise; hence it is $1/\sqrt{\log(T)}$-subgaussian. By Hoeffding’s Inequality (Lemma A.10),

$$\Pr \left[ \sum_{i=1}^{k} X_i \geq 3k/4 \right] \leq 2 \cdot \exp(-ck \log(T))$$

for some absolute constant $c > 0$. Thus, we conclude that there is some set $\mathcal{X}$ with $|\mathcal{X}| \geq 2^{\Omega(k \log(T))}$ and $d_H(x, y) \geq k/4$ for any pair of distinct $x, y \in \mathcal{X}$. \hfill \Box

We now prove Theorem 5.5 using Lemma 5.1 again. The proof of Theorem 5.2 promises a collection of $T = 2^{\Omega(d^2)}$ matrices $\Sigma_1, \ldots, \Sigma_T \prec 2I_d$ with $\text{KL}(\mathcal{N}(0, \Sigma_i || \mathcal{N}(0, \Sigma_d))) \leq O(\varepsilon^2 \log^2(1/\varepsilon))$ and $\text{TV}(\mathcal{N}(0, \Sigma_i), \mathcal{N}(0, \Sigma_d)) \geq \Omega(\varepsilon)$ for $i \neq i'$. Choose $\mu_1, \ldots, \mu_k \in \mathbb{R}^d$ such that $\|\mu_i - \mu_{i'}\|_2 \geq C(kd/\varepsilon)^{10}$ for all $i \neq i'$, where $C$ is a large enough constant. By Lemma E.3 there exists a set $\mathcal{X} \subseteq [T]^k$ of size $2^{\Omega(k \log(T))} = 2^{\Omega(kd^2)}$ such that $d_H(x, y) \geq k/4$ for any distinct $x, y \in \mathcal{X}$. We now define a set of mixture distributions as

$$\mathcal{F} = \left\{ f_x := \frac{1}{k} \left( \mathcal{N}(\mu_1, \Sigma_{x_1}) + \ldots + \mathcal{N}(\mu_k, \Sigma_{x_k}) \right) : x \in \mathcal{X} \right\}.$$

We shall show that for any $x \neq y$, we have $\text{KL}(f_x || f_y) \leq O(\varepsilon^2 \log^2(1/\varepsilon))$ and $\text{TV}(f_x, f_y) \geq \Omega(\varepsilon)$. Lemma 5.1 will then conclude the proof.

The proof of upper bound for KL divergence simply follows from convexity of KL divergence (see Cover and Thomas, 2006, Theorem 2.7.2). and the fact that, for each $i$,

$$\text{KL}(\mathcal{N}(\mu_i, \Sigma_{x_i}, || \mathcal{N}(\mu_i, \Sigma_{y_i})) = \text{KL}(\mathcal{N}(0, \Sigma_{x_i}, || \mathcal{N}(0, \Sigma_{y_i})) \leq O(\varepsilon^2 \log^2(1/\varepsilon)).$$

Next we show that $\text{TV}(f_x, f_y) \geq \Omega(\varepsilon)$. Let

$$A_j' \in \text{argmax}_{A \subseteq \mathbb{R}^d} \left\{ \Pr_{g \sim \mathcal{N}(\mu_j, \Sigma_{x_j})}[g \in A] - \Pr_{g \sim \mathcal{N}(\mu_j, \Sigma_{y_j})}[g \in A] \right\}.$$

Since $\Sigma_i \prec 2I_d$ for all $i$, we have

$$\Pr_{g \sim \mathcal{N}(\mu_j, \Sigma_{x_j})}[\|g - \mu_j\|_2^2 \geq 2d + O(d \log(k/\varepsilon))] \leq \varepsilon^2/k^2.$$
and a similar bound holds for $\mathcal{N}(\mu_j, \Sigma_{y_j})$. Now define $A_j = A_j' \cap B^d_2(\mu_j, O(\sqrt{d \log(k/\varepsilon)})$, where $B^d_2(\mu, r)$ denotes the $\ell_2$ ball centered at $\mu$ with radius $r$. Then,
\[
\Pr_{g \sim \mathcal{N}(\mu_j, \Sigma_{x_j})}[g \in A_j] - \Pr_{g \sim \mathcal{N}(\mu_j, \Sigma_{y_j})}[g \in A_j] \geq \text{TV}(\mathcal{N}(\mu_j, \Sigma_{x_j}), \mathcal{N}(\mu_j, \Sigma_{y_j})) - \varepsilon^2/k^2.
\]
Note that the separation of $\mu_1, \ldots, \mu_k$ implies that $A_1, \ldots, A_k$ are disjoint sets and $\Pr_{g \sim \mathcal{N}(\mu_i, \Sigma_{y_j})}[g \in A_j] \leq \varepsilon^2/k^2$ for $i \neq j$. Let $A = \cup_j A_j$. Finally, to lower bound the total variation distance, we have
\[
\text{TV}(f_x, f_y) \geq \Pr_{g \sim f_x}[g \in A] - \Pr_{g \sim f_y}[g \in A]
\]
\[
= \sum_{j=1}^k \left[ \Pr_{g \sim f_x}[g \in A_j] - \Pr_{g \sim f_y}[g \in A_j] \right]
\]
\[
= \frac{1}{k} \sum_{j=1}^k \sum_{i=1}^k \left[ \Pr_{g \sim \mathcal{N}(\mu_i, \Sigma_{x_j})}[g \in A_j] - \Pr_{g \sim \mathcal{N}(\mu_i, \Sigma_{y_j})}[g \in A_j] \right]
\]
\[
= \frac{1}{k} \sum_{j=1}^k \left[ \Pr_{g \sim \mathcal{N}(\mu_j, \Sigma_{x_j})}[g \in A_j] - \Pr_{g \sim \mathcal{N}(\mu_j, \Sigma_{y_j})}[g \in A_j] \right]
\]
\[
+ \frac{1}{k} \sum_{j=1}^k \sum_{i \neq j} \left[ \Pr_{g \sim \mathcal{N}(\mu_i, \Sigma_{x_j})}[g \in A_j] - \Pr_{g \sim \mathcal{N}(\mu_i, \Sigma_{y_j})}[g \in A_j] \right]
\]
\[
\geq \frac{1}{k} \sum_{j=1}^k \left[ \Pr_{g \sim \mathcal{N}(\mu_j, \Sigma_{x_j})}[g \in A_j] - \Pr_{g \sim \mathcal{N}(\mu_j, \Sigma_{y_j})}[g \in A_j] \right] - \varepsilon^2
\]
\[
\geq \frac{1}{k} \sum_{j=1}^k \left[ \text{TV}(\mathcal{N}(\mu_j, \Sigma_{x_j}), \mathcal{N}(\mu_j, \Sigma_{y_j})) - \varepsilon^2/k^2 \right] - \varepsilon^2
\]
\[
\geq \frac{1}{k} (k/4) \Omega(\varepsilon) - 2\varepsilon^2 \geq \Omega(\varepsilon),
\]
where the last inequality is because $\text{TV}(\mathcal{N}(\mu_j, \Sigma_{x_j}), \mathcal{N}(\mu_j, \Sigma_{y_j})) \geq \Omega(\varepsilon)$ whenever $x_j \neq y_j$ which is the case for at least $k/4$ of the indices $j$.

References


