Near-optimal Sample Complexity Bounds for Robust Learning of Gaussian Mixtures via Compression Schemes

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July 20, 2019

Abstract

We prove that $\tilde{\Theta}(kd^2/\varepsilon^2)$ samples are necessary and sufficient for learning a mixture of $k$ Gaussians in $\mathbb{R}^d$, up to error $\varepsilon$ in total variation distance. This improves both the known upper bounds and lower bounds for this problem. For mixtures of axis-aligned Gaussians, we show that $\tilde{O}(kd/\varepsilon^2)$ samples suffice, matching a known lower bound.

Moreover, these results hold in an agnostic learning (or robust estimation) setting, in which the target distribution is only approximately a mixture of Gaussians.

The upper bound is shown using a novel technique for distribution learning based on a notion of compression. Any class of distributions that allows such a compression scheme can also be learned with few samples. Moreover, if a class of distributions has such a compression scheme, then so do the classes of products and mixtures of those distributions. Our main result is proven by showing that the class of Gaussians in $\mathbb{R}^d$ admits a small compression scheme.

Keywords: density estimation, distribution learning, mixtures of Gaussians, compression schemes

1 Introduction

Estimating distributions from observed data is a fundamental task in statistics that has been studied for over a century. This task frequently arises in applied machine learning, commonly assuming that the distribution can be modeled approximately with a mixture of Gaussians. Popular software packages have implemented heuristics, such as the EM algorithm, for learning a mixture of Gaussians. The theoretical machine learning community also has a rich

*A preliminary version of this paper appeared in the proceedings of NeurIPS 2018 [5]. In this full version, we have strengthened the results from realizable learning to agnostic (robust) learning, improved polylogarithmic factors in the lower bound, and included all the proofs.
literature on distribution learning. For example, the recent survey [19] considers learning structured distributions, and the survey [20] focuses on mixtures of Gaussians.

This paper develops a general technique for distribution learning, then employs this technique in the canonical setting of Gaussian mixtures. The learning model we adopt is density estimation: given i.i.d. samples from an unknown target distribution, find a distribution that is close to the target in total variation (TV) distance. Our analysis focuses on sample complexity rather than computational complexity. That is, a learning algorithm must obtain a good estimate of the target distribution while using as few samples as possible, but ignoring the time to compute this estimate. For background on this model see, e.g., [16, Chapter 5] and [19].

Our new technique for proving upper bounds on the sample complexity involves a form of sample compression. If it is possible to “encode” members of a class of distributions using a carefully chosen subset of the samples, then this yields an upper bound on the sample complexity of distribution learning for that class. In particular, by constructing compression schemes for mixtures of axis-aligned Gaussians and general Gaussians, we obtain new upper bounds on the sample complexity of learning with respect to these classes. Furthermore, we prove that these new bounds are optimal up to polylogarithmic factors.

1.1 The distribution learning framework

A distribution learning method or density estimation method is an algorithm that takes as input a sequence of i.i.d. samples generated from a distribution \( g \), then outputs (a description of) a distribution \( \hat{g} \) to serve as an estimate of \( g \). We work with continuous distributions in this paper (i.e., distributions that have a density with respect to the Lebesgue measure), so we identify a probability distribution by its probability density function. Let \( f_1 \) and \( f_2 \) be two probability distributions defined over \( \mathbb{R}^d \) and let \( \mathcal{B} \) be the Borel sigma algebra on \( \mathbb{R}^d \). Their total variation (TV) distance is defined by

\[
TV(f_1, f_2) := \sup_{B \in \mathcal{B}} \int_B (f_1(x) - f_2(x)) \, dx = \frac{1}{2} \|f_1 - f_2\|_1,
\]

where \( \|f\|_1 := \int_{\mathbb{R}^d} |f(x)| \, dx \) is the \( L^1 \) norm of \( f \). Sometimes we write \( TV(X, Y) \), where \( X \) and \( Y \) are random variables rather than distributions. In the following definitions, \( \mathcal{F} \) is a class of probability distributions, and \( g \) is a distribution not necessarily in \( \mathcal{F} \).

**Definition 1.1** (\( \varepsilon \)-approximation, \( \varepsilon \)-close, \((\varepsilon, C)\)-approximation). A distribution \( \hat{g} \) is an \( \varepsilon \)-approximation for \( g \) if \( \|\hat{g} - g\|_1 \leq \varepsilon \). We also say that \( \hat{g} \) is \( \varepsilon \)-close to \( g \). A distribution \( \hat{g} \) is an \((\varepsilon, C)\)-approximation for \( g \) with respect to \( \mathcal{F} \) if

\[
\|\hat{g} - g\|_1 \leq C \cdot \inf_{f \in \mathcal{F}} \|f - g\|_1 + \varepsilon
\]

**Definition 1.2** (PAC-learning distributions, realizable setting). A distribution learning method is called a (realizable) PAC-learner for \( \mathcal{F} \) with sample complexity \( m_{\mathcal{F}}(\varepsilon, \delta) \) if, for all distributions \( g \in \mathcal{F} \) and all \( \varepsilon, \delta \in (0, 1) \), given \( \varepsilon, \delta \), and an i.i.d. sample of size \( m_{\mathcal{F}}(\varepsilon, \delta) \) from \( g \), with probability at least \( 1 - \delta \) (over the samples) the method outputs an \( \varepsilon \)-approximation of \( g \).

**Definition 1.3** (PAC-learning distributions, agnostic setting). For \( C > 0 \), a distribution learning method is called a \( C \)-agnostic PAC-learner for \( \mathcal{F} \) with sample complexity \( m^C_{\mathcal{F}}(\varepsilon, \delta) \) if, for all distributions \( g \in \mathcal{F} \) and all \( \varepsilon, \delta \in (0, 1) \), given \( \varepsilon, \delta \), and a sample of size \( m^C_{\mathcal{F}}(\varepsilon, \delta) \) generated i.i.d. from \( g \), with probability at least \( 1 - \delta \) the method outputs an \((\varepsilon, C)\)-approximation of \( g \) with respect to \( \mathcal{F} \).

The statement that a class can be “\( C \)-learned in the agnostic setting” means the existence of a \( C \)-agnostic PAC-learner for the class. The case \( C > 1 \) is sometimes called semi-agnostic learning in the learning theory literature. Let \( \Delta_n := \{ (w_1, \ldots, w_n) \in \mathbb{R}^n : w_i \geq 0, \sum w_i = 1 \} \) denote the \( n \)-dimensional simplex.

**Definition 1.4** (\( k \)-mix(\( \mathcal{F} \))). Let \( \mathcal{F} \) be a class of probability distributions. Then the class of \( k \)-mixtures of \( \mathcal{F} \), written \( k\text{-mix}(\mathcal{F}) \), is defined as

\[
k\text{-mix}(\mathcal{F}) := \{ \sum_{i=1}^k w_i f_i : (w_1, \ldots, w_k) \in \Delta_k, f_1, \ldots, f_k \in \mathcal{F} \}.
\]

1.2 Main results

Our first main result is an upper bound for learning mixtures of multivariate Gaussians. This bound is tight up to logarithmic factors. Henceforth, the notations \( \tilde{O}(\cdot) \) and \( \tilde{\Omega}(\cdot) \) suppress \( polylog(kd/\varepsilon\delta) \) factors, and all results hold non-asymptotically, meaning that they hold for all admissible parameters.
Theorem 1.5. The class of $k$-mixtures of $d$-dimensional Gaussians can be learned in the realizable setting, and can be $9$-learned in the agnostic setting, using $O(kd^2/\varepsilon^4)$ samples.

We emphasize that the $\tilde{O}(\cdot)$ notation has no dependence whatsoever on the scaling, condition number, separation, or any other structural property of the distribution. Previously, the best known upper bounds on the sample complexity of this problem were $\tilde{O}(kd^2/\varepsilon^4)$, due to [6], and $O((kd^4 + k^3d^3)/\varepsilon^2)$, based on a VC-dimension bound that we discuss below. For the case of a single Gaussian (i.e., $k = 1$), a sample complexity bound of $O(d^2/\varepsilon^2)$ is known, again using a VC-dimension bound discussed below.

Our second main result is a lower bound matching Theorem 1.5 up to logarithmic factors.

Theorem 1.6. Any method for learning the class of $k$-mixtures of $d$-dimensional Gaussians in the realizable setting has sample complexity $\Omega(kd^2/\varepsilon^2)$.

Note that this is a worst-case (i.e., minimax) lower bound: for any estimation method, there exists at least one distribution that requires that many samples. Previously, the best known lower bound on the sample complexity was $\tilde{O}(kd/\varepsilon^2)$ [40]. Even for a single Gaussian (i.e., $k = 1$), an $\tilde{O}(d^2/\varepsilon^2)$ lower bound was not known prior to this work.

Our third main result is an upper bound for learning mixtures of axis-aligned Gaussians, i.e., Gaussians with diagonal covariance matrices. This bound is also tight up to logarithmic factors.

Theorem 1.7. The class of $k$-mixtures of axis-aligned $d$-dimensional Gaussians can be learned in the realizable setting, and can be $9$-learned in the agnostic setting, using $O(kd/\varepsilon^2)$ samples.

A matching lower bound of $\tilde{\Omega}(kd/\varepsilon^2)$ was proved in [40]. Previously, the best known upper bounds were $\tilde{O}(kd/\varepsilon^4)$, due to [6], and $O((kd^4 + k^3d^3)/\varepsilon^2)$, based on a VC-dimension bound that we discuss below.

In the agnostic results of Theorem 1.5 and Theorem 1.7, the constant $9$ can be decreased to any constant larger than $6$. One may verify this statement through a detailed inspection of our proofs. We omit a full derivation of this improved constant in order to avoid tedious details in the proofs.

Minimax estimation rates. Our results are stated in terms of sample complexity, which is the terminology mostly used in the machine learning literature. It is also possible to state our results in terms of minimax estimation rates, which are often used in the statistics literature. There is a direct connection between sample complexity bounds and estimation rates, although translating between them sometimes incurs logarithmic factors.

We recall some definitions from the minimax estimation framework (see, e.g., [41, Chapter 2]). Let $F$ be a class of probability distributions defined on domain $\mathcal{X}$. The risk of a density estimation method $\hat{f} : \mathcal{X}^n \to F$ for this class in total variation distance is defined as

$$R_n(\hat{f}, F) := \sup_{f \in F} \mathbb{E}_{X_1, \ldots, X_n} [TV(\hat{f}(X_1, \ldots, X_n), f)],$$

where the expectation is over the i.i.d. samples $X_1, \ldots, X_n$ from $f$, and possible randomization of the estimator. The minimax estimation rate for $F$ is the smallest risk over all possible estimators $\hat{f} : \mathcal{X}^n \to F$, i.e.,

$$R_n(F) := \inf_{\hat{f}} R_n(\hat{f}, F) = \inf_{\hat{f}} \sup_{f \in F} \mathbb{E}_{X_1, \ldots, X_n} [TV(\hat{f}(X_1, \ldots, X_n), f)].$$

Let $G_{d,k}$ denote the class of $k$-mixtures of $d$-dimensional Gaussian distributions, and $A_{d,k}$ denote the class of $k$-mixtures of $d$-dimensional axis-aligned Gaussian distributions. Then Theorem 1.5 implies the minimax estimation rate of $G_{d,k}$ is $O(\sqrt{kd}/\sqrt{n})$, and Theorem 1.7 implies the minimax estimation rate of $A_{d,k}$ is $O(\sqrt{kd}/n)$. Note that these theorems indeed imply stronger statements than these risk bounds, since they give guarantees for the case where the target distribution does not necessarily belong to the known class $F$, a setting not captured by the minimax framework.

Finally, the proof of Theorem 1.6 (see Theorem 5.3 below) implies that the minimax rate of $G_{d,k}$ is $\Omega(\sqrt{kd^2/n})$, improving the $\Omega(\sqrt{kd^2}/\sqrt{n} \log n)$ lower bound proved in the preliminary version of this paper [5].

Computational efficiency. Although our approach for proving sample complexity upper bounds is algorithmic, our focus is not on computational efficiency. The resulting algorithms have nearly optimal sample complexities, but their running times are exponential in the dimension $d$ and the number of mixture components $k$. More precisely, the running time is $2^{kd^2 \cdot \text{polylog}(d,k,1/\varepsilon,1/\delta)}$ for mixtures of general Gaussians, and $2^{kd \cdot \text{polylog}(d,k,1/\varepsilon,1/\delta)}$ for mixtures of
axis-aligned Gaussians. The existence of an algorithm for density estimation that runs in time \(\text{poly}(k,d)\) is unknown even for the class of mixtures of axis-aligned Gaussians, see [21] Question 1.1.

Even for the case of a single Gaussian, the published proofs of the \(O(d^2/\varepsilon^2)\) bound, of which we are aware, are not algorithmically efficient, e.g., [6] Theorem 13. Adopting ideas from our proof of Theorem 1.5, an algorithmically efficient learner for a single Gaussian can be obtained simply by computing the empirical mean and (an appropriate estimate of the) covariance matrix using \(O(d^2/\varepsilon^2)\) samples. The details appear in our technical report [7].

1.3 Related work

Distribution learning is a vast topic and many approaches have been considered in the literature. This section reviews the approaches that are particularly relevant to our work.

For parametric families of distributions, a common approach is to use the samples to estimate the parameters of the distribution, possibly in a maximum likelihood sense, or possibly aiming to approximate the true parameters. For the specific case of mixtures of Gaussians, there is a substantial theoretical literature on algorithms that approximate the mixing weights, means and covariances (e.g., [4] [5] [13, 34]); see [26] for a survey. The strictness of this objective cuts both ways. On the one hand, a successful learner uncovers substantial structure of the target distribution. On the other hand, this objective is impossible when the means and covariances are extremely close. Thus, algorithms for parameter estimation of mixtures necessarily require some separation assumptions on the target parameters.

Density estimation has a long history in the statistics literature, where the focus is on the sample complexity question; see [13] [16] [38] for general background. It was first studied in the computational learning theory community under the name PAC learning of distributions by [27], whose focus is on the computational complexity of the learning problem.

Various measures of dissimilarity between distributions have been considered in existing density estimation schemes. For example, one natural measure is the TV distance [16] Chapter 5; this has been used by several existing algorithms for mixtures of Gaussians [6] [11] [14]. Another natural measure is the Kullback-Leibler (KL) divergence, which has also been used for mixtures of Gaussians [23]. Yet another natural measure is the \(L^p\) distance for \(p > 1\); for example, some prior work has used the \(L^2\) distance for density estimation [2] [20]. (The \(L^p\) distance between densities \(f\) and \(g\) is defined as \(\|f - g\|_p := (\int_{\mathbb{R}^d} |f(x) - g(x)|^p \, dx)^{1/p}\).) This paper focuses on the TV distance (i.e., the \(L^1\) distance), and we provide justification for this choice in Section 2.

A popular method for distribution learning in practice is kernel density estimation (see, e.g., [16] Chapter 9). The rigorously proven sample complexity/estimation rate upper bounds for this method require either smoothness assumptions (e.g., [16] Theorem 9.5) or boundedness assumptions (e.g., [25] Theorem 2.2) on the class of densities. The class of Gaussians is not universally Lipschitz or universally bounded, so those results do not apply to the problems we consider. Moreover, numerical calculations demonstrate that the number of samples required to estimate a standard Gaussian (within \(L^2\) distance 0.1 using Gaussian kernels) grow exponentially with the dimension (see [35] Table 4.2 on page 94), which hints that this method suffers from the curse of dimensionality.

Another elementary method for density estimation is using histogram estimators (see, e.g., [16] Section 10.3). Straightforward calculations show that the sample complexity of histogram estimators is exponential in the dimension even for learning a single Gaussian.

The minimum distance estimate [16] Section 6.8] is another approach for deriving sample complexity upper bounds for distribution learning. This approach is based on uniform convergence theory. In particular, an upper bound for any class of distributions can be achieved by bounding the VC-dimension of an associated set system, called the Yatracos class (see [16] page 58 for the definition). For example, [22] used this approach to bound the sample complexity of learning high-dimensional log-concave distributions. For the class of single Gaussians in \(d\) dimensions, this approach leads to the optimal sample complexity upper bound of \(O(d^2/\varepsilon^2)\). However, for mixtures of Gaussians and axis-aligned Gaussians in \(\mathbb{R}^d\), the best known VC-dimension bounds (see [16] Section 8.5 and [6] Theorem 8.14]) result in loose upper bounds of \(O(k^3d^3/\varepsilon^2)\) and \(O((k^3d^2+k^3d^3)/\varepsilon^2)\), respectively.

Another approach is to first approximate the mixture class using a more manageable class such as piecewise polynomials, and then study the associated Yatracos class; see, e.g., [11]. However, piecewise polynomials do a poor job in approximating \(d\)-dimensional Gaussians, resulting in an exponential dependence on \(d\).

For density estimation of mixtures of Gaussians using the TV distance, the current best sample complexity upper bounds (in terms of \(k\) and \(d\)) are \(\tilde{O}(kd^2/\varepsilon^4)\) for general Gaussians and \(\tilde{O}(kd/\varepsilon^4)\) for axis-aligned Gaussians, both due to [6]. For the general Gaussian case, their method takes an i.i.d. sample of size \(\tilde{O}(kd^2/\varepsilon^2)\) and partitions this sample in every possible way into \(k\) subsets. Based on those partitions, \(\tilde{O}(kd^2/\varepsilon^2)\) “candidate distributions” are generated. The problem is then reduced to learning with respect to this finite class of candidates. Their sample complexity has a
Table 1: Bounds on the sample complexities of learning Gaussian mixtures and their subclasses. The lower bounds are minimax (i.e., worst-case). The bounds in the first two rows are well known; proofs can be found in [6].

<table>
<thead>
<tr>
<th>Number of Gaussians</th>
<th>Dimension</th>
<th>Axis-aligned</th>
<th>Sample complexity</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$d$</td>
<td>no</td>
<td>$O(d^2/\varepsilon^2)$</td>
<td>standard</td>
</tr>
<tr>
<td>1</td>
<td>$d$</td>
<td>yes</td>
<td>$O(d/\varepsilon^2)$</td>
<td>standard</td>
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<tr>
<td>$k$</td>
<td>1</td>
<td>n/a</td>
<td>$	ilde{O}(k/\varepsilon^2)$</td>
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<tr>
<td>$k$</td>
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<td>1</td>
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<td>[40]</td>
</tr>
</tbody>
</table>

suboptimal factor of $1/\varepsilon^4$, of which $1/\varepsilon^2$ arises in their approach for choosing the best candidate, and another factor $1/\varepsilon^2$ is due to the exponent in the number of candidates.

Our approach via compression schemes also ultimately reduces the problem to learning with respect to finite classes, although yielding a more refined bound than previous work. In the case of mixtures of Gaussians, one factor of $1/\varepsilon^2$ is again incurred due to learning with respect to finite classes. The key is that the number of compressed samples is only $O_d(1)$, so the overall sample complexity bound has only an $\tilde{O}(1/\varepsilon^2)$ dependence on $\varepsilon$.

As for lower bounds on the sample complexity for learning mixtures of Gaussians under the TV distance, much fewer results are known. The only lower bound prior to this work is due to [40], which shows a bound of $\tilde{\Omega}(kd/\varepsilon^2)$ for learning mixtures of axis-aligned Gaussians (and hence for general Gaussians as well). This bound is tight for the axis-aligned case, as we show in Theorem 1.7, but loose in the general case, as we show in Theorem 1.6. After the preliminary version of this paper was completed [5], an alternative construction was provided in [17] giving the same lower bound as ours using a deterministic construction.

A summary of bounds on the sample complexity for learning Gaussian mixtures and their subclasses is presented in Table 1.

1.4 Our techniques

We introduce a method for learning distributions via a novel form of compression. Given a class of distributions, suppose there is a method for “compressing” information about the true distribution using a subset of samples from that distribution and some additional bits. Further, suppose there exists a fixed (deterministic) decoder for the class, such that given the subset of samples and the additional bits, it approximately recovers the original distribution. In this case, if the size of the subset and the number of bits is guaranteed to be small, we show that the sample complexity of learning that class is small as well.

More precisely, we say a class of distributions admits $(\tau, t, m)$ compression if there exists a decoder function such that upon generating $m$ i.i.d. samples from any distribution in the class, we are guaranteed, with probability at least $2/3$, to have a sequence of length at most $\tau$ from that sample, and a sequence of at most $t$ bits, on which the decoder outputs a distribution which is within total variation distance of $\varepsilon$ to the original distribution. Note that $\tau, t$, and $m$ may be functions of $\varepsilon$, the accuracy parameter.

This definition is further generalized to a stronger notion of robust compression, where the target distribution is to be encoded using samples that are not necessarily generated from the target itself, but are generated from a distribution that is close to the target. We prove that robust compression implies agnostic learning. In particular, if a class admits $(\tau, t, m)$ robust compression, then the sample complexity of agnostic learning with respect to this class is bounded by $\tilde{O}(m + (\tau + t)/\varepsilon^2)$ (Theorem 3.5).

An attractive property of compression is that it enjoys two closure properties. Specifically, if a base class admits
compression, then the class of mixtures of that base class, as well as the class of products of the base class, are compressible (lemmata 3.6 and 3.7).

Consequently, it suffices to provide a compression scheme for the class of single Gaussian distributions in order to obtain a compression scheme (and a sample complexity bound) for the class of mixtures of Gaussians. We prove that the class of \(d\)-dimensional Gaussian distributions admits \(O(d)\) robust compression (Lemma 4.2). The high level idea is that by generating \(O(d)\) samples from a Gaussian, one can get a rough sketch of the geometry of the Gaussian. In particular, the points drawn from a Gaussian concentrate around an ellipsoid centered at the mean and whose principal axes are the eigenvectors of the covariance matrix. Using ideas from convex geometry and random matrix theory, we show one can in fact encode the center of the ellipsoid and the principal axes using a linear combination of these samples. Then we discretize the coefficients and obtain an approximate encoding.

One can view an algorithm based on compression schemes as a data-dependent covering-number-based algorithm, working like this: form an \(\varepsilon\)-net (in the space of distributions) of the target class \(F\) in TV distance; then choose the best one among the finitely many distributions in the net. The issue with a naive implementation of this idea is that the size of the \(\varepsilon\)-net for Gaussians is infinite. In other words, without assuming a bound on the mean and the variance, the metric entropy of Gaussians (with respect to the TV distance) is infinite. In fact, even for the class of mean-zero Gaussians with bounded (entry-wise) covariance matrices the metric entropy is infinite (unless we assume a bound on the condition number of the covariance matrices; see Proposition 2.4). The power of compression schemes is that they take a data-dependent approach: a first round of sampling is used to shrink the space of feasible distributions significantly, making the metric entropy of the resulting feasible set finite. Therefore, for the case of Gaussians we achieve bounds that are independent of the condition number or the size of the parameters.

The above results together imply tight (up to logarithmic factors) upper bounds of \(\tilde{O}(kd^2/\varepsilon^2)\) for mixtures of \(k\) Gaussians, and \(\tilde{O}(kd/\varepsilon^2)\) for mixtures of \(k\) axis-aligned Gaussians over \(\mathbb{R}^d\). The compression framework we introduce is quite flexible, and can be used to prove sample complexity upper bounds for other distribution classes as well. This is left for future work.

**Lower bound.** Before proving our lower bound for mixtures of Gaussians, we first prove a lower bound of \(\tilde{O}(d^2/\varepsilon^2)\) for learning a single Gaussian. Although the approach is quite intuitive, the details are intricate and much care is required to make a formal proof. The main step is to construct a large family (of size \(2^{O(d^2)}\)) of covariance matrices such that the associated Gaussian distributions are well-separated in terms of their total variation distance, while simultaneously ensuring that their Kullback-Leibler divergences are small. Once this is established, we can then apply a generalized version of Fano’s inequality to complete the proof.

To construct this family of covariance matrices, we sample \(2^{O(d^2)}\) matrices from the following probabilistic process: start with an identity covariance matrix; then choose a random subspace of dimension \(d/9\) and slightly increase the eigenvalues corresponding to this eigenspace. It is easy to bound the KL divergences between the constructed Gaussians. To lower bound the TV distance, we show that for every pair of these distributions, there is some subspace for which a vector drawn from one Gaussian will have slightly larger projection than a vector drawn from the other Gaussian. Quantifying this gap will then give us the desired lower bound on the total variation distance.

**Paper outline.** We introduce some notation in Section 1.5. In Section 2 we provide justification for our learning model. In Section 3 we define compression schemes for distributions, prove their closure properties, and show their connection with density estimation. Theorems 1.6 and 1.7 are proved in Section 4. Theorem 1.6 is proven in Section 6. Some standard but useful results and all omitted proofs appear in the appendices.

### 1.5 Terminology

Let \(d\) denote the dimension. A Gaussian distribution with mean \(\mu \in \mathbb{R}^d\) and covariance matrix \(\Sigma \in \mathbb{R}^{d \times d}\) is denoted by \(\mathcal{N}(\mu, \Sigma)\). If \(\Sigma\) is a diagonal matrix, then \(\mathcal{N}(\mu, \Sigma)\) is called an *axis-aligned* Gaussian. For a distribution \(g\), we write \(X \sim g\) to mean \(X\) is a random variable with distribution \(g\), and we write \(S \sim g^m\) to mean that \(S\) is an i.i.d. sample of size \(m\) generated from \(g\).

**Definition 1.8** (Kullback-Leibler (KL) divergence). The Kullback-Leibler (KL) divergence between densities \(f_1\) and \(f_2\) is defined by

\[
\text{KL} (f_1 \parallel f_2) := \int_{\mathbb{R}^d} f_1(x) \log \frac{f_1(x)}{f_2(x)} \, dx,
\]

where we define \(\text{KL} (f_1 \parallel f_2) = +\infty\) if the set \(\{ x : f_2(x) = 0 < f_1(x) \}\) has positive Lebesgue measure.
We will use $\|v\|$ or $\|v\|_2$ to denote the Euclidean norm of a vector $v$, $\|A\|$ or $\|A\|_2$ to denote the operator norm of a matrix $A$, and $\|A\|_F := \sqrt{\text{Tr}(A^TA)}$ to denote the Frobenius norm of a matrix $A$. We write $B \preceq A$ if $A - B$ is a positive semidefinite matrix, and $B \prec A$ if $A - B$ is a positive definite matrix. For $x \in \mathbb{R}$, we will write $(x)_+ := \max\{0,x\}$.

For random variables $X$ and $Y$, the notation $X \overset{d}{=} Y$ means that $X$ and $Y$ have the same distribution. The notation $\log(\cdot)$ denotes logarithm in the natural base. Denote by $[M]$ the set $\{1,2,\ldots,M\}$. Throughout the paper, $a/bc$ always means $a/(bc)$.

## 2 Justification for our model

Several of the existing models for learning mixtures of Gaussians need some structural assumption on the distribution. For example, learning under the parameter estimation model requires that the means are sufficiently separated and that the mixing weights are not too small, see, e.g., [20, Definition 1].

A key motivation for our work is to study a model for learning mixtures of Gaussians that requires no structural assumptions at all. Specifically, we would like to identify a model in which Gaussians can be learned up to error $\varepsilon$ with sample complexity depending only on $k$, $d$ and $\varepsilon$, then derive optimal sample complexity bounds in that model. It is known that density estimation under the TV distance is one such model; this follows from prior work, e.g., [6], or from Theorem 1.5. In this section we provide further justification for using this particular model.

In Section 2.1 we argue that the TV distance is not an arbitrary choice. If instead we had used the KL divergence or any $L^p$ distance, with $p > 1$, then the sample complexity must necessarily depend on the structural properties of the distribution. Thus, TV distance is a very natural choice. It is also natural to wonder whether some of our results could be derived from existing results on parameter estimation. In Section 2.2 we show that this is not the case: entry-wise estimation of the covariance matrices is quite unrelated to density estimation under the TV distance. Thus our model is interesting and not subsumed by previous work.

### 2.1 Comparison to KL divergence and $L^p$ distances

In this section we consider the problem of density estimation for a mixture of Gaussians, using a distance measure that is either the $L^1$ divergence or an $L^p$ distance with $p > 1$. Under these distance measures, we show that the sample complexity of this problem must necessarily depend on structural properties of the distribution — that is, it cannot be bounded purely as a function of $k$, $d$ and $\varepsilon$.

First we consider using the KL divergence. We show that no algorithm can guarantee that the KL divergence of the true distribution and the output distribution is smaller than any finite number with a uniformly bounded assumption at all. Specifically, we would like to identify a model in which Gaussians can be learned up to error $\varepsilon$ with sample complexity depending only on $k$, $d$ and $\varepsilon$, then derive optimal sample complexity bounds in that model. It is known that density estimation under the TV distance is one such model; this follows from prior work, e.g., [6], or from Theorem 1.5. In this section we provide further justification for using this particular model.

In Section 2.1 we argue that the TV distance is not an arbitrary choice. If instead we had used the KL divergence or any $L^p$ distance, with $p > 1$, then the sample complexity must necessarily depend on the structural properties of the distribution. Thus, TV distance is a very natural choice. It is also natural to wonder whether some of our results could be derived from existing results on parameter estimation. In Section 2.2 we show that this is not the case: entry-wise estimation of the covariance matrices is quite unrelated to density estimation under the TV distance. Thus our model is interesting and not subsumed by previous work.

#### Theorem 2.1.

Let $F$ be the class of mixtures of two Gaussians in $\mathbb{R}$, both of which have unit variance. Let $A$ be any function that maps a finite-length sequence of real numbers to a (Lebesgue) measurable density function. Then for every $m \in \mathbb{N}$ and every $\tau > 0$, there exists a density $f \in F$ such that if $X'_1, \ldots, X'_m \sim f$ then $\text{KL}(f \parallel A(X'_1, \ldots, X'_m)) \geq \tau$ with probability at least 0.98.

The intuition behind the theorem is as follows. Let $a \in \mathbb{N}$ and consider the set of distributions $(1 - \delta) \cdot \mathcal{N}(0,1) + \delta \cdot \mathcal{N}(a,1)$ where $\delta \ll 1/m$. Any algorithm that draws $m$ samples from such a distribution will likely have all of its samples come from $\mathcal{N}(0,1)$. However, the only way for the KL divergence to be small is if the distribution returned by $A$ has non-negligible mass near the $\mathcal{N}(a,1)$ distribution, which is impossible since the samples provide no information about $a$.

Next we consider $L^p$ distances, and prove a result analogous to Theorem 2.1. The main difference is that the argument uses Gaussians with non-unit variance, which can strongly influence the $L^p$ distance.

#### Theorem 2.2.

Let $F$ be the class of mixtures of two Gaussians in $\mathbb{R}$. Let $A$ be any function that maps a finite-length sequence of real numbers to a (Lebesgue) measurable density function. Then for every $p > 1$, every $m \in \mathbb{N}$, and every $\tau > 0$, there exists a density $f \in F$ such that if $X'_1, \ldots, X'_m \sim f$ then $\|f - A(X'_1, \ldots, X'_m)\|_p \geq \tau$ with probability at least 0.98.

The proofs of Theorems 2.1 and 2.2 appear in Appendix B.1.

#### Remark 2.3.

Theorem 2.1 and Theorem 2.2 are stated for deterministic mappings but it is straightforward to infer from the proofs that the statements also hold for randomized mappings.

---

1 Recall that KL divergence is not symmetric; we only consider using KL divergence in one direction.
2.2 Comparison to parameter estimation

In this section, we observe that neither our upper bound (Theorem 1.5) nor our lower bound (Theorem 1.6) can directly follow from results about parameter estimation for Gaussian mixtures. First, recall that our sample complexity upper bound in Theorem 1.5 has no dependence on the structural properties of the Gaussians in the mixture. Next consider an algorithm that learns a single $d$-dimensional Gaussian, and provides an entrywise guarantee on the covariance matrix. If we use this entrywise guarantee to infer closeness in either KL divergence or TV distance, then we argue that the error must depend on the condition number of the covariance matrix.

We use the notation $\kappa(\Sigma)$ to denote the condition number of the covariance matrix $\Sigma$, i.e., the ratio of the maximum and minimum eigenvalues. See Section 1.5 for other relevant definitions. The proof of the following proposition can be found in Appendix B.2.

Proposition 2.4. Set $\varepsilon = \frac{2}{\kappa(\Sigma)+1}$. There exist two covariance matrices $\Sigma$ and $\hat{\Sigma}$ that are good entrywise approximations (both additively and multiplicatively):

$$|\Sigma_{i,j} - \hat{\Sigma}_{i,j}| \leq \varepsilon \quad \text{and} \quad \hat{\Sigma}_{i,j} \in [1, 1 + 2\varepsilon] \cdot \Sigma_{i,j} \quad \forall i, j,$$

but the corresponding Gaussian distributions are as far as they can get, i.e.,

$$\text{KL}(\mathcal{N}(0, \Sigma) \parallel \mathcal{N}(0, \hat{\Sigma})) = \infty \quad \text{and} \quad \text{TV}(\mathcal{N}(0, \Sigma), \mathcal{N}(0, \hat{\Sigma})) = 1.$$

Thus, given a black-box algorithm that provides an entrywise $\varepsilon$-approximation to the true covariance matrix $\Sigma$, if $\kappa(\Sigma) \geq 2/\varepsilon$, it might output $\hat{\Sigma}$, which does not approximate $\Sigma$ in KL divergence or total variation distance. Thus Theorem 1.5 is not a direct consequence of any parameter estimation algorithm with entrywise covariance guarantees.

One might wonder instead if our lower bound is a direct consequence of existing lower bounds on parameter estimation. We show that this is also not the case: the next proposition shows that there exist Gaussians that are close in TV distance but whose covariance matrices do not satisfy any (multiplicative) entrywise guarantee. Thus, even if a lower bound concludes that a class of algorithms cannot provide entrywise covariance guarantees, it is still possible that that an algorithm in that class can provide guarantees on TV distance. The proof of the proposition can be found in Appendix B.2.

Proposition 2.5. For any $\varepsilon \in (0, 1/2)$, there exist two covariance matrices $\Sigma$ and $\hat{\Sigma}$ such that $\text{TV}(\mathcal{N}(0, \Sigma), \mathcal{N}(0, \hat{\Sigma})) \leq \varepsilon$, but there exist $i, j$ such that, for any $c \geq 1$, $\hat{\Sigma}_{i,j} \notin [1/c, c] \cdot \Sigma_{i,j}$.

3 Compression schemes

The main technique introduced in this paper is using compression for density estimation. An overview of this technique was given in Section 1.4. In this section we provide formal definitions of compression schemes and their usage.

3.1 Definitions of compression schemes

Let $\mathcal{F}$ be a class of distributions over a domain $Z$. Intuitively, a compression scheme for $\mathcal{F}$ involves two agents: an encoder and a decoder.

- The encoder knows a distribution $g \in \mathcal{F}$, and receives $m$ samples from this distribution. She uses her knowledge of $g$ to construct a small message that will be provided to the decoder, and which will suffice for him to construct a distribution that is close to $g$. From the $m$ samples she selects a sequence of length $\tau$, which somehow are representative of $g$. This sequence, together with $t$ additional bits, constitutes the message provided to the decoder.

- The decoder receives the message provided by the encoder—the $\tau$ samples and $t$ bits—and constructs a distribution that is close to $g$.

Of course, there is some probability that the samples are not at all representative of the distribution $g$, in which case the compression scheme will fail. Thus, we only require that the decoding succeed with constant probability.

Observe that the use of samples in the message is critical and they cannot be simply substituted by a finite number of bits (e.g., encoding a real number has infinite bit complexity in general). In fact, if we allow the use of bits only, then the resulting compression schemes will be quite weak and not applicable to classes with infinite metric entropy.

A formal definition of a decoder is as follows.
Definition 3.1 (decoder). A decoder for $\mathcal{F}$ is a deterministic function $\mathcal{J} : \bigcup_{n=0}^{\infty} \mathbb{Z}^n \times \bigcup_{n=0}^{\infty} \{0,1\}^n \rightarrow \mathcal{F}$, which takes a finite sequence of elements of $Z$ and a finite sequence of bits, and outputs a member of $\mathcal{F}$.

A formal definition of a compression scheme is as follows.

Definition 3.2 (robust compression schemes). Let $\tau, t, m : (0,1) \rightarrow \mathbb{Z}_{\geq 0}$ be functions, and let $r \geq 0$. We say $\mathcal{F}$ admits $(\tau, t, m)$ $r$-robust compression if there exists a decoder $\mathcal{J}$ for $\mathcal{F}$ such that for any distribution $g \in \mathcal{F}$, and for any distribution $q$ on $Z$ with $\|g - q\|_1 \leq r$, the following holds:

- For any $\varepsilon \in (0, 1)$, if a sample $S$ is drawn from $q^{m(\varepsilon)}$, then with probability at least $2/3$, there exists a sequence $L$ of at most $\tau(\varepsilon)$ elements of $S$, and a sequence $B$ of at most $t(\varepsilon)$ bits, such that $\|\mathcal{J}(L, B) - g\|_1 \leq \varepsilon$.

Note that $S$ and $L$ are sequences rather than sets; in particular, they can contain repetitions. Lastly, note that $m(\varepsilon)$ is a lower bound on the number of samples needed, whereas $\tau(\varepsilon), t(\varepsilon)$ are upper bounds on the size of compression and the number of bits.

To summarize, the definition asserts that with probability $2/3$, there is a (short) sequence $L$ of elements from $S$ and a (short) sequence $B$ of additional bits, from which $g$ can be approximately reconstructed. We say that the distribution $g$ is “encoded” by the message $(L, B)$. This compression scheme is called “robust” since it requires $g$ to be approximately reconstructed from a sample generated from $q$ rather than $g$ itself. A 0-robust compression scheme is called a (non-robust) compression scheme.

Remark 3.3. In the preceding definition we required that $L$ and $B$ exist with probability only $2/3$. Naturally, one can boost this probability to $1 - \delta$ by generating a sample of size $m(\varepsilon) \log_3(1/\delta)$.

3.2 Connection between compression and learning

We now show that if a class of distributions has a (robust) compression scheme, then it can be learned in the (agnostic) density estimation model.

The main idea is as follows. An encoder cannot be implemented in the density estimation model because she requires knowledge of the distribution $g$. However, since her interaction with the decoder only amounts to sending a short message, we can explore all possible behaviors of the encoder by brute-force search over all possible messages that she could have sent. When any such message is provided as input to the decoder, he will output some distribution that she could have sent. When any such message is provided as input to the decoder, he will output some distribution $f$. Moreover, for some message (the one that would have been produced by the encoder), the decoder will output an $f$ that is guaranteed to be close to $g$.

Thus, if we collect all possible distributions produced by the decoder on all possible input messages, then the only remaining task is to select the distribution from that collection that is closest to $g$. Fortunately, this task has a known solution in the setting where the distance measure is the TV distance. Formally, we will use the following result, which states that a finite class of size $M$ can be 3-learned in the agnostic setting using $O(\log(M/\delta)/\varepsilon^2)$ samples.

Theorem 3.4. There exists a deterministic algorithm that, given candidate distributions $f_1, \ldots, f_M$, a parameter $\varepsilon > 0$, and $\log(3M^2/\delta)/2\varepsilon^2$ i.i.d. samples from an unknown distribution $g$, outputs an index $j \in [M]$ such that

$$\|f_j - g\|_1 \leq 3 \min_{i \in [M]} \|f_i - g\|_1 + 4\varepsilon,$$

with probability at least $1 - \delta/3$.

This result is essentially proven in [45, Theorem 1]. It immediately follows from [16, Theorem 6.3] and a standard Chernoff bound.

Our approach for relating compression schemes and density estimation, described informally above, is made formal by the following theorem. It uses Theorem 3.4 to select the best distribution that the decoder could output. The proof appears in Appendix C.1.

Theorem 3.5 (compression implies learning). Suppose $\mathcal{F}$ admits $(\tau, t, m)$ $r$-robust compression. Let $\tau'(\varepsilon) := \tau(\varepsilon) + t(\varepsilon)$. Then $\mathcal{F}$ can be max$\{3, 2/r\}$-learned in the agnostic setting using

$$O \left( m\left(\frac{\varepsilon}{6}\right) \log \left(\frac{1}{\delta}\right) + \tau'(\varepsilon/6) \log(m(\varepsilon) \log_3(1/\delta)) + (\log(1/\delta)) \right) = \tilde{O} \left( m\left(\frac{\varepsilon}{6}\right) + \frac{\tau'(\varepsilon/6)}{\varepsilon^2} \right)$$

samples.

If $\mathcal{F}$ admits $(\tau, t, m)$ non-robust compression, then $\mathcal{F}$ can be learned in the realizable setting using the same number of samples.
3.3 Combining compression schemes

To conclude this section, we state a few results showing that compression schemes can be combined in useful ways. These results concern product distributions (which will be useful for axis-aligned Gaussians) and mixture distributions (which will be useful for mixtures of Gaussians).

First, Lemma 3.6 below states that if a class $\mathcal{F}$ of distributions can be robustly compressed, then the class of distributions that are formed by taking products of members of $\mathcal{F}$ can also be robustly compressed. If $p_1,\ldots,p_d$ are distributions over domains $Z_1,\ldots,Z_d$, then $\prod_{i=1}^d p_i$ denotes the standard product distribution over $\prod_{i=1}^d Z_i$. For a class $\mathcal{F}$ of distributions, define

$$\mathcal{F}^d := \left\{ \prod_{i=1}^d p_i : p_1,\ldots,p_d \in \mathcal{F} \right\}.$$  

The following lemma is proven in Appendix C.2.

**Lemma 3.6** (compressing product distributions).

If $\mathcal{F}$ admits $(\tau(\varepsilon), t(\varepsilon), m(\varepsilon))$ r-robust compression then $\mathcal{F}^d$ admits $(d \cdot \tau(\varepsilon/d), d \cdot t(\varepsilon/d), \log_d(3d) \cdot m(\varepsilon/d))$ r-robust compression.

Our next lemma states that if a class $\mathcal{F}$ of distributions can be compressed, then the class of distributions that are formed by taking mixtures of members of $\mathcal{F}$ can also be compressed. The proof can be found in Appendix C.3.

**Lemma 3.7** (compressing mixtures, non-robustly).

If $\mathcal{F}$ admits $(\tau(\varepsilon), t(\varepsilon), m(\varepsilon))$ (non-robust) compression then $k$-mix($\mathcal{F}$) admits $(k \cdot \tau(\varepsilon/3), k \cdot t(\varepsilon/3) + k \log_2(3k/\varepsilon), \frac{48k \log(6k)}{\varepsilon} \cdot m(\varepsilon/3))$ (non-robust) compression.

The preceding lemma shows that non-robust compression of $\mathcal{F}$ implies non-robust compression of $k$-mix($\mathcal{F}$). We do not know whether an analogous statement holds for robust compression. That is, does robust compression of $\mathcal{F}$ imply robust compression of $k$-mix($\mathcal{F}$), for a general class $\mathcal{F}$? Nevertheless, in the next lemma we show that if $\mathcal{F}$ can be robustly compressed, then $k$-mix($\mathcal{F}$) can be learned in the agnostic setting. The proof appears in Appendix C.4.

**Lemma 3.8** (learning mixtures, robustly). Suppose $\mathcal{F}$ admits $(\tau(\varepsilon), t(\varepsilon), m(\varepsilon))$ r-robust compression, and let $\tau'(\varepsilon) := \tau(\varepsilon) + t(\varepsilon)$. Then $k$-mix($\mathcal{F}$) admits $\frac{8}{\epsilon}$-agnostic learning with sample complexity

$$\tilde{O}\left(\frac{km(\varepsilon/10)}{\epsilon} + \frac{k\tau'(\varepsilon/10)\log m(\varepsilon/10)}{\epsilon^2}\right).$$

4 Upper bound: learning mixtures of Gaussians by compression schemes

The main positive results of this paper are sample complexity bounds for learning mixtures of Gaussians (Theorems 1.5 and 1.7). In this section we prove these results by describing compression schemes for a single Gaussian, then applying the techniques of the previous section. To begin, we illustrate the techniques by considering the simple setting of mixtures of axis-aligned Gaussians.

4.1 A simple example: mixtures of axis-aligned Gaussians, non-robustly

In this short section, we give an illustrative use of our compression framework to prove an upper bound of $\tilde{O}(kd/\epsilon^2)$ for the sample complexity of learning mixtures of $k$ axis-aligned Gaussians in the realizability setting. The next section gives a much more general argument that works for general Gaussians in the agnostic setting.

**Lemma 4.1.** The class of single-dimensional Gaussians admits a $(3, O(\log(1/\epsilon)), 3)$ non-robust compression scheme.

**Proof.** Let $c < 1 < C$ be such that $\Pr_{X \sim \mathcal{N}(0,1)}[c < |X| < C] \geq 0.99$. Let $\mathcal{N}(\mu, \sigma^2)$ be the target distribution. We first show how to encode $\sigma$. Let $g_1, g_2 \sim \mathcal{N}(\mu, \sigma^2)$. Then $g = \frac{1}{\sqrt{2}}(g_1 - g_2) \sim \mathcal{N}(0, \sigma^2)$. So with probability at least 0.99, we have $\sigma c < |g| < \sigma C$. Conditioned on this event, we have $\lambda := \sigma/g \in [-1/c, 1/c]$. We now choose...
\[ \hat{\lambda} \in \{0, \pm \varepsilon / 2C^2, \pm 2\varepsilon / 2C^2, \pm 3\varepsilon / 2C^2, \ldots, \pm 1/c \} \] satisfying \(|\hat{\lambda} - \lambda| \leq \varepsilon / (4C^2)\), and encode the standard deviation by \((g_1, g_2, \hat{\lambda})\). The decoder then estimates \(\hat{\sigma} = \lambda g_1 g_2 / \sqrt{2}\). Note that \(|\hat{\sigma} - \sigma| \leq |\hat{\lambda} - \lambda| \leq \varepsilon / (4C)\) and that the encoding requires two sample points and \(O(\log(C^2 / \varepsilon)) = O(\log(1 / \varepsilon))\) bits (for encoding \(\hat{\lambda}\)).

Now we turn to encoding \(\mu\). Let \(g_3 \sim \mathcal{N}(\mu, \sigma^2)\). Then \(|g_3 - \mu| \leq C\sigma\) with probability at least 0.99. We will condition on this event, which implies existence of some \(\eta \in [-C, C]\) such that \(g_3 + \sigma \eta = \mu\). We choose \(\hat{\eta} \in \{0, \pm \varepsilon / 2, \pm 2\varepsilon / 2, \pm 3\varepsilon / 2, \ldots, \pm C\}\) such that \(|\hat{\eta} - \eta| \leq \varepsilon / 4\), and encode the mean by \((g_3, \hat{\eta})\). The decoder estimates \(\hat{\mu} := g_3 + \sigma \hat{\eta}\). Again, note that \(|\hat{\mu} - \mu| = |\sigma \eta - \hat{\eta}| \leq |\sigma \eta - \hat{\eta}| + |\hat{\eta} - \eta| \leq \varepsilon / 2\). Moreover, encoding the mean requires one sample point and \(O(\log(1 / \varepsilon))\) bits.

To summarize, the decoder has \(|\hat{\mu} - \mu| \leq \varepsilon / 2\) and \(|\hat{\sigma} - \sigma| \leq \varepsilon / 2\). Plugging these bounds into Lemma 4.7 gives \(\|\mathcal{N}(\mu, \sigma^2) - \mathcal{N}(\hat{\mu}, \hat{\sigma}^2)\|_1 \leq \varepsilon\), as required.

To complete the proof of Theorem 1.7 in the realizable setting, we note that Lemma 4.1 combined with Lemma 3.6 implies that the class of axis-aligned Gaussians in \(\mathbb{R}^d\) admits an
\[
(O(d), O(d \log(d / \varepsilon)), O(\log(3d))
\]
non-robust compression scheme. (Note that any axis-aligned Gaussian is a product of one-dimensional Gaussians.) Then, by Lemma 3.7, the class of mixtures of \(k\) axis-aligned Gaussians admits an
\[
(O(kd), O(kd \log(d / \varepsilon) + k \log(k / \varepsilon)), O(k \log(k) \log(d / \varepsilon))
\]
non-robust compression scheme. Theorem 3.5 now implies that the class of \(k\)-mixtures of axis-aligned Gaussians in \(\mathbb{R}^d\) can be learned using \(\tilde{O}(kd / \varepsilon^2)\) samples in the realizable setting.

### 4.2 Learning axis-aligned and general Gaussians in the agnostic setting

We now turn to the general case and prove an upper bound of \(\tilde{O}(kd^2 / \varepsilon^2)\) for the sample complexity of learning mixtures of \(k\) Gaussians in \(d\) dimensions, and an upper bound of \(\tilde{O}(kd / \varepsilon^2)\) for the sample complexity of learning mixtures of \(k\) axis-aligned Gaussians, both in the agnostic sense. The heart of the proof is to show that Gaussians have robust compression schemes in any dimension.

**Lemma 4.2.** For any positive integer \(d\), the class \(\mathcal{G}^d\) of \(d\)-dimensional Gaussians admits an
\[
(O(d \log(2d)), O(d^2 \log(2d) \log(d / \varepsilon)), O(d \log(2d))
\]
2/3-robust compression scheme.

**Remark 4.3.** In the special case \(d = 1\), there also exists a \((4, 1, O(1 / \varepsilon))\) (i.e., constant size) \(0.773\)-robust compression scheme using completely different ideas. The proof appears in our technical report. Surprisingly, this compression scheme has constant size, as the value of \(\tau + t\) is independent of \(\varepsilon\) (unlike Lemma 4.2). This scheme could be used instead of Lemma 4.7 in the proof of Theorem 1.7, although it would not improve the sample complexity bound asymptotically.

**Remark 4.4.** The proof of Lemma 4.2 can be amended to give an \(r\)-robust compression schemes for any \(r < 1\), which will change the constant 9 in the above proofs to any constant larger than 6, at the expense of worse constants for \(\tau\), \(t\) and \(m\). This is straightforward but creates additional cumbersome notation, hence we omit the details.

Before proving Lemma 4.2 we show how it can be combined with the previous lemmata to prove our main upper bounds.

**Proof of Theorem 1.5.** Combining Lemma 4.2 with Lemma 3.8 shows that the class of \(k\)-mixtures of \(d\)-dimensional Gaussians is \(9\)-agnostically learnable with sample complexity \(O(kd^2 / \varepsilon^2)\).

**Proof of Theorem 1.7.** Recall that \(\mathcal{G}^d\) denote the class of \(d\)-dimensional Gaussian distributions. Applying Lemma 4.2 for the case \(d = 1\), then Lemma 3.6 shows that \(\mathcal{G}^d\) admits \((O(d), O(d \log(d / \varepsilon)), O(\log(3d)))\) 2/3-robust compression. Lemma 3.8 then implies that the class \(k\text{-mix}(\mathcal{G}^d)\) is \(9\)-agnostically learnable with sample complexity \(\tilde{O}(kd / \varepsilon^2)\), completing the proof.

\[
\]
4.3 Proof of Lemma 4.2

The goal is to obtain a 2/3-robust compression scheme for $G^d$. Accordingly, we consider any target distribution $Q$ for which there exists a Gaussian $\mathcal{N}(\mu, \Sigma)$ satisfying $\|Q - \mathcal{N}(\mu, \Sigma)\|_1 \leq 2/3$. Recall, from [1], that this implies that $\text{TV}(Q, \mathcal{N}(\mu, \Sigma)) \leq 1/3$.

We may assume that $\Sigma$ has full rank, since there is a reduction from the case of rank-deficient $\Sigma$. If the rank of $\Sigma$ is $\rho < d$, then any $X \sim \mathcal{N}(\mu, \Sigma)$ lies in some affine subspace $S$ of dimension $\rho$. Thus, by [1], any $X \sim Q$ lies in $S$ with probability at least 2/3. With high probability, after seeing 10$d$ samples from $Q$, at least $\rho + 1$ points from $S$ will appear in the sample. We encode $S$ using these samples, and for the rest of the process we work in this affine subspace, and discard outside points.

**Definition of $v_1, \ldots, v_d, \Psi$.** Since $\Sigma$ has full rank, we may find an orthogonal set of vectors $v_1, \ldots, v_d$ satisfying $\Sigma = \sum_{i=1}^d v_iv_i^T$. For convenience, let $\Psi = \Sigma^{1/2}$ be the unique positive definite square root of $\Sigma$. Observe that

$$\Psi = \sum_{i=1}^d \frac{v_iv_i^T}{\|v_i\|}, \quad \Sigma^{-1} = \sum_{i=1}^d \frac{v_iv_i^T}{\|v_i\|^2}, \quad \text{and} \quad \Psi^{-1} = \sum_{i=1}^d \frac{v_iv_i^T}{\|v_i\|^3}. \quad (2)$$

We first prove a lemma that is similar to known results in random matrix theory [cf. 32, Corollary 4.1], but is tailored for our purposes. Its proof appears in Appendix D.1. The notation $\text{TV}(Q)$ is defined as the total variation distance between two probability measures. Its proof appears in Appendix D.1. The notation $\text{TV}(Q)$ is defined as the total variation distance between two probability measures.

**Lemma 4.5.** Let $q_1, \ldots, q_m \in \mathbb{R}^d$ be i.i.d. samples from a distribution $Q$ where $\text{TV}(Q, \mathcal{N}(0, I_d)) \leq 2/3$. Let

$$T := \{ \pm q_i : \|q_i\| \leq 4\sqrt{d} \}.\]$$

Then for a large enough absolute constant $C$, if $m \geq Cd(1 + \log d)$ then

$$\Pr \left[ \frac{1}{20} \mathcal{B}_2^d \not\subseteq \text{conv}(T) \right] \leq 1/6.$$

The following lemma is proven in Appendix D.2.

**Lemma 4.6.** Let $C$ be a sufficiently large absolute constant. Suppose that $S$ contains $2m = 2Cd(1 + \log d)$ samples drawn from $Q$, where $\text{TV}(Q, \mathcal{N}(\mu, \Sigma)) \leq 1/3$. Then, with probability at least 2/3, one can encode vectors $\hat{v}_1, \ldots, \hat{v}_d, \hat{\mu} \in \mathbb{R}^d$ satisfying

$$\|\Psi^{-1}(\hat{v}_j - v_j)\| \leq \varepsilon/2d^2 \quad \forall j \in [d], \quad (3)$$

$$\|\Psi^{-1}(\hat{\mu} - \mu)\| \leq \varepsilon/2, \quad (4)$$

using $O(d^3 \log(2d) \log(2d/\varepsilon))$ bits and the points in $S$.

**Lemma 4.2 now follows immediately from the following lemma, which is proven in Appendix D.3.**

**Lemma 4.7.** Suppose that the vectors $\hat{v}_1, \ldots, \hat{v}_d, \hat{\mu} \in \mathbb{R}^d$ satisfy

$$\|\Psi^{-1}(\hat{v}_j - v_j)\| \leq \rho \leq 1/6d \quad \forall j \in [d], \quad (5)$$

$$\|\Psi^{-1}(\hat{\mu} - \mu)\| \leq \zeta. \quad (6)$$

Then

$$\text{TV} \left( \mathcal{N}(\mu, \sum_{i \in [d]} v_iv_i^T), \mathcal{N}(\hat{\mu}, \sum_{i \in [d]} \hat{v}_i\hat{v}_i^T) \right) \leq \frac{\sqrt{9d^3 \rho^2 + \zeta^2}}{2}. \quad (7)$$

5 The lower bound for Gaussians and their mixtures

In this section, we establish a lower bound of $\Omega(d^2/\varepsilon^2)$ for learning a single Gaussian, and then lift it to obtain a lower bound of $\Omega(kd^2/\varepsilon^2)$ for learning mixtures of $k$ Gaussians in $d$ dimensions. Both our lower bounds are for the realizable setting, and therefore also hold in the agnostic setting.

The high-level strategy for our lower bound follows a strategy adopted in earlier work for mixtures of spherical Gaussians [40], The idea is to create a large number of distributions that are pairwise close in KL divergence (roughly $\varepsilon^2$) but pairwise far in TV distance (roughly $\varepsilon$). An application of the following lemma will then yield the desired sample complexity bound.
Lemma 5.1. Let \( \mathcal{F} \) be a class of distributions such that, for all small enough \( \varepsilon > 0 \), there exist distributions \( f_1, \ldots, f_M \in \mathcal{F} \) with
\[
\text{KL} (f_i \parallel f_j) \leq \kappa(\varepsilon) \quad \text{and} \quad \text{TV} (f_i, f_j) > 2\varepsilon \quad \forall i \neq j \in [M].
\]
Then any method that learns \( \mathcal{F} \) to within total variation distance \( \varepsilon \) with success probability at least 2/3 has sample complexity \( \Omega \left( \frac{\log M}{\kappa(\varepsilon) \log (1/\varepsilon)} \right) \).

The preceding lemma is a straightforward consequence of the following result, which is a generalized form of Fano’s inequality. It may be found in [36] Lemma 3.

Lemma 5.2 (Generalized Fano inequality). Let the distributions \( f_1, \ldots, f_M \) satisfy
\[
\text{KL} (f_i \parallel f_j) \leq \beta \quad \text{and} \quad \|f_i - f_j\|_1 > \alpha \quad \forall i \neq j \in [M].
\]
Consider any density estimation method that has an explicit description of the method receives samples from \( f_i \) (although \( i \) is unknown), then outputs an estimate \( \hat{f} \). For each \( i \), define \( e_i := \mathbb{E}\|f_i - \hat{f}\|_1 \) for the case in which the method receives samples from \( f_i \). Then
\[
\max_i e_i \geq \frac{\alpha \log M - n\beta + \log 2}{2\log M}.
\]

Proof (of Lemma 5.1). Consider a distribution learning method for learning \( \mathcal{F} \) with sample complexity \( m(\varepsilon) \), and consider \( M \) distributions \( f_1, \ldots, f_M \) satisfying the hypotheses. The method will receive samples from \( f_j \), where \( j \in [M] \) is unknown. We will amplify its success probability by running it \( k \) times, then apply the generalized Fano inequality.

Perform a sequence of \( k \) trials as follows. In each trial, the method receives \( m(\varepsilon) \) samples from (the same) \( f_j \). The trial is a success if the method outputs some density \( g \) whose TV distance from \( f_j \) is at most \( \varepsilon \). Since the method’s sample complexity is \( m(\varepsilon) \), each trial is a success with probability at least 2/3. After performing the \( k \) trials, there have been \( k \) densities \( g_1, \ldots, g_k \) produced as output. If some \( f_i \) is within TV distance \( \varepsilon \) from at least \( k/2 \) of these outputs, then this \( f_i \) is used as the overall output \( \hat{f} \) of this amplified method; otherwise, \( \hat{f} = f_1 \) is the overall output.

Let \( \mathcal{E} \) be the event that at least \( k/2 \) of the trials were a success. By a standard Chernoff bound, \( \text{Pr}[\mathcal{E}] \geq 1 - \exp(-\Omega(\varepsilon)) \). When event \( \mathcal{E} \) occurs, then at least \( k/2 \) of \( g_1, \ldots, g_k \) have TV distance at most \( \varepsilon \) from the target density \( f_j \), so the overall output must be \( \hat{f} = f_j \), so \( \|f_j - \hat{f}\|_1 = 0 \). Thus, the expected error is
\[
e_j = \mathbb{E}\|f_j - \hat{f}\|_1 \leq \text{Pr}[\mathcal{E}^c] \cdot 2 \leq \exp(-\Omega(\varepsilon)) \quad \forall j \in [M].
\]
The total number of samples is \( n = km(\varepsilon) \), so Lemma 5.2 gives
\[
\max_i e_i \geq \frac{\alpha \log M - n\beta + \log 2}{2\log M} \leq \exp(-\Omega(\varepsilon)).
\]
Choose \( k = \Theta(\log(1/\varepsilon)) \) to be sufficiently large. Rearranging gives \( m(\varepsilon) = \Omega(\log M / \kappa(\varepsilon) \log (1/\varepsilon)) \), as required.

Our main lower bound for learning a single Gaussian is the following result.

Theorem 5.3. Any algorithm that learns the class of \( d \)-dimensional Gaussians in \( \mathbb{R}^d \) in the realizable setting within total variation distance \( \varepsilon \) and with success probability 2/3 has sample complexity \( \Omega \left( \frac{d^2}{\varepsilon^2 \log (1/\varepsilon)} \right) \).

Proof. In order to apply Lemma 5.1, we must create a large number \( M \) of Gaussian distributions whose pairwise KL divergence is at most \( \kappa \), and whose pairwise TV distance is at least \( 2\varepsilon \). We will accomplish this with parameters \( M = 2^{\Omega(d^2)} \) and \( \kappa = O(\varepsilon^2) \), so Lemma 5.1 will yield the desired lower bound.

The existence of these \( M \) distributions will be shown using the probabilistic method. Specifically, let us fix parameters \( r = 9 \) and \( \lambda = \Theta(\varepsilon d^{-1/2}) \). For each \( a \in [M] \), we pick \( U_a \) to be a random matrix of size \( d \times d/r \) with orthonormal columns. From this, we create the distribution
\[
f_a := \mathcal{N}(0, \Sigma_a) \quad \text{where} \quad \Sigma_a = I_d + \lambda U_a U_a^T \quad \forall a \in [M].
\]
To apply Lemma 5.1, we must analyze the pairwise KL divergences and TV distances between \( f_1, \ldots, f_M \).
Bound on KL divergences. This analysis is straightforward since there is a closed-form expression for the KL divergence between any two Gaussians. First, observe that any two $\Sigma_a$ and $\Sigma_b$ have the same spectrum: there are $d/r$ eigenvalues equal to $1 + \lambda$ and the remaining eigenvalues equal 1. Consequently,

$$\log \det (\Sigma_b \Sigma_a^{-1}) = \log (\det \Sigma_b \cdot \det \Sigma_a^{-1}) = 0.$$  \hspace{1cm} (7)

Next observe that

$$\Sigma_a^{-1} = I - \frac{\lambda}{1 + \lambda} U_a U_a^T,$$  \hspace{1cm} (8)

this may be verified simply by multiplying by $\Sigma_a$. Thus

$$2 \cdot \text{KL}(f_a \parallel f_b) = \text{Tr}(\Sigma_a^{-1} \Sigma_b - I) \quad \text{(by (7) and Lemma A.3)}$$

$$= \text{Tr} \left( \left( I - \frac{\lambda}{1 + \lambda} U_a U_a^T \right)(I + \lambda U_b U_b^T) - I \right) \quad \text{(by (8))}$$

$$= \text{Tr} \left( \lambda U_b U_b^T - \frac{\lambda}{1 + \lambda} U_a U_a^T - \frac{\lambda^2}{1 + \lambda} U_a U_a^T U_b U_b^T \right)$$

$$= \lambda \cdot \frac{d}{r} - \frac{\lambda}{1 + \lambda} \cdot \frac{d}{r} - \frac{\lambda^2}{1 + \lambda} \cdot \|U_a U_b\|_F^2$$

$$\leq \frac{\lambda^2 d}{(1 + \lambda)r} \leq \frac{\lambda^2 d}{r} = O(\varepsilon^2).$$

This bound holds with probability 1.

Bound on TV distances. The remaining step is to show that TV$(f_a, f_b) = \Omega(\varepsilon)$ for all $a \neq b$. Then, by scaling $\varepsilon$ by a constant factor, we may apply Lemma 5.1 and complete the proof.

First we provide some intuition on why such an inequality should hold. Let $S_a$ be the subspace spanned by the columns of $U_a$. One would expect that a vector drawn from $\mathcal{N}(0, \Sigma_a)$ should have a slightly larger projection onto $S_a$ than under the latter. Recalling the definition of the TV distance as a supremum over events (see (11)), such an argument would give the desired lower bound on the TV distance. This approach was used in a preliminary version of this work [12, version 2], but it is fairly technical.

Here we use a simpler argument, formulated as Lemma 5.5, which shows that a lower bound on TV$(f_a, f_b)$ can be obtained if $\|U_a^T U_b\|_F^2$ is small. This would hold if the columns of $U_a$ are nearly pairwise orthogonal to the columns of $U_b$, which intuitively should hold since $U_a$ and $U_b$ are chosen randomly. This is formalized in Lemma 5.4 below, which shows that, with positive probability, $\|U_a^T U_b\|_F^2 \leq d/2r$ for all $a \neq b$. Then Lemma 5.5 implies that, for all $a \neq b$,

$$\text{TV}(f_a, f_b) = \Omega \left( \min\{1, \lambda \sqrt{d/r}\} \right) = \Omega(\varepsilon),$$

by our choice of parameters.

The main technical lemma underlying our lower bound is Lemma 5.4. Its proof appears in Appendix E.1

Lemma 5.4. Suppose $d \geq r \geq 9$. There exists $M = 2^{O(d^2/r)}$ such that the following holds. Let the matrices $U_a$, for $a \in [M]$, be independently chosen with size $d \times d/r$ and with orthonormal columns. Then, with positive probability, we have $\|U_a^T U_b\|_F^2 \leq d/2r$ for all $a \neq b$.

Lemma 5.5. Suppose that $\lambda \leq 1/4$. If $\|U_a^T U_b\|_F^2 \leq d/2r$, then TV$(f_a, f_b) = \Omega \left( \min\{1, \lambda \sqrt{d/r}\} \right)$.

Proof. This proof relies on the following approximate characterization of the TV distance between two zero-mean Gaussians. For any two symmetric positive definite matrices $\Sigma_a$ and $\Sigma_b$ of the same size,

$$\text{TV}(\mathcal{N}(0, \Sigma_a), \mathcal{N}(0, \Sigma_b)) = \Theta \left( \min\{1, \|\Sigma_a^{-1} \Sigma_b - I\|_F\} \right).$$

This result appears in [28, Theorem 1.1]; see also [3, Corollary 2]. Hence to complete the proof it suffices to show that $\|\Sigma_a^{-1} \Sigma_b - I\|_F \geq \lambda \sqrt{d/r}/2$. From (9) we have $\Sigma_a^{-1} \Sigma_b - I = \lambda U_b U_b^T - \frac{\lambda}{1 + \lambda} U_a U_a^T - \frac{\lambda^2}{1 + \lambda} U_a U_a^T U_b U_b^T$, and thus by the triangle inequality,

$$\|\Sigma_a^{-1} \Sigma_b - I\|_F \geq \lambda \|U_b U_b^T - U_a U_a^T\|_F - \frac{\lambda^2}{1 + \lambda} \|U_a U_a^T\|_F - \frac{\lambda^2}{1 + \lambda} \|U_a U_a^T U_b U_b^T\|_F.$$  \hspace{1cm} (10)
We control each term on the right-hand-side here. Recall that for any matrix $A$, we have $\|A\|_F^2 = \text{Tr}(AA^T) = \text{Tr}(A^TA) = \|A^T\|_F^2$, so in particular $\|U_a\|_F^2 = \text{Tr}(U_aU_a^T) = \text{Tr}(U_a^TU_a) = \text{Tr}(I_{d/r}) = d/r$. For the first term in (10), since $U_bU_b^T - U_aU_a^T$ is symmetric we have

$$\|U_bU_b^T - U_aU_a^T\|_F^2 = \text{Tr}((U_bU_b^T - U_aU_a^T)(U_bU_b^T - U_aU_a^T))$$

$$= \text{Tr}(U_bU_b^TU_bU_b^T) + \text{Tr}(U_aU_a^TU_aU_a^T) - \text{Tr}(U_bU_b^TU_aU_a^T) - \text{Tr}(U_aU_a^TU_bU_b^T)$$

$$\leq \text{Tr}(U_bU_b^TU_bU_b^T) + \text{Tr}(U_aU_a^TU_aU_a^T) - \text{Tr}(U_bU_b^TU_bU_b^T) - \text{Tr}(U_aU_a^TU_aU_a^T)$$

$$= d/r + d/r - \|U_aU_a^T\|_F^2 - \|U_aU_a^T\|_F^2$$

$$\geq 2d/r - 2d/2r = d/r.$$ 

Second, we have

$$\|U_aU_a^T\|_F^2 = \|U_a^TU_a\|_F^2 = \|I_{d/r}\|_F^2 = d/r.$$ 

And finally, using the cyclic property of the trace,

$$\|U_aU_a^T\|_F^2 = \text{Tr}(U_aU_a^TU_aU_a^T) = \text{Tr}(U_aU_b(U_bU_b^TU_aU_a^T) = \text{Tr}(U_a^TU_aU_a^TU_a) = \|U_aU_a^T\|_F^2 \leq d/2r.$$ 

Plugging these back into (10) and noting $\lambda \leq 1/4$, we find

$$\|\Sigma_a^{-1}\Sigma_b - I\|_F \geq \lambda \sqrt{d/r} - \lambda^2 \sqrt{d/r} - \lambda^2 \sqrt{d/2r} \geq \lambda \sqrt{d/r}/2,$$

completing the proof of the lemma.

Finally, in Appendix E.2 we prove our lower bound for mixtures.

**Theorem 5.6.** Any algorithm that learns the class of mixtures of $k$ Gaussians in $\mathbb{R}^d$ in the realizable setting within total variation distance $\varepsilon$ and with success probability at least $2/3$ has sample complexity $\Omega\left(\frac{kd^2}{\varepsilon^4 \log(1/\varepsilon)}\right)$.

# 6 Discussion and open problems

This work has drawn a connection between distribution learning and compression. Another concept related to compression is that of core-sets. The idea of core-sets is to “summarize” the training data (using a small subset of them) by a factor of $\frac{d}{r}$, so in particular $\|U_a\|_F^2 = \text{Tr}(U_aU_a^T) = \text{Tr}(U_a^TU_a) = \text{Tr}(I_{d/r}) = d/r$. For the first term in (10), since $U_bU_b^T - U_aU_a^T$ is symmetric we have

$$\|U_bU_b^T - U_aU_a^T\|_F^2 = \text{Tr}((U_bU_b^T - U_aU_a^T)(U_bU_b^T - U_aU_a^T))$$

$$= \text{Tr}(U_bU_b^TU_bU_b^T) + \text{Tr}(U_aU_a^TU_aU_a^T) - \text{Tr}(U_bU_b^TU_aU_a^T) - \text{Tr}(U_aU_a^TU_bU_b^T)$$

$$\leq \text{Tr}(U_bU_b^TU_bU_b^T) + \text{Tr}(U_aU_a^TU_aU_a^T) - \text{Tr}(U_bU_b^TU_bU_b^T) - \text{Tr}(U_aU_a^TU_aU_a^T)$$

$$= d/r + d/r - \|U_aU_a^T\|_F^2 - \|U_aU_a^T\|_F^2$$

$$\geq 2d/r - 2d/2r = d/r.$$ 

And finally, using the cyclic property of the trace,

$$\|U_aU_a^T\|_F^2 = \text{Tr}(U_aU_a^TU_aU_a^T) = \text{Tr}(U_a^TU_aU_a^TU_a) = \|U_aU_a^T\|_F^2 \leq d/2r.$$ 

Plugging these back into (10) and noting $\lambda \leq 1/4$, we find

$$\|\Sigma_a^{-1}\Sigma_b - I\|_F \geq \lambda \sqrt{d/r} - \lambda^2 \sqrt{d/r} - \lambda^2 \sqrt{d/2r} \geq \lambda \sqrt{d/r}/2,$$

completing the proof of the lemma.

| Page 3 for the relevant definitions. |

**Conjecture 6.1.** The minimax estimation rate for $\mathcal{G}_{d,k}$ is $\Theta(\sqrt{kd^2/n})$ and that for $\mathcal{A}_{d,k}$ is $\Theta(\sqrt{kd/n})$.

Note that this conjecture is true for the case $k = 1$ (see [17]). For general $k$, the lower bound for $\mathcal{G}_{d,k}$ follows from the proof of Theorem 1.6 (see Theorem 5.3), and for $\mathcal{A}_{d,k}$ it follows from the lower bound proof in [40]. The upper bounds hold up to polylogarithmic factors (see Theorem 1.5 and Theorem 1.7).

**Learning mixtures of sparse Gaussians**

Our sample complexity lower and upper bounds (in both axis-aligned and general cases) differ by multiplicative polylogarithmic factors. Can one remove these factors? In this direction, we propose the following conjecture. (See Page 3 for the relevant definitions.)

**Conjecture 6.1.** The minimax estimation rate for $\mathcal{G}_{d,k}$ is $\Theta(\sqrt{kd^2/n})$ and that for $\mathcal{A}_{d,k}$ is $\Theta(\sqrt{kd/n})$.

Note that this conjecture is true for the case $k = 1$ (see [17]). For general $k$, the lower bound for $\mathcal{G}_{d,k}$ follows from the proof of Theorem 1.6 (see Theorem 5.3), and for $\mathcal{A}_{d,k}$ it follows from the lower bound proof in [40]. The upper bounds hold up to polylogarithmic factors (see Theorem 1.5 and Theorem 1.7).

Our main results imply that the sample complexity for learning mixtures of *axis-aligned* Gaussians is smaller than that of mixtures of general Gaussians by a factor of $\Theta(d)$. Can one interpolate between these two extremes by exploiting some notion of sparsity of the target distribution?

Consider the class of $d$-dimensional Gaussians whose inverse covariance matrices have at most $m$ off-diagonal nonzero entries. The sample complexity of learning with respect to this class is known to be $\tilde{\Theta}((m+d)/\varepsilon^2)$ (see [17]). Note
that \( m \) measures the amount of correlation between the Gaussian components: if we build a probabilistic graphical model (also known as a Markov random field) whose nodes are the Gaussian components, then an axis-aligned Gaussian corresponds to an empty graph with no correlation between the components, in which case \( m = 0 \), and a Gaussian with fully correlated components corresponds to the complete graph, in which case \( m = \binom{d}{2} \). In general, \( m \) counts the number of edges in this graph (see, e.g., [30, Section 5.1.3] for a proof).

This result on learning single Gaussians and the fact that in some applications the underlying Gaussians have some sparsity structure, motivates the following question: can one extend the bound of [17] from single Gaussians to mixtures of Gaussians, obtaining sample complexity bounds that depend on some notion of sparsity of the mixture components?

**Polynomial time algorithms for learning mixtures of Gaussians**

The running time of our density estimation algorithm is \( 2^{kd^2} \text{polylog}(d,k,1/\varepsilon,1/\delta) \), which is not polynomial in the problem parameters. An important open question is whether or not there exists an algorithm for learning mixtures of Gaussians that runs in time \( \text{poly}(k,d,1/\varepsilon) \) (see also [19, Open Problem 15.5]). If the covariance matrices for all the Gaussians are multiples of the identity matrix (known as spherical Gaussians), [40] gives an algorithm with running time that is polynomial in \( d \) and \( 1/\varepsilon \), but exponential in \( k \). On the other hand, for mixtures of general Gaussians, it is shown in [21] that no polynomial time (in all the parameters) algorithm exists for the case that the learner has access to the distribution only via *statistical queries*. (See [21] for the definition of this model.)

**What if \( k \) is not known?**

Our density estimation algorithms assume that \( k \) is given as input, while in some applications \( k \) might be unknown. One approach to solve this issue is to do a binary-search type algorithm: run our algorithm for \( k = 1, 2, 4, 8, \ldots \), and stop as soon as the output of our algorithm has total variation distance less than \( \varepsilon \) with the target distribution. The trouble with this approach is that it is not clear how to compute or approximate this total variation distance. Moreover, it is not clear how to apply this approach to the robust learning scenario.

**Is robust compression closed under taking mixtures?**

Lemma 3.7 states that for any distribution class \( \mathcal{F} \), non-robust compression of \( \mathcal{F} \) implies non-robust compression of \( k\text{-mix}(\mathcal{F}) \). Does an analogous statement hold for robust compression? That is, does robust compression of \( \mathcal{F} \) imply robust compression of \( k\text{-mix}(\mathcal{F}) \), for a general class \( \mathcal{F} \)?

**Sample complexity for learning with respect to the KL divergence**

Our Theorem 2.1 states that there does not exist a function \( g(k,d,\varepsilon) \) such that there exists an algorithm that upon receiving \( g(k,d,\varepsilon) \) i.i.d. samples from an unknown \( k\)-mixture of \( d\)-dimensional Gaussians \( f \), outputs \( \hat{f} \) such that \( \text{KL}(f \parallel \hat{f}) \leq \varepsilon \) with probability more than 0.02. Recalling that the KL divergence is non-symmetric, we pose the following open question: what is the smallest function \( g(k,d,\varepsilon,\delta) \) such that there exists an algorithm that upon receiving \( g(k,d,\varepsilon,\delta) \) i.i.d. samples from an unknown \( k\)-mixture of \( d\)-dimensional Gaussians \( f \), outputs \( \hat{f} \) such that \( \text{KL}(\hat{f} \parallel f) \leq \varepsilon \) with probability at least \( 1 - \delta \)?

**Characterizing the sample complexity of learning a class of distributions**

A central open problem in distribution learning and density estimation is characterizing the sample complexity of learning a distribution class ([19, Open Problem 15.1]). An insight from supervised learning theory is that the sample complexity of learning a class (of concepts, functions, or distributions) is typically proportional, up to logarithmic factors, to (some notion of) intrinsic dimension of that class divided by \( \varepsilon^2 \), where \( \varepsilon \) is the error tolerance. For the case of binary classification, the intrinsic dimension is captured by the VC-dimension of the concept class (see [12, 10]). For the case of distribution learning with respect to ‘natural’ parametric classes, we expect this dimension to be equal to the number of parameters. This is indeed true for the class of Gaussians (which have \( d^2 \) parameters) and axis-aligned Gaussians (which have \( d \) parameters), and we showed in this paper that it holds for their mixtures as well (which have \( kd^2 \) and \( kd \) parameters, respectively).
While it may first seem that the VC-dimension of the Yatracos set associated with a class of distributions can characterize its sample complexity, it is not hard to come up with examples where this VC-dimension is infinite while the class can be learned with finite samples. Covering numbers do not characterize the sample complexity either: for instance, the class of Gaussians does not have a finite covering number in the TV metric, nevertheless it is learnable with finitely many samples. Thus, we leave characterizing the sample complexity of learning a class of distributions as an important open problem.

Do learnable classes have bounded compression schemes?

In binary classification, the combinatorial notion of Littlestone-Warmuth compression has been shown to be sufficient and necessary for learning. In this work, we showed that the new but related notion of distribution compression is sufficient for distribution learning (Theorem 3.5). Whether the existence of compression schemes is necessary for learning an arbitrary class of distributions remains an intriguing open problem. In this direction, we conjecture the following converse for Theorem 3.5. Let \( m_F(\varepsilon, \delta) \) denote the sample complexity function associated with learning the class \( F \) of distributions (see Definition 1.2).

**Conjecture 6.2.** There exists an absolute constant \( C \) such that any class \( F \) admits an \( (C\varepsilon^2 m(\varepsilon))^{2/3}C\varepsilon^2 m(\varepsilon)\log(m(\varepsilon))^2/C\varepsilon m(\varepsilon) \log(m(\varepsilon))^2) \) (non-robust) compression, where \( m(\varepsilon) := m_F(\varepsilon, \frac{1}{2}) \).

The value \( \varepsilon^2 m(\varepsilon) \) is a candidate for the notion of ‘intrinsic dimension’ of the class. We also propose the following weaker conjecture.

**Conjecture 6.3.** There exists fixed polynomials \( P, Q \) and \( R \) such that any class \( F \) admits a \( (P(\varepsilon^2 m(\varepsilon), \log(1/\varepsilon)), Q(\varepsilon^2 m(\varepsilon), \log(1/\varepsilon)), R(m(\varepsilon), 1/\varepsilon)) \) (non-robust) compression, where \( m(\varepsilon) := m_F(\varepsilon, \frac{1}{2}) \).

**Acknowledgments**

We thank Yaoliang Yu for pointing out a mistake in an earlier version of this paper, and Luc Devroye for fruitful discussions. Abbas Mehrabian was supported by a CRM-ISM postdoctoral fellowship and an IVADO-Apogée-CFREF postdoctoral fellowship. Nicholas Harvey and Hassan Ashtiani were supported by NSERC Discovery Grants. Christopher Liaw was supported by a NSERC graduate award. Yaniv Plan was supported by NSERC grant 22R23068.

**A Standard results**

**Definition A.1.** Let \( A \) and \( B \) be symmetric, positive definite matrices of the same size. The log-det divergence of \( A \) and \( B \) is defined as \( \text{LD}(A, B) := \text{Tr}(B^{-1}A) - \log \det(B^{-1}A) \).

**Claim A.2.** Let \( A, B \) and \( C \) be square matrices of the same size. Suppose that \( A \) and \( B \) are symmetric, positive definite and \( C \) is invertible. Then \( \text{LD}(A, B) = \text{LD}(CAC, CBC) \).

**Proof.** From the definition it is apparent that \( \text{LD}(A, B) \) only depends on the spectrum of \( B^{-1}A \). So the claim follows from the fact that \( B^{-1}A \) and \( (CBC)^{-1}CAC \) have the same spectrum. This fact holds because \( v \) is an eigenvector for \( B^{-1}A \) of eigenvalue \( \lambda \) if and only if \( C^{-1}v \) is an eigenvector for \( (CBC)^{-1}CAC \) of eigenvalue \( \lambda \).

**Lemma A.3** (Rasmussen and Williams [36 Equation A.23]). For two full-rank Gaussians \( \mathcal{N}(\mu_0, \Sigma_0) \) and \( \mathcal{N}(\mu_1, \Sigma_1) \), their KL divergence is

\[
\text{KL} (\mathcal{N}(\mu_0, \Sigma_0) \parallel \mathcal{N}(\mu_1, \Sigma_1)) = \frac{1}{2} \left( \text{Tr}(\Sigma_1^{-1} \Sigma_0) + (\mu_0 - \mu_1)^T \Sigma_1^{-1} (\mu_0 - \mu_1) - \log \det(\Sigma_0 \Sigma_1^{-1}) \right)
\]

**Lemma A.4.** Let \( A, B \) be symmetric, positive definite matrices, satisfying \( (1 - \alpha)B \leq A \leq (1 + \alpha)B \) for some \( \alpha \in [0, 1/2] \). Then \( \text{LD}(A, B) \leq \alpha a^2 \).
Proof. Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of $B^{-1}A$. By the hypothesis, each $\lambda_i \in [1 - \alpha, 1 + \alpha]$. So,

$$\text{LD} (A, B) = \text{Tr}(B^{-1}A - I) - \log \det(B^{-1}A) = \sum_{i=1}^{d} (\lambda_i - 1) - \log \prod_{i=1}^{d} \lambda_i$$

$$= \sum_{i=1}^{d} (\lambda_i - 1 - \log(\lambda_i)) \leq \sum_{i=1}^{d} (\lambda_i - 1)^2 \leq d\alpha^2.$$ 

The first inequality follows from $x - 1 - \log x \leq (x - 1)^2$, valid for any $x \geq 1/2$. □

**Lemma A.5** (Pinsker's Inequality [[1], Lemma 2.5]). For any two distributions $A$ and $B$, we have $2\text{TV} (A, B)^2 \leq \text{KL} (A \parallel B)$.

**Lemma A.6.** For two full-rank Gaussians $\mathcal{N}(\mu, \Sigma)$ and $\mathcal{N}(\mu', \Sigma')$, their total variation distance is bounded by

$$2\text{TV} (\mathcal{N}(\mu_0, \Sigma_0), \mathcal{N}(\mu_1, \Sigma_1))^2 \leq \text{KL} (\mathcal{N}(\mu_0, \Sigma_0) \parallel \mathcal{N}(\mu_1, \Sigma_1))$$

$$= \frac{1}{2} \left( \text{LD} (\Sigma_0, \Sigma_1) + (\mu_0 - \mu_1)^T \Sigma_1^{-1} (\mu_0 - \mu_1) \right).$$

Proof. Follows from Lemma [[A.3] and Lemma [A.3]. □

**Lemma A.7.** For any $\mu, \sigma, \tilde{\mu}, \tilde{\sigma} \in \mathbb{R}$ with $|\tilde{\mu} - \mu| \leq \varepsilon \sigma$ and $|\tilde{\sigma} - \sigma| \leq \varepsilon \sigma$ and $\varepsilon \in [0, 2/3]$ we have

$$\|\mathcal{N}(\mu, \sigma^2) - \mathcal{N}(\tilde{\mu}, \tilde{\sigma}^2)\|_1 \leq 2\varepsilon.$$ 

Proof. By Lemma [[A.6]

$$4\text{TV} (\mathcal{N}(\tilde{\mu}, \tilde{\sigma}^2), \mathcal{N}(\mu, \sigma^2))^2 \leq \frac{\sigma^2}{\tilde{\sigma}^2} - 1 - \log \left( \frac{\sigma^2}{\tilde{\sigma}^2} \right) + \frac{|\mu - \tilde{\mu}|^2}{\sigma^2} \leq \left( \frac{\sigma}{\tilde{\sigma}} \right)^2 - 1 - \log \left( \frac{\sigma}{\tilde{\sigma}} \right)^2 + \varepsilon^2.$$ 

Since $z := \sigma/\tilde{\sigma} \in [1 - \varepsilon, 1 + \varepsilon]$ and $\varepsilon \leq 2/3$, using the inequality $x^2 - 1 - \log(x^2) \leq 3(x - 1)^2$ valid for all $|x - 1| \leq 2/3$, we find

$$\text{TV} (\mathcal{N}(\tilde{\mu}, \tilde{\sigma}^2), \mathcal{N}(\mu, \sigma^2))^2 \leq \frac{1}{4} (3(z - 1)^2 + \varepsilon^2) \leq \frac{1}{4} (4\varepsilon^2) = \varepsilon^2.$$ 

The lemma follows since the $L^1$ distance is symmetric and is equal to twice the TV distance. □

**Fact A.8.** Let $X$ and $Y$ be arbitrary random variables on the same space. For any function $f$, we have

$$\text{TV} (f(X), f(Y)) \leq \text{TV} (X, Y).$$

Proof. This follows from the observation that $\Pr [f(X) \in A] - \Pr [f(Y) \in A] = \Pr [X \in f^{-1}(A)] - \Pr [Y \in f^{-1}(A)] \leq \text{TV} (X, Y)$, so taking supremum of the left-hand side gives the result. □

**Lemma A.9** ([29], Lemma 1]). Let $X$ have the chi-squared distribution with parameter $d$; that is, $X = \sum_{i=1}^{d} X_i^2$ where the $X_i$ are i.i.d. standard normal. Then,

$$\Pr[X - d \geq 2\sqrt{dt} + 2t] \leq \exp(-t)$$

and

$$\Pr[d - X \geq 2\sqrt{dt}] \leq \exp(-t).$$

The first inequality above implies, in particular, that $\Pr[X \geq 16d] \leq \exp(-3)$ for any positive integer $d$.

**Lemma A.10.** Let $g_1, \ldots, g_m \in \mathbb{R}^d$ be independent samples from $\mathcal{N}(0, I)$. For $\theta \in [0, 1]$,

$$\Pr \left[ \frac{1}{m} \sum_{i=1}^{m} g_i \right]^2 \geq (1 + \theta)d/m \right] \leq \exp(-\theta^2d/9).$$

Proof. Note that $X = \left\| \frac{1}{\sqrt{m}} \sum_{i=1}^{m} g_i \right\|^2$ has the chi-squared distribution with parameter $d$. Applying Lemma [A.9] with $t = \theta^2d/9$ shows that $\Pr[X \geq (1 + \theta)d] \leq \exp(-\theta^2d/9)$. □

**Definition A.11.** A random variable $X$ is said to be $\sigma$-subgaussian if $\Pr[|X| \geq t] \leq 2 \exp(-t^2/\sigma^2)$ for all $t > 0$.
For instance if $X \sim \mathcal{N}(0,1)$ then $X$ is $\sqrt{2}$-subgaussian, see, e.g., Abramowitz and Stegun \cite[formula (7.1.13)]{11}.

**Lemma A.12** (Theorem 3.1.1 in \cite{11}). Let $g \sim \mathcal{N}(0, I_d)$. Then $(\|g\|_2 - \sqrt{d})$ is $O(1)$-subgaussian. Consequently, $(\|g\|_2 - \sqrt{d})_+$ is also $O(1)$-subgaussian.

**Lemma A.13** (Proposition 2.5.2 in \cite{11}). There exist absolute positive constants $C_1, C_2$ with the following properties. A random variable $X$ is $\sigma$-subgaussian if $\sup_{p \geq 1} p^{1/2} (E|X|^p)^{1/p} \leq C_1 \sigma$. Conversely, if $\sup_{p \geq 1} p^{1/2} (E|X|^p)^{1/p} \leq C_2 \sigma$ then $X$ is $\sigma$-subgaussian.

**Lemma A.14** (Hoeffding’s Inequality, Proposition 2.6.1 in \cite{11}). Let $X_1, \ldots, X_n$ be independent, mean-zero random variables and suppose $X_i$ is $\sigma_i$-subgaussian. Then, for some global constant $c > 0$ and any $t \geq 0$,

$$
\Pr \left[ \left\| \sum_{i=1}^{n} X_i \right\| > t \right] \leq 2 \exp \left( -\frac{ct^2}{\sum_{i=1}^{n} \sigma_i^2} \right).
$$

**Lemma A.15.** Let $g_1, \ldots, g_n \sim \mathcal{N}(0,1)$ and $a_1, \ldots, a_n > 0$. Then, there is a global constant $c > 0$ such that for every $t \geq 0$,

$$
\Pr \left[ \left| \sum_{i=1}^{n} a_i g_i^2 - E \left( \sum_{i=1}^{n} a_i g_i^2 \right) \right| > t \right] \leq 2 \exp \left( -c \min \left\{ \frac{t^2}{\sum_{i=1}^{n} a_i^2}, \frac{t}{\max_i a_i} \right\} \right).
$$

**Proof.** Follows from Bernstein’s inequality for subexponential random variables (Theorem 2.8.1 in \cite{11}).

**Theorem A.16** (Gordon’s Theorem, Theorem 5.32 in \cite{11}). For a matrix $A$, $\sigma_{\min}(A)$ denotes the smallest positive singular value of $A$. Let $G$ be an $m \times n$ matrix with entries independently drawn from $\mathcal{N}(0,1)$. Then $\sigma_{\min}(G) \geq \sqrt{m} - \sqrt{n}$.

**Lemma A.17** (Corollary 5.50 in \cite{11}). There exist an absolute constant $C$ with the following property. Let $X_1, \ldots, X_m \sim \mathcal{N}(0, I_d)$, and let $0 < \varepsilon < 1 < t$. If $m \geq C(t/\varepsilon)^2 d$, then we have

$$
\Pr \left[ \left\| \frac{1}{m} \sum_{i=1}^{m} X_i X_i^T - I_d \right\| > \varepsilon \right] < 2 \exp(-t^2 d).
$$

**Definition A.18** ($\varepsilon$-net). Let $\varepsilon \geq 0$. We say $N \subseteq X$ is an $\varepsilon$-net for $X$ in metric $d$ if for each $x \in X$ there exists some $y \in N$ such that $d(x,y) \leq \varepsilon$.

**Lemma A.19** (Corollary 4.2.13 in \cite{11}). For any $\varepsilon \in (0,1]$, there exists an $\varepsilon$-net for $B_2^d$ in $\ell_2$ metric of size $(3/\varepsilon)^d$.

Recall the $\ell_\infty$ metric between $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ is defined as $\max_i |x_i - y_i|$.

**Lemma A.20.** For any $\varepsilon \in (0,1]$ there exists an $\varepsilon$-net for $[-1,1]^d$ in $\ell_\infty$ metric of size $\varepsilon^{-d}$.

**Proof.** Consider a partition of $[-1,1]^d$ into $\varepsilon^{-d}$ cubes of side-length $2\varepsilon$. The cube centers form an $\varepsilon$-net for $[-1,1]^d$ in $\ell_\infty$.

\section{Omitted proofs from Section 2}

\subsection{Omitted proofs from Section 2.1}

In this section, let $\nu$ denote the Lebesgue measure on $\mathbb{R}$. We will first prove \textbf{Theorem 2.1}. To that end, we begin with a simple calculation that will be useful later.

**Claim B.1.** Suppose $I \subseteq \mathbb{R}$ satisfies $\nu(I) \geq \gamma$. Moreover, let $f, h: \mathbb{R} \to \mathbb{R}_{\geq 0}$ be measurable density functions such that $f(x) \geq \beta, h(x) \leq \alpha$ for all $x \in I$ and $f(x) > 0$ for all $x \in \mathbb{R}$. Then $\text{KL}(f \parallel h) \geq \gamma \beta \log(\beta/\alpha) - 1/\varepsilon$.

**Proof.** First, let us write

$$
\text{KL}(f \parallel h) = \int_I f(x) \log \frac{f(x)}{h(x)} \, dx + \int_{I^c} f(x) \log \frac{f(x)}{h(x)} \, dx.
$$

For the first integral, we have

$$
\int_I f(x) \log \frac{f(x)}{h(x)} \, dx \geq \int_I \beta \log \frac{\beta}{\alpha} \, dx \geq \gamma \beta \log(\beta/\alpha).
$$

19
Next, we bound the second integral and show that it has value at least $-1/e$ which completes the proof. Let $F = \int_{I^c} f(x) \, dx$ and $H = \int_{I} h(x) \, dx$. Note that $F > 0$ as $f(x) > 0$ for all $x \in \mathbb{R}$. If $H = 0$ then $h(x) = 0$ almost everywhere on $I^c$ so the second integral is $+\infty$.

So assume that $H > 0$. Then $f/F$ and $h/H$ are densities on $I^c$. Hence, we have

$$\int_{I^c} f(x) \log \frac{f(x)}{h(x)} \, dx = F \int_{I^c} \frac{f(x)}{F} \log \frac{f(x)}{H} \, dx + F \int_{I^c} \frac{f(x)}{F} \log \frac{F}{H} \, dx \geq F \log(F/H),$$

where the inequality is because the KL divergence of two densities is always non-negative. Since $H \leq 1$, we have $-\log(H) \geq 0$ so $F \log(F/H) = F \log F - F \log H \geq F \log F \geq -1/e$.

Combining the two bounds gives the claim. \(\square\)

**Proof of Theorem 2.1.** Fix the function $A$ and view it as an algorithm whose input is the sampled data from the ‘true’ distribution, and whose output is a density function. We will first analyze the behavior of the algorithm when the true distribution is $\mathcal{N}(0,1)$ and show that there exists some $a' \in \mathbb{R}$ for which the algorithm’s output puts almost no probability mass on around $a'$. We then show that if the true distribution is a carefully chosen mixture of $\mathcal{N}(0,1)$ and $\mathcal{N}(a',1)$, then the algorithm’s output does not change with high probability, so it still puts almost no mass on $\mathcal{N}(a',1)$; hence the KL divergence of the output and the true distribution is large.

Define the parameters $\delta = \frac{0.01}{m}$, $\beta = \frac{\delta}{\sqrt{2\pi}} \exp(-1/32)$, and $\alpha = \beta \exp \left(\frac{-4\pi - 4/\beta}{\beta} \right)$.

Let $X_1, \ldots, X_m \sim \mathcal{N}(0,1)$ and set $h = A(X_1, \ldots, X_m)$. Note that $h$ is random. Define the (random) set $H = \{ x \in \mathbb{R} : h(x) > \alpha \}$. Then $\nu(H) \leq 1/\alpha$. For $a \in \mathbb{Z}$, define $I_a = [a - 1/4, a + 1/4]$. Note that the $I_a$ are disjoint intervals. Hence $\sum_{a \in \mathbb{Z}} \nu(I_a \cap H) \leq 1/\alpha$ deterministically so $\mathbb{E} \left[ \sum_{a \in \mathbb{Z}} \nu(I_a \cap H) \right] \leq 1/\alpha$. Note that the left hand side of the inequality is an infinite sum while the right hand side is a finite number. Since expectation is linear, we can find $a' \in \mathbb{Z}$ such that $\mathbb{E} \left[ \nu(I_{a'} \cap H) \right] \leq 1/400$. By Markov’s Inequality, $\nu(I_{a'} \cap H) \leq 1/4$ with probability at least 0.99. We condition on this event.

Define $f = (1 - \delta)\mathcal{N}(0,1) + \delta \cdot \mathcal{N}(a',1)$ and note that for all $x \in I_{a'}$ we have $f(x) \geq \frac{\delta}{\sqrt{2\pi}} \exp(-1/32) = \beta$, and $f$ is positive everywhere. Let $J_{a'} = I_{a'} \setminus H$. Then $\nu(J_{a'}) \geq \nu(I_{a'}) - \nu(I_{a'} \cap H) \geq 1/4$, and for all $x \in J_{a'}$ we have $f(x) \geq \beta$ and $h(x) < \alpha$. So

$$\text{KL}(f \parallel h) \geq \beta \log(\beta/\alpha)/4 - 1/e = \tau,$$

where the inequality is by Claim B.1 and the equality is by definition of $\alpha$. Hence, $\text{KL}(f \parallel h) \geq \tau$ with probability at least 0.99.

Note that $TV(f,\mathcal{N}(0,1)) \leq \delta$. If $S = (X_1, \ldots, X_m)$ and $S' = (X'_1, \ldots, X'_m)$ where $X_i \sim \mathcal{N}(0,1)$ and $X'_i \sim f$ then $TV(S,S') \leq m\delta = 0.01$. Hence, if $h = A(S)$ and $h' = A(S')$ then $TV(h,h') \leq 0.01$ so $\mathbb{P}[\text{KL}(f \parallel h') \geq \tau] \geq \mathbb{P}[\text{KL}(f \parallel h) \geq \tau] - 0.01 \geq 0.98$, completing the proof. \(\square\)

Next, we prove Theorem 2.2.

**Proof of Theorem 2.2.** Define the parameters $\delta = \frac{0.01}{m}$, $\sigma^{p-1} = \frac{\delta^p}{\sqrt{2\pi}} \sqrt{\ln(9/2\pi)}$, and $M = 4\sigma \sqrt{\ln(9/2\pi)}$.

Let $X_1, \ldots, X_m \sim \mathcal{N}(0,1)$ and set $h = A(X_1, \ldots, X_m)$. Note that $h$ is random. Define $H = \{ x \in \mathbb{R} : h(x) \geq \delta/6\alpha \}$. Then $\nu(H) \leq 6\alpha/\delta$. For $a \in \mathbb{Z}$, define the intervals $I_a = [aM - M/4, aM + M/4]$ and note that $I_a$ are disjoint intervals. Hence, $\sum_{a \in \mathbb{Z}} \nu(I_a \cap H) \leq 6\sigma/\delta$ deterministically so $\mathbb{E} \left[ \sum_{a \in \mathbb{Z}} \nu(I_a \cap H) \right] \leq 6\sigma/\delta$. Note that the left hand side of the inequality is an infinite sum while the right hand side of the inequality is a finite number. Since expectation is linear, we can find $a' \in \mathbb{Z}$ such that $\mathbb{E} \left[ \nu(I_{a'} \cap H) \right] \leq M/400$. By Markov’s Inequality, $\nu(I_{a'} \cap H) \leq M/4$ with probability at least 0.99. We condition on this event.

Define $f = (1 - \delta)\mathcal{N}(0,1) + \delta \cdot \mathcal{N}(a',\sigma^2)$. Note that for $x \in I_{a'}$, we have $f(x) \geq \frac{\delta}{\sqrt{2\pi}} \exp(-(M/4)^2/2\sigma^2) = \delta/3\sigma$. Let $J_{a'} = I_{a'} \setminus H$. Then $\nu(J_{a'}) \geq M/2 - M/4 = M/4 = \sigma \sqrt{\ln(9/2\pi)}$ and for all $x \in J_{a'}$, we have $f(x) \geq \delta/3\sigma$ and $h(x) \leq \delta/6\sigma$. So

$$\|f - h\|_p^p \geq \int_{J_{a'}} |f(x) - h(x)|^p \, dx \geq \frac{\delta^p}{(6\alpha)^p \sigma^p \sqrt{\ln(9/2\pi)}} = \frac{\delta^p}{6\sigma \sqrt{2\pi}} = \tau^p,$$

where the last equality is by definition of $\sigma$. Hence, $\|f - h\|_p \geq \tau$ with probability at least 0.99.

Note that $TV(f,\mathcal{N}(0,1)) \leq \delta$. If $S = (X_1, \ldots, X_m)$ and $S' = (X'_1, \ldots, X'_m)$ where $X_i \sim \mathcal{N}(0,1)$ and $X'_i \sim f$ then $TV(S,S') \leq m\delta = 0.01$. Hence, if $h = A(S)$ and $h' = A(S')$ then $TV(h,h') \leq 0.01$ so $\mathbb{P}[\|f - h\|_p \geq c] \geq \mathbb{P}[\|f - h\|_p \geq c] - 0.01 \geq 0.98$. \(\square\)
B.2 Omitted proofs from Section 2.2

Proof of Proposition 2.4. Define

\[ \Sigma = \begin{pmatrix} 1 & -(1 - \varepsilon) \\ -(1 - \varepsilon) & 1 \end{pmatrix} \quad \text{and} \quad \hat{\Sigma} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \]

where \( \varepsilon \in (0, 1/2) \). The eigenvalues of \( \Sigma \) are \( 2 - \varepsilon \) and \( \varepsilon \), so \( \kappa(\Sigma) = \frac{2}{\varepsilon} - 1 \), satisfying the stated condition for \( \varepsilon \). Observe that \( \Sigma \) and \( \hat{\Sigma} \) satisfy the entrywise approximation guarantees in the statement of the theorem. However \( \Sigma \) is non-singular and \( \hat{\Sigma} \) is singular. Thus \( \text{TV} \left( \mathcal{N}(0, \Sigma), \mathcal{N}(0, \hat{\Sigma}) \right) = 1 \) (consider the event that \( X \) lies in the range of \( \hat{\Sigma} \)). Furthermore \( \text{KL} \left( \mathcal{N}(0, \Sigma) \parallel \mathcal{N}(0, \hat{\Sigma}) \right) = \infty \) (consider the formula in Lemma A.3).

Proof of Proposition 2.5. Define

\[ \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \hat{\Sigma} = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}, \]

where \( \varepsilon \in [0, 1/2] \). By Lemma A.3

\[ \text{KL} \left( \mathcal{N}(0, \Sigma) \parallel \mathcal{N}(0, \hat{\Sigma}) \right) = -\log(1 - \varepsilon^2)/2 \leq \varepsilon^2. \]

So \( \text{TV} \left( \mathcal{N}(0, \Sigma), \mathcal{N}(0, \hat{\Sigma}) \right) \leq \varepsilon \) by Pinsker’s inequality (Lemma A.5). But, \( \hat{\Sigma} \) is not an entrywise multiplicative approximation of \( \Sigma \) as \( \Sigma_{1,2} = 0 \) but \( \hat{\Sigma}_{1,2} \neq 0 \).

C Omitted proofs from Section 3

C.1 Proof of Theorem 3.5

We give the proof for the agnostic case. The proof for the realizable case is similar. Let \( q \) be the target distribution from which the samples are being generated. Let \( \alpha = \inf_{f \in \mathcal{F}} \| f - q \|_1 \) be the approximation error of \( q \) with respect to \( \mathcal{F} \). The goal of the learner is to find a distribution \( \hat{h} \) such that \( \| \hat{h} - q \|_1 \leq \max\{3, 2/r\} \cdot \alpha + \varepsilon \).

First, consider the case \( \alpha \leq r \). In this case, we develop a learner that finds a distribution \( \hat{h} \) such that \( \| \hat{h} - q \|_1 \leq 3\alpha + \varepsilon \). Let \( g \in \mathcal{F} \) be a distribution such that

\[ \| g - q \|_1 \leq \frac{\alpha + \varepsilon}{12}. \]  \hfill (11)

Such a \( g \) exists by the definition of \( \alpha \). By assumption, \( \mathcal{F} \) admits \((\tau, t, m)\) compression. Let \( \mathcal{J} \) denote the corresponding decoder. Given \( \varepsilon \), the learner first asks for an i.i.d. sample \( S \sim q^m(\varepsilon/6) \cdot \log_3(2/\delta) \). Recall the definition of robust compression and Remark 3.3 which allows us to amplify the success probability of the decoder. Then, with probability at least \( 1 - \delta/2 \), there exist \( L \in S^{\tau(\varepsilon/6)} \) and \( B \in \{0, 1\}^{t(\varepsilon/6)} \) satisfying the following guarantee: letting \( h^* := \mathcal{J}(L, B) \), we have

\[ \| h^* - g \|_1 \leq \frac{\varepsilon}{6}. \]  \hfill (12)

The learner is of course unaware of \( L \) and \( B \). However, given the sample \( S \), it can try all of the possibilities for \( L \) and \( B \) and create a candidate set of distributions. More concretely, let

\[ H = \{ \mathcal{J}(L, B) : L \in S^{\tau(\varepsilon/6)}, B \in \{0, 1\}^{t(\varepsilon/6)} \}. \]

Note that

\[ |H| \leq \left( m(\varepsilon/6) \log_3(2/\delta) \right)^{\tau(\varepsilon/6)} 2^{t(\varepsilon/6)} \leq \left( m(\varepsilon/6) \log_3(2/\delta) \right)^{\tau(\varepsilon/6)}. \]

Since \( H \) is finite, we can use the algorithm of Theorem 3.4 to find a good candidate \( \hat{h} \) from \( H \). In particular, we set the accuracy parameter in Theorem 3.4 to be \( \varepsilon/16 \) and the confidence parameter to be \( \delta/2 \). In this case, Theorem 3.4 requires

\[ \frac{\log(6|H|^2/\delta)}{2(\varepsilon/16)^2} = O \left( \frac{\tau(\varepsilon/6) \log(m(\varepsilon/6) \log_3(1/\varepsilon)) + \log(1/\varepsilon)}{\varepsilon^2} \right) = O(\tau(\varepsilon/6)/\varepsilon^2) \]
additional samples, and its output $\hat{h}$ satisfies the following guarantee:

$$\|\hat{h} - q\|_1 \leq 3\|h^* - q\|_1 + 4\frac{\epsilon}{16} \quad \text{(by Theorem 3.4)}$$

$$\leq 3\|h^* - q\|_1 + \|q - g\|_1 + \frac{\epsilon}{4}$$

$$\leq 3\left(\frac{\epsilon}{6} + \left(\alpha + \frac{\epsilon}{12}\right)\right) + \frac{\epsilon}{4} \quad \text{(by (11) and (12))}$$

$$= 3\alpha + \epsilon.$$

Note that the above procedure uses $O(m(\epsilon/6) + \tau(\epsilon/6)/\epsilon^2)$ samples, and the probability of failure is at most $\delta$. That is, the probability of either $H$ not containing a good $h^*$, or the failure of Theorem 3.4 in choosing a good candidate among $H$, is bounded by $\delta/2 + \delta/2 = \delta$.

The other case, $\alpha > r$, is trivial: the learner outputs some distribution $\hat{h}$. Since $\hat{h}$ and $q$ are density functions, we have $\|\hat{h} - q\|_1 \leq 2 < \frac{\epsilon}{r} \cdot \alpha < \max\{3, 2/r\} \cdot \alpha + \epsilon$.

## C.2 Proof of Lemma 3.6

The following proposition is standard and can be proved, e.g., using the coupling characterization of the total variation distance.

**Proposition C.1** (Lemma 3.3.7 in [37]). For $i \in [d]$, let $p_i$ and $q_i$ be probability distributions over the same domain $Z$. Then $\|\Pi_{i=1}^d p_i - \Pi_{i=1}^d q_i\|_1 \leq \frac{\epsilon}{d}$.

**Proof of Lemma C.1** Let $G = \Pi_{i=1}^d g_i$ be an arbitrary element of $\mathcal{F}^d$, with all $g_i \in \mathcal{F}$. Let $Q$ be an arbitrary distribution over $Z^d$, subject to $\|G - Q\|_1 \leq \epsilon$. Let $q_1, \ldots, q_d$ be the marginal distributions of $Q$ on the $d$ components. Observe that $\|q_j - g_j\|_1 \leq r$ for each $j \in [d]$, since Fact A.8 implies that projection onto a coordinate cannot increase the total variation distance.

The lemma’s hypothesis is that $\mathcal{F}$ admits $(\tau, \epsilon, \alpha, \beta)$ $r$-robust compression. Let $J$ denote the corresponding decoder, let $m_0 := m(\epsilon/d)\log_2(3d)$, and $S \sim Q^m$. To prove the lemma we must encode an $\epsilon$-approximation of $G$ using $d \cdot \tau(\epsilon/d)$ elements of $S$ and $d \cdot \tau(\epsilon/d)$ bits.

Since $S$ contains $m_0$ samples, each of which is a $d$-dimensional vector, we may think of $S$ as a $d \times m_0$ matrix over $Z$. Let $S_i$ denote the $i$th row of this matrix. That is, for $i \in [d]$, let $S_i \in Z^{m_0}$ be the vector of the $i$th components of all elements of $S$. By definition of $q_i$, we have $S_i \sim q_i^{m_0}$ for each $i$. As observed above, we have $\|q_i - q_i\|_1 \leq r$.

Apply Remark 3.3 with parameters $\epsilon/d$ and $\delta = 1/3d$ for each $i \in [d]$. Then, for each $i$, the following statement holds with probability at least $1 - 1/3d$: there exists a sequence $L_i$ of at most $\tau(\epsilon/d)$ elements of $S_i$, and a sequence $B_i$ of at most $\tau(\epsilon/d)$ bits, such that $\|\mathcal{J}(L_i, B_i) - g_i\|_1 \leq \epsilon/d$. By the union bound, this statement holds simultaneously for all $i \in [d]$ with probability at least $2/3$. We may encode these $L_1, \ldots, L_d, B_1, \ldots, B_d$ using $d \cdot \tau(\epsilon/d)$ samples from $S$ and $d \cdot \tau(\epsilon/d)$ bits. Our decoder for $\mathcal{F}^d$ then extracts $L_1, \ldots, L_d, B_1, \ldots, B_d$ from these samples and bits, and then outputs $\prod_{i=1}^d \mathcal{J}(L_i, B_i) \in \mathcal{F}^d$. Finally, Proposition C.1 gives

$$\|\Pi_{i=1}^d \mathcal{J}(L_i, B_i) - G\|_1 \leq \sum_{i=1}^d \|\mathcal{J}(L_i, B_i) - g_i\|_1 \leq d \cdot \epsilon/d \leq \epsilon,$$

completing the proof.

## C.3 Proof of Lemma 3.7

Consider any $g \in k$-mix($\mathcal{F}$), so $g = \sum_{i \in [k]} w_i f_i$ for some distributions $f_1, \ldots, f_k \in \mathcal{F}$ and mixing weights $w_1, \ldots, w_k$. Define $m_0 := 48m(\epsilon/3k)k \log(6k)/\epsilon$, and draw $S \sim g^{m_0}$. Then $S$ has the same distribution as the process that performs $m_0$ independent trials as follows: select a component $i$ according to the weights $w_i$, then draw a sample from $f_i$. In the latter process, we may define $S_i$ to be the sequence of samples that were generated using $f_i$. Our encoder for $g$ will discretize the mixing weights, then use the compression scheme for $\mathcal{F}$ to separately encode each $S_i$.

**Encoding the mixing weights.** We encode $w_1, \ldots, w_k$ using bits as follows. Consider an $(\epsilon/3k)$-net in $\ell_\infty$ for $\Delta_k$ of size $(3k/\epsilon)^k$ (see Lemma A.20). Let $(\hat{w}_1, \ldots, \hat{w}_k)$ be an element in the net that has

$$\|w_1, \ldots, w_k\|_\infty \leq \epsilon/3k.$$ (13)

Encoding the element $(\hat{w}_1, \ldots, \hat{w}_k)$ from the net requires only $k \log_2(3k/\epsilon)$ bits.
Encoding $S_i$. For any $i \in [k]$, we say that index $i$ is negligible if $w_i \leq \varepsilon/(6k)$. For any negligible index we will approximate $f_i$ by an arbitrary distribution $\hat{f}_i$. For any non-negligible index we will likely have enough samples from $f_i$ to use the compression scheme for $F$ to encode a distribution $\hat{f}_i$ that approximates $f_i$.

Define $m_1 = m(\varepsilon/3) \log(6k)$. For each non-negligible index $i$, a standard Chernoff bound shows that, with probability at least $1-1/6k$, we have $|S_i| \geq m_1$. By a union bound, this statement holds simultaneously for all non-negligible $i \in [k]$ with probability at least 5/6.

Apply Remark 3.3 with parameters $\varepsilon/3$ and $\delta = 1/6k$ for each non-negligible index $i$. Then, for each such $i$, the following statement holds with probability at least $1-1/6k$: there exist $\tau(\varepsilon/3)$ samples from $S_i$ and $t(\varepsilon/3)$ bits from which the decoder can construct a distribution $\hat{f}_i$ with

$$
\|f_i - \hat{f}_i\|_1 \leq \varepsilon/3.
$$

(14)

By the union bound, this statement holds simultaneously for all non-negligible indices with probability at least 5/6. The encoding consists of these samples and bits for each non-negligible $i$, whereas for negligible $i$ we use the same number of samples and bits, chosen arbitrarily.

By a union bound, the failure probability of the encoding is at most $2 \cdot (1-5/6) = 1/3$.

Complexity of the encoding. The discretized weights require $k \log_2(3k/\varepsilon)$ bits. For each index $i \in [k]$, we use at most $\tau(\varepsilon/3)$ samples and $t(\varepsilon/3)$ bits. Thus, the total number of bits is $k \cdot t(\varepsilon/3) + k \log_2(3k/\varepsilon)$, and the total number of samples is $k \cdot \tau(\varepsilon/3)$.

Decoding. The decoder for $k$-mix($F$) is explicitly given the discretized weights $\hat{w}_1, \ldots, \hat{w}_k$. It is also given, for each index $i$, $\tau(\varepsilon/3)$ samples and $t(\varepsilon/3)$ bits, which it provides to the decoder for $F$, yielding the distribution $\hat{f}_i$. (Recall that, for a negligible index $i$, the distribution $\hat{f}_i$ can be arbitrary.) The decoder outputs the distribution $\sum_i \hat{w}_i \hat{f}_i$.

To complete the proof of the lemma, we will show that $\|\sum_i w_i f_i - \sum_i \hat{w}_i \hat{f}_i\|_1 \leq \varepsilon$ with probability at least 2/3. Let $N \subseteq [k]$ denote the set of negligible components. Recall that the encoder succeeds with probability at least 2/3, in which case the decoded distributions $\hat{f}_i$ will satisfy (14) for each $i \notin N$. We then have

$$
\left\| \sum_{i \in [k]} (\hat{w}_i \hat{f}_i - w_i f_i) \right\|_1 \leq \left\| \sum_{i \in [k]} w_i (\hat{f}_i - f_i) \right\|_1 + \left\| \sum_{i \in [k]} (\hat{w}_i - w_i) \hat{f}_i \right\|_1
$$

$$
\leq \left\| \sum_{i \in N} w_i (\hat{f}_i - f_i) \right\|_1 + \left\| \sum_{i \notin N} w_i (\hat{f}_i - f_i) \right\|_1 + \left\| \sum_{i \in [k]} \hat{w}_i - w_i \right\|_1 \cdot \left\| \hat{f}_i \right\|_1
$$

$$
\leq \sum_{i \in N} w_i \cdot 2 + \sum_{i \notin N} w_i \cdot \frac{\varepsilon}{3} + \sum_{i \in [k]} \varepsilon \cdot 1
$$

$$
\leq k \cdot \frac{\varepsilon}{6k} \cdot 2 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
$$

(by definition of $N$).

This completes the analysis of the compression scheme for $k$-mix($F$).

C.4 Proof of Lemma 3.8

We first give a high-level idea of the proof. Let $g$ be the target distribution and suppose there exists $\rho \geq 0$ and $f \in k$-mix($F$) such that $\|g - f\|_1 \leq \rho$. Since $f \in k$-mix($F$), we can write $f = \sum_{i \in [k]} w_i f_i$, where $f_i \in F$, $w_i \geq 0$, and $\sum_{i \in [k]} w_i = 1$. A first attempt would be to try to write $g = \sum_{i \in [k]} w_i g_i$ such that each $\|g_i - f_i\| \leq r$; if this were true, then given a sufficient number of samples from $f$, we would have sufficient samples from each $f_i$, and then we could use an $r$-robust compression scheme for each $f_i$ to output some $\hat{g}_i$ satisfying $\|g_i - \hat{g}_i\|_1 \leq \varepsilon$. Alas, it is not clear whether we can ensure that $\|g_i - f_i\|_1 \leq r$ for all $i$. However, Lemma C.2 below asserts that we can write $g = \sum_{i \in [k]} w_i g_i$ in such a way that for each $i$, either $\|g_i - f_i\| \leq r$ or $w_i$ is small (in fact, the sum of all such weights is small) and, hence, their contribution to the TV distance is small. Thus, we will only need to deal with the case where $\|g_i - f_i\| \leq r$, a task for which $r$-robust compression is well-suited.

Lemma C.2. Let $g$ be a density and suppose there exists $f = \sum_{i \in [k]} w_i f_i$ with $(w_1, \ldots, w_k) \in \Delta_k$ and each $f_i \in F$ such that $\|g - f\|_1 \leq \rho$ for some $\rho \geq 0$. Then we can write $g = \sum_{i \in [k]} w_i g_i$ such that, for any $r > 0$,

$$
\sum_{i : \|g_i - f_i\|_1 > r} w_i < \rho/r.
$$
Proof. Let $\mathcal{X} := \{x : g(x) < f(x)\}$. Our goal is to “transform” each $f_i$ into another density $g_i$ so that $g = \sum_{i \in [k]} w_i g_i$. To that end, we define

$$g_i(x) := \begin{cases} f_i(x)g(x)/f(x) & \text{for } x \in \mathcal{X}, \\ f_i(x) + \Delta_i(x) & \text{for } x \notin \mathcal{X}, \end{cases}$$

where

$$\Delta_i(x) := (g(x) - f(x)) \left( \int_{\mathcal{X}} f_i(y) \cdot \frac{f(y) - g(y)}{f(y)} \, dy \right) \int_{\mathcal{X}} (f(y) - g(y)) \, dy.$$

Recall that $Z$ is the domain of $g$ and the densities in $\mathcal{F}$. We now check that each $g_i$ is a density and that $g = \sum_{i \in [k]} w_i g_i$.

**Claim C.3.** For all $i \in [k]$, $g_i$ is a density on $Z$.

**Proof.** We first check that $g_i(x) \geq 0$ for all $x$. If $x \in \mathcal{X}$, then $g_i(x) \geq 0$ because $f_i, g, f$ are all densities and hence non-negative. If $x \notin \mathcal{X}$, then $\Delta_i(x) \geq 0$ because $g(x) - f(x) \geq 0$ on $\mathcal{X}^c$ and $f(x) - g(x) \geq 0$ on $\mathcal{X}$. We now check that $\int_Z g_i(x) \, dx = 1$. Since both $g$ and $f$ are densities, both integrate to 1 over $Z$, and therefore

$$\int_{\mathcal{X}^c} (g(x) - f(x)) \, dx = \int_{\mathcal{X}} (f(x) - g(x)) \, dx.$$  \hfill (15)

The following calculation completes the proof.

$$\begin{align*}
\int_{\mathcal{X}^c} g_i(x) \, dx &= \int_{\mathcal{X}^c} (\Delta_i(x) + f_i(x)) \, dx \\
&= \int_{\mathcal{X}^c} (g(x) - f(x)) \, dx \int_{\mathcal{X}} f_i(y) \cdot \frac{f(y) - g(y)}{f(y)} \, dy + \int_{\mathcal{X}^c} f_i(x) \, dx \quad \text{(definition of $\Delta_i$)} \\
&= \int_{\mathcal{X}} f_i(y) \cdot \left( 1 - \frac{g(y)}{f(y)} \right) \, dy + \int_{\mathcal{X}^c} f_i(x) \, dx \quad \text{(by (15))} \\
&= \int_{\mathcal{X}} f_i(y) \, dy - \int_{\mathcal{X}} f_i(y) \cdot \left( \frac{g(y)}{f(y)} \right) \, dy + \int_{\mathcal{X}^c} f_i(x) \, dy \\
&= 1 - \int_{\mathcal{X}} f_i(y) \cdot \left( \frac{g(y)}{f(y)} \right) \, dy \\
&= 1 - \int_{\mathcal{X}} g_i(y) \, dy \quad \text{(definition of $g_i$)}. \quad \square
\end{align*}$$

**Claim C.4.** $g = \sum_{i \in [k]} w_i g_i$.

**Proof.** First suppose $x \in \mathcal{X}$. Since $\sum_{i \in [k]} w_if_i = f$, we have

$$\sum_{i \in [k]} w_i g_i(x) = \sum_{i \in [k]} w_i f_i(x) \frac{g(x)}{f(x)} = g(x).$$

On the other hand, for $x \notin \mathcal{X}$ we have

$$\sum_{i \in [k]} w_ig_i(x) = \sum_{i \in [k]} w_i \Delta_i(x) + w_if_i(x) \quad \text{(definition of $g_i$)}$$

$$= \sum_{i \in [k]} w_i \left( \frac{g(x) - f(x)}{f(y) - g(y)} \right) \int_{\mathcal{X}} f_i(y) \cdot \frac{f(y) - g(y)}{f(y)} \, dy + \sum_{i \in [k]} w_if_i(x) \quad \text{(definition of $\Delta_i$)}$$

$$= \frac{(g(x) - f(x))}{\int_{\mathcal{X}} (f(y) - g(y)) \, dy} \cdot \int_{\mathcal{X}} \left( \sum_{i \in [k]} w_if_i(y) \cdot \frac{f(y) - g(y)}{f(y)} \right) \, dy + f(x)$$

$$= \frac{(g(x) - f(x))}{\int_{\mathcal{X}} (f(y) - g(y)) \, dy} \cdot \int_{\mathcal{X}} (f(y) - g(y)) \, dy + f(x) \quad \text{(since $\sum_{i \in [k]} w_if_i = f$)}$$

$$= g(x) - f(x) + f(x) = g(x). \quad \square$$
Let $I := \{ i \in [k] : \|f_i - g_i\|_1 > r \}$. It remains to show that $\sum_{i \in I} w_i < \rho/r$. Observe from the definition of the $g_i$ that we also have $X = \{ x : g_i(x) < f_i(x) \}$ for each $i \in [k]$. Thus, using Claim C.4

$$\|f - g\|_1 = 2 \int_X (f(x) - g(x)) \, dx = 2 \sum_{i \in [k]} w_i \int_X (f_i(x) - g_i(x)) \, dx = \sum_{i \in [k]} w_i \|f_i - g_i\|_1.$$ 

Thus, from the hypothesis of the lemma,

$$\rho \geq \|f - g\|_1 = \sum_{i \in [k]} w_i \|f_i - g_i\|_1 \geq \sum_{i \in I} w_i \|f_i - g_i\|_1 > \sum_{i \in I} w_i r,$$

by definition of $I$. This gives $\sum_{i \in I} w_i < \rho/r$, as required.

We now turn to proving Lemma 3.8.

**Proof of Lemma 3.8** Let $g$ be the target distribution. Let $f \in k\text{-}mix(F)$ such that $\|f - g\|_1 \leq \rho$ for some $\rho \geq 0$. Let $g = \sum_{i \in [k]} w_i g_i$ be the representation given by Lemma C.2. The learner first takes $M = 160m(\varepsilon/10) \log(3k/\delta)k/\varepsilon$ samples from $g$. Let $S$ be the set of these samples. We view $g$ as a mixture of the $g_i$, so $S$ can be partitioned into $k$ subsets such that the $i$th subset has distribution $g_i$. We learn each of the components individually. The learner does not know which sample point comes from which component, but it can try all possible ways of partitioning $S$ into $k$ subsets, hence generating several ‘candidate distributions’, such that at least one of them is close to $g$. Moreover, the learner also ‘guesses’ the weights $w_i$ as follows: let $W$ be an $(\varepsilon/10k)$-net in $\ell_\infty$ for $\Delta_k$ of size $(10k/\varepsilon)^k$ (see Lemma A.20). So there exists some point $(\hat{w}_1, \ldots, \hat{w}_k) \in W$ such that $\max_i |w_i - \hat{w}_i| \leq \varepsilon/10k$.

We say component $i$ is tiny if $w_i < \varepsilon/20k$, and we say component $i$ is far if $\|g_i - F\|_1 > r$. We say a component is nice if it is neither far nor tiny. The sum of weights of tiny components is at most $\varepsilon/20$, and the sum of weights of far components is at most $\rho/r$ by Lemma C.2.

The number of samples from component $i$ is binomial with mean $Mw_i$. By a Chernoff bound and a union bound over nice components, with probability at least $1 - \delta/3$, there are at least $m(\varepsilon/10) \log(3k/\delta)$ points from each nice component. If this is the case, then the definition of robust compression implies that for each such component $g_i$, with probability at least $1 - \delta/3k$ there exists a sequence $L_i \in S^\tau(\varepsilon/10)$ and a sequence $B_i \in \{0, 1\}^t(\varepsilon/10)$ such that $\|J(L_i, B_i) - g_i\|_1 \leq \varepsilon/10$, where $J$ is the corresponding decoder. By a union bound over nice components, this is simultaneously true for all nice components, with probability at least $1 - \delta/3$.

Thus far we have proved that with probability at least $1 - 2\delta/3$ there exist sequences $L_1, \ldots, L_k \in S^\tau(\varepsilon/10)$ and $B_1, \ldots, B_k \in \{0, 1\}^t(\varepsilon/10)$ such that $\|J(L_i, B_i) - g_i\|_1 \leq \varepsilon/10$ for each nice component $i$. The learner builds the following set of candidate distributions:

$$\mathcal{C} := \left\{ \sum_{i=1}^k \hat{w}_i J(L_i, B_i) : L_1, \ldots, L_k \in S^\tau(\varepsilon/10), B_1, \ldots, B_k \in \{0, 1\}^t(\varepsilon/10), (\hat{w}_1, \ldots, \hat{w}_k) \in W \right\}.$$ 

We claim that with probability at least $1 - 2\delta/3$ at least one of the distributions in $\mathcal{C}$ is $(3\varepsilon/10 + 2\rho/r)$-close to $g$. This corresponds to the ‘correct’ sequences $L_i, B_i$, and $\hat{w}_i$. To show this, set $T$ denote the set of tiny components, let $F$ denote the set of far components, and let $N$ denote the nice components. Then we have

$$\left\| \sum_{i \in [k]} \hat{w}_i J(L_i, B_i) - w_i g_i \right\|_1 \leq \left\| \sum_{i \in [k]} w_i (J(L_i, B_i) - g_i) \right\|_1 + \sum_{i \in [k]} (\hat{w}_i - w_i) J(L_i, B_i) \right\|_1 \leq \sum_{i \in T \cup F} w_i \|J(L_i, B_i) - g_i\|_1 + \sum_{i \in N} w_i \|J(L_i, B_i) - g_i\|_1 + \sum_{i \in [k]} |\hat{w}_i - w_i| \cdot \|J(L_i, B_i)\|_1 \leq (\varepsilon/10 + 2\rho/r) + \varepsilon/10 + \varepsilon/10,$$

by the definitions of tiny and far. This proves the claim.

Next the learner applies the algorithm of Theorem 3.4 (with error parameter $\varepsilon/40$) to obtain a member of $\mathcal{C}$ whose distance from $g$ is bounded by $3 \cdot (3\varepsilon/10 + 2\rho/r) + 4(\varepsilon/40) \leq \varepsilon + 6\rho/r$, as required. The overall failure probability is bounded by $2\delta/3$ (probability of the claim failing) plus $\delta/3$ (the probability that algorithm of Theorem 3.4 fails).
The sample complexity of the algorithm is bounded as follows. The number of candidate distributions can be bounded by
\[ |C| \leq \left(M^{\tau(\varepsilon/10)\log(\varepsilon/10)}\right)^k \cdot (10k/\varepsilon)^k \leq M^{k\tau(\varepsilon/10)} \cdot (10k/\varepsilon)^k, \]
whence the total sample complexity can be bounded by
\[ M + \frac{\log(3|C|^2/\delta)}{2\varepsilon^2} = O\left(m\left(\frac{\varepsilon}{10}\right) \log\left(\frac{k}{\delta}\right) + \log(1/\delta) + k\log(k/\varepsilon) + k\tau(\varepsilon/10) \log\left(m(\frac{\varepsilon}{10}) \log(k/\varepsilon)k\right)\right) \]
\[ = O\left(\frac{km(\varepsilon/10)}{\varepsilon} + \frac{k\tau(\varepsilon/10)\log m(\varepsilon/10)}{\varepsilon^2}\right), \]
completing the proof.

\[ \square \]

## D Omitted proofs from Section 4

### D.1 Proof of Lemma 4.5

Let \( S^{d-1} := \{ y \in \mathbb{R}^d : \|y\| = 1 \} \). Consider the following statement:
\[ \max_{q \in T} |\langle y, q \rangle| \geq \frac{1}{20} \quad \forall y \in S^{d-1}. \] (16)

We first show that (16) implies that \( \frac{1}{30}B_2^d \subseteq \text{conv}(T) \), which is the event that we wish to analyze. Let \( P := \text{conv}(T) \). Its polar is \( P^o = \{ y \in \mathbb{R}^d : |\langle y, q \rangle| \leq 1 \ \forall q \in T \} \). So (16) implies \( P^o \subseteq 20B_2^d \). As polarity reverses containment and the polar of \( B_2^d \) is itself, we obtain \( P \supseteq (20B_2^d)^o = \left(\frac{1}{20}\right)B_2^d \).

We now bound the probability that (16) fails using an \( \varepsilon \)-net argument. For this, fix some \( y \in S^{d-1} \) and let \( g \sim \mathcal{N}(0, I_d) \) and let \( Y_y := \langle y, g \rangle \). Notice that \( Y_y \sim \mathcal{N}(0, 1) \). Since the pdf of \( Y_y \) is bounded above by \( \frac{1}{\sqrt{2\pi}} < 1 \), we have \( \Pr[|Y_y| \leq \frac{1}{10}] \leq 1/5 \). Moreover, by Lemma A.9 \( \Pr[\|g\|_2 > 4\sqrt{d}] \leq \exp(-3) \). Hence by the union bound,
\[ \Pr[|Y_y| \leq \frac{1}{10} \lor \|g\|_2 > 4\sqrt{d}] \leq 1/5 + \exp(-3) < 0.25. \]

Now let
\[ Y_{y,i} := \langle y, q_i \rangle \quad \forall y \in S^{d-1}, i \in [m], \]
and let \( E_{y,i} \) be the event \( \{|Y_{y,i}| \leq \frac{1}{10} \lor \|q_i\| > 4\sqrt{d}\} \). As \( \text{TV}(Q, \mathcal{N}(0, I_d)) \leq 2/3 \), we have \( \Pr[E_{y,i}] \leq 0.25 + 2/3 < 0.92 \). Thus
\[ \Pr\left[\bigwedge_{i \in [m]} E_{y,i}\right] < (0.92)^m. \]

Let \( N \) be an \( (1/80\sqrt{d})\)-net of \( S^{d-1} \) in \( \ell_2 \) with \( |N| \leq (240\sqrt{d})^d \) (see Lemma A.19). By the union bound, since \( m \geq Cd(1 + \log d) \) for \( C \) large enough, with probability at least \( 1 - (240\sqrt{d})^d(0.92)^m \geq 5/6 \), for all \( y \in N \) there exists \( i \in [m] \) such that \( |Y_{y,i}| \geq \frac{1}{10} \) and \( \|q_i\| \leq 4\sqrt{d} \).

To complete the proof, we suppose that this event holds, and show that (16) also holds. Consider any \( y \in S^{d-1} \), and let \( y' \in N \) satisfy \( \|y - y'\|_2 \leq 1/80\sqrt{d} \). Let \( q_i \) be such that \( \|q_i\| \leq 4\sqrt{d} \) and \( |Y_{y',i}| \geq \frac{1}{10} \). These imply that \( \pm q_i \in T \) and that
\[ |Y_{y,i}| \geq |Y_{y',i}| - \frac{|q_i|}{80\sqrt{d}} \geq \frac{1}{10} - \frac{1}{20} = \frac{1}{20}. \]
Thus \( |\langle y, q_i \rangle| \geq 1/20 \), as required.

### D.2 Proof of Lemma 4.6

The samples in \( S \) will be denoted \( X_1, \ldots, X_{2m} \).
Encoding $\hat{v}_j$. We define “normalized” samples

$$Y_i := \frac{1}{\sqrt{2}} \Psi^{-1}(X_{2i} - X_{2i-1}) \quad \forall i \in [m].$$

If we were in the non-robust case, then $X_{2i}$ and $X_{2i-1}$ would both have distribution $\mathcal{N}(\mu, \Sigma)$, so $Y_i$ would have distribution $\mathcal{N}(0, I)$. Instead, both $X_{2i}$ and $X_{2i-1}$ have TV distance at most $1/3$ from $\mathcal{N}(\mu, \Sigma)$. It follows that $Y_i$ has TV distance at most $2/3$ from $\mathcal{N}(0, I)$. (This may be seen, for example, by the coupling definition of TV distance; see [28, Eq. (18.10)].) Define the event

$$\mathcal{E} := \left\{ \frac{1}{C} \mathcal{B}_2 \subseteq \text{conv} \left\{ \pm Y_i : i \in \mathcal{I} \right\} \right\} \quad \text{where} \quad \mathcal{I} := \left\{ i \in [m] : \|Y_i\| \leq 4\sqrt{d} \right\}.$$  

Since $C$ is large, and in particular $C \geq 20$, by Lemma 4.5 we have $\Pr[\mathcal{E}] \geq 5/6$. Our encoding will assume that the event $\mathcal{E}$ occurs.

Fix some $j \in [d]$. Referring to (2), we see that $\Psi^{-1} v_j = v_j/\|v_j\|$ has unit norm. Since $\mathcal{E}$ occurs, we can write

$$\frac{\Psi^{-1} v_j}{C} = \sum_{i \in \mathcal{I}} \theta_{j,i} Y_i$$

for some vector $\theta_j \in [-1, 1]^m$ supported on $\mathcal{I}$. Applying $\Psi$ to both sides, we obtain

$$v_j = \frac{C}{\sqrt{2}} \sum_{i \in \mathcal{I}} \theta_{j,i} (X_{2i} - X_{2i-1}).$$

Consider the natural $(\varepsilon/96Cd^3)$-net for $[-1, 1]^m$ in the $\ell_\infty$ norm, formed by the Cartesian product of 1-dimensional nets (see Lemma A.20). This net has size at most $(96Cd^3/\varepsilon)^m$. Recalling that $m = O(d(1 + \log d))$, it follows that any element of the net can be described using $O(m \log(2d/\varepsilon))$ bits. Let $\hat{\theta}_j$ be an element in the net that is closest to $\theta_j$. Since each $\theta_j$ is supported on $\mathcal{I}$, and the net has the Cartesian product structure, we may choose $\hat{\theta}_j$ also to be supported on $\mathcal{I}$. Define

$$\hat{v}_j := \frac{C}{\sqrt{2}} \sum_{i \in \mathcal{I}} \hat{\theta}_{j,i} (X_{2i} - X_{2i-1}).$$

The error of this encoding is

$$\left\| \Psi^{-1}(\hat{v}_j - v_j) \right\| = \frac{C}{\sqrt{2}} \left\| \sum_{i \in \mathcal{I}} (\theta_{j,i} - \hat{\theta}_{j,i}) \Psi^{-1}(X_{2i} - X_{2i-1}) \right\| \leq \frac{C}{\sqrt{2}} |\mathcal{I}| \left( \max_{i \in \mathcal{I}} |\theta_{j,i} - \hat{\theta}_{j,i}| \right) \left( \max_{i \in \mathcal{I}} \sqrt{2} \|Y_i\| \right).$$

By definition of $\hat{\theta}_j$, we have $\left\| \theta_j - \hat{\theta}_j \right\|_\infty \leq \varepsilon/96Cd^3$. By definition of $\mathcal{I}$, we have $\|Y_i\| \leq 4\sqrt{d}$, leading to the bound

$$\left\| \Psi^{-1}(\hat{v}_j - v_j) \right\| \leq \frac{C}{\sqrt{2}} m \left( \frac{\varepsilon}{96Cd^3} \right) \left( 4\sqrt{2} \sqrt{d} \right) \leq \frac{\varepsilon}{24d^2},$$

establishing (3). The vectors $\hat{v}_1, \ldots, \hat{v}_d$ are encoded simply using $\hat{\theta}_1, \ldots, \hat{\theta}_d$. Each $\hat{\theta}_i$ requires $O(m \log(2d/\varepsilon))$ bits. Recalling that $m = O(d \log(2d))$, the total number of bits required is $O(d^2 \log(2d) \log(2d/\varepsilon))$.

Encoding $\hat{\mu}$. Let $Z_i := \Psi^{-1}(X_i - \mu)$ and observe that $Z_i$ has a distribution with TV distance at most $1/3$ to $\mathcal{N}(0, I)$. Define the event

$$\mathcal{E}' := \left\{ \min \{\|Z_1\|, \|Z_2\| \} \leq 4\sqrt{d} \right\}.$$  

Lemma A.9 implies that

$$\Pr[\|Z_i\| \geq 4\sqrt{d}] \leq \exp(-3) + 1/3 < \sqrt{1/6}.$$  

Thus $\Pr[\mathcal{E}'] \geq 5/6$. Our encoding will assume that the event $\mathcal{E}'$ occurs.

By symmetry assume $\|Z_1\| \leq 4\sqrt{d}$, and suppose $Z_1 = \sum_{j \in [d]} \lambda_j v_j/\|v_j\|$. Thus $\sum \lambda_j^2 \leq 16d^2$. Furthermore, from the definitions of $Z_1$ and $\Psi$ we have

$$\mu = X_1 - \Psi Z_1 = X_1 - \sum_{j \in [d]} \lambda_j v_j.$$
Consider an \((\varepsilon/3d)\)-net for \(4\sqrt{d}B^d_2\) of size \(O(d^{1.5}/\varepsilon)^d\) (see Lemma A.19). Observe that \(\lambda \in 4\sqrt{d}B^d_2\), and let \(\hat{\lambda}\) be the closest element to \(\lambda\) in this net. The encoding is

\[ \hat{\mu} := X_1 - \sum_{j \in [d]} \hat{\lambda}_j \hat{v}_j. \]

The error of this encoding is

\[
\|\Psi^{-1} (\mu - \hat{\mu})\| = \left\| \sum_{j \in [d]} \Psi^{-1} (\lambda_j v_j - \hat{\lambda}_j \hat{v}_j) \right\|
\leq d \cdot \max_{j \in [d]} \left\{ \left| \lambda_j \right| \cdot \|\Psi^{-1} v_j - \Psi^{-1} \hat{v}_j\| + \left| \lambda_j - \hat{\lambda}_j \right| \cdot \|\Psi^{-1} v_j\| \right\},
\]

By definition of \(\hat{\lambda}\), we have \(\left\| \hat{\lambda} \right\| \leq 4\sqrt{d}\) and \(\left\| \lambda - \hat{\lambda} \right\| \leq \varepsilon/3d\). From (2) we have \(\|\Psi^{-1} v_j\| \leq 1\). Lastly, using (17) we have \(\|\Psi^{-1} (\hat{v}_j - v_j)\| \leq \varepsilon/24d^2\), leading to the bound

\[
\|\Psi^{-1} (\mu - \hat{\mu})\| \leq d \cdot \left( 4\sqrt{d} \cdot \frac{\varepsilon}{24d^2} + \frac{\varepsilon}{3d} \cdot 1 \right) \leq \varepsilon/2,
\]

establishing (4). The encoding for \(\hat{\mu}\) consists only of \(\hat{\lambda}\). Since \(\hat{\lambda}\) comes from a net of size \(O(d^{1.5}/\varepsilon)^d\), the number of bits required for the encoding is \(O(d \log(d/\varepsilon))\).

All encodings will succeed so long as both \(E\) and \(E'\) occur, which happens with probability at least \(2/3\).

### D.3 Proof of Lemma 4.7

In this proof, we will use the log-det divergence, which is defined in Definition A.1. Define \(\hat{\Sigma} := \sum_i \hat{v}_i \hat{v}_i^T\). We will show that

\[ \text{LD} \left( \hat{\Sigma}, \Sigma \right) \leq 9d^3 \rho^2. \]  

(18)

If this is true, then Lemma A.6 and (6) yield

\[ \text{TV} \left( \mathcal{N}(\mu, \Sigma), \mathcal{N}(\hat{\mu}, \hat{\Sigma}) \right)^2 \leq \frac{1}{4} \left( \text{LD} \left( \hat{\Sigma}, \Sigma \right) + (\mu - \hat{\mu})^T \Sigma^{-1} (\mu - \hat{\mu}) \right) \leq \frac{1}{4} (9d^3 \rho^2 + \varepsilon^2), \]

which completes the proof of the lemma.

Thus, we focus on (18). Recalling from (2) that \(\Psi = \Sigma^{1/2}\), from Claim A.2 we have

\[ \text{LD} \left( \hat{\Sigma}, \Sigma \right) = \text{LD} \left( \Psi^{-1} \hat{\Sigma} \Psi^{-1}, \Psi^{-1} \Sigma \Psi^{-1} \right) = \text{LD} \left( B, I \right) \]

where \( B := \sum_{i=1}^d \Psi^{-1} \hat{v}_i \hat{v}_i^T \Psi^{-1} \).

We will show that \(\|B - I\| \leq 3d\rho\), or equivalently \(-3d I \preceq B - I \preceq 3d I\). Then Lemma A.4 will imply that \(\text{LD} \left( \hat{\Sigma}, \Sigma \right) = \text{LD} \left( B, I \right) \leq 9d^3 \rho^2\), which establishes (18).

To complete the proof, we have

\[ \|B - I\| = \left\| \sum_{i=1}^d (\Psi^{-1} \hat{v}_i \hat{v}_i^T \Psi^{-1} - \Psi^{-1} v_i v_i^T \Psi^{-1}) \right\| \leq \sum_{i=1}^d \left\| \Psi^{-1} \hat{v}_i \hat{v}_i^T \Psi^{-1} - \Psi^{-1} v_i v_i^T \Psi^{-1} \right\| \leq \sum_{i=1}^d \|x_i x_i^T - y_i y_i^T\|, \]

with \(x_i := \Psi^{-1} \hat{v}_i\) and \(y_i := \Psi^{-1} v_i\). Referring to (2) we see that \(\|y_i\| = \|\Psi^{-1} v_i\| = 1\). By the lemma’s hypothesis, \(\|x_i - y_i\| \leq \rho\). By applying the following simple lemma, we conclude that \(\|B - I\| \leq 3d\rho\).

**Lemma D.1.** Suppose \(x, y\) satisfy \(\|y\| = 1\) and \(\|x - y\| \leq \varepsilon \leq 1\). Then we have \(\|xx^T - yy^T\| \leq 3\varepsilon\).

**Proof.** Suppose \(x = y + z\) with \(\|z\| \leq \varepsilon\). Then,

\[ \|xx^T - yy^T\| = \|yz^T + zy^T + zz^T\| \leq \|yz^T\| + \|zy^T\| + \|zz^T\| \leq \varepsilon + \varepsilon + \varepsilon^2 \leq 3\varepsilon, \]

where we have used the facts that \(\|AB\| \leq \|A\| \cdot \|B\|\) for any two size-compatible matrices \(A\) and \(B\), and that for any column or row vector \(v\), the operator norm of \(v\) as a matrix coincides with its Euclidean norm as a vector. \(\square\)
E Omitted proofs from Section 5

E.1 Proof of Lemma 5.4

The columns of each matrix $U_a$ are chosen to be the first $d/r$ vectors of a uniformly random orthonormal basis of $\mathbb{R}^d$. We will show that, for any two such matrices $U_a$ and $U_b$, with probability $1 - 2^{-\Omega(d^2/r)}$ we have $\|U_a^TU_b\|_F^2 \leq d/2r$. The lemma then follows by a union bound.

Fix $a, b \in [M]$ with $a \neq b$. By rotational invariance, we may assume without loss of generality that $U_a = (\mathbb{1}_d)$. Thus $\|U_a^TU_b\|_F^2 \overset{d}{=} \|U_{d/r}\|_F^2$, where $U_{d/r}$ is a $d/r \times d/r$ principal submatrix of a uniformly random orthogonal matrix $U$. (Alternatively, the columns of $U_{d/r}$ are the first $d/r$ coordinates of $d/r$ orthonormal vectors in $\mathbb{R}^d$ chosen uniformly at random.) Hence, it suffices to show that $\|U_{d/r}\|_F^2 \leq d/2r$ with probability at least $1 - 2^{-\Omega(d^2/r)}$. The main difficulty is that $U_{d/r}$ does not have independent entries, due to the orthonormality, but intuitively it should behave very similarly to a matrix with independent Gaussian entries.

Relating to a Gaussian matrix. The matrix $U$ is naturally related to the Gaussian matrix $G \in \mathbb{R}^{d \times d/r}$ with i.i.d. $\mathcal{N}(0, 1)$ entries. Similarly, the matrix $U_{d/r}$ is naturally related to the Gaussian matrix $G_{d/r} \in \mathbb{R}^{d/r \times d/r}$ comprised of the first $d/r$ rows of $G$. To see this, let $G = U_G \Sigma_G V_G^T$ be the singular value decomposition of $G$, where $U_G \in \mathbb{R}^{d \times d/r}$ and $\Sigma_G, V_G \in \mathbb{R}^{d/r \times d/r}$. Observe that, by rotational invariance, the columns of $U_G$ are $d/r$ uniformly random orthonormal vectors, and therefore the top $d/r$ rows of $U_G$ (which we denote, slightly awkwardly, by $(U_G)_{d/r}$) have the same distribution as $U_{d/r}$. More precisely, since $U_G$ is independent of $\Sigma_G, V_G$, we have

$$G_{d/r} = (U_G)_{d/r} \Sigma_G V_G^T \overset{d}{=} U_{d/r} \Sigma_G V_G^T,$$  

Observe that $\mathbb{E}[\|G_{d/r}\|_F^2] = (d/r)^2 \cdot (1/d) = d/r^2$, so it remains to show that $\|U_{d/r}\|_F^2$ is unlikely to exceed this by a factor $r/2$.

The Frobenius norms $\|G_{d/r}\|_F$ and $\|U_{d/r}\|_F$ can be related as follows. By (19),

$$\|G_{d/r}\|_F = \sqrt{\text{Tr}(G_{d/r}^T \cdot G_{d/r})} = \sqrt{\text{Tr}(U_{d/r}^T \Sigma_G \cdot \Sigma_G U_{d/r}^T)} = \|U_{d/r} \Sigma_G\|_F \geq \sigma_{\min}(\Sigma_G) \|U_{d/r}\|_F,$$

where $\sigma_{\min}(\Sigma_G)$ denotes the smallest singular value of $\Sigma_G$.

Moments of $\|U_{d/r}\|_F$. Intuitively, (20) should show that $\|U_{d/r}\|_F$ is unlikely to deviate significantly above $\mathbb{E}[\|G_{d/r}\|_F]$, since $\mathbb{E}[\sigma_{\min}(\Sigma_G)] \geq 1 - 1/\sqrt{r}$ (by Theorem A.16) and since $\|G_{d/r}\|_F^2$ concentrates sharply around its mean (as it is a sum of i.i.d. random variables). To make this precise, we will bound the (suitably modified) $p$th moment of (20) for any $p \geq 1$.

Since an upper bound on $\|U_{d/r}\|_F$ is desired, it will be convenient, and sufficient, to consider only the moments of positive deviations. To formalize this idea, recall the notation $(x)_+ := \max\{0, x\}$, and observe that the map $x \mapsto (x)_+$ is monotone and convex on $\mathbb{R}$ for $p \geq 1$. The bound on the (modified) moments proceeds as follows:

$$\mathbb{E}[\|G_{d/r}\|_F - \sqrt{d/r})_+^p] \geq \mathbb{E}[(\sigma_{\min}(\Sigma_G)) \cdot \|U_{d/r}\|_F - \sqrt{d/r})_+^p] \geq \mathbb{E}[\mathbb{E}[(\sigma_{\min}(\Sigma_G)) \cdot \|U_{d/r}\|_F - \sqrt{d/r})_+^p | U_{d/r}]]$$

The next step uses Jensen’s inequality for the conditional expectation $\mathbb{E}[\cdot | U_{d/r}]$, and convexity of $x \mapsto (x)_+$ to obtain

$$\geq \mathbb{E}\left[\mathbb{E}[\sigma_{\min}(\Sigma_G) \cdot \|U_{d/r}\|_F - \sqrt{d/r})_+^p | U_{d/r}]\right]$$

by monotonicity of $(\cdot)_+$, and by applying Theorem A.16 to the matrix $\sqrt{d}G$ (whose entries are i.i.d. $\mathcal{N}(0, 1)$), which yields $\mathbb{E}\sigma_{\min}(\sqrt{d}G) \geq \sqrt{d} - \sqrt{d/r}$ and therefore $\mathbb{E}\sigma_{\min}(G) \geq 1 - 1/\sqrt{r}$.
High-probability bound on $\|U_{d/r}\|_F$. All that remains is the routine task of deriving a high-probability bound from moment bounds. Observe that $\|G_{d/r}\|_F \overset{d}{\sim} \|g\|_2 / \sqrt{d}$, where $g \sim \mathcal{N}(0, I_{(d/r)^2})$. Lemma A.12 states that $\|g\|_2 - d/r$ is $O(1)$-subgaussian; by scaling, $\|G_{d/r}\|_F - \sqrt{d}/r$ is $O(1/\sqrt{d})$-subgaussian. Since the property of being $O(\sigma)$-subgaussian can be characterized via moments (see Lemma A.13), and since inequality (21) shows that the $p$th moments of $((1 - 1/\sqrt{d}) \cdot \|U_{d/r}\|_F - \sqrt{d}/r)$ are bounded by the moments of $\|G_{d/r}\|_F - \sqrt{d}/r$, for all $p \geq 1$, we conclude that $((1 - 1/\sqrt{d}) \cdot \|U_{d/r}\|_F - \sqrt{d}/r)$ is also $O(1/\sqrt{d})$-subgaussian. This allows us to bound the right tail of $(1 - 1/\sqrt{d}) \cdot \|U_{d/r}\|_F - \sqrt{d}/r$ (but not the left tail, due to the $(\cdot)_+$). Recalling the definition of a subgaussian random variable (Definition A.11), we have

$$\Pr \left[ \left(1 - \frac{1}{\sqrt{d}}\right) \cdot \|U_{d/r}\|_F - \sqrt{d}/r \leq t \right] \geq 1 - 2^{-\Omega(t^2d)} \quad \forall t > 0. \quad (22)$$

Fix $t = \sqrt{d}/(12\sqrt{r})$. By simple manipulations, the event in (22) is equivalent to

$$\|U_{d/r}\|_F^2 \leq \frac{d}{r} \left( \frac{\frac{1}{\sqrt{r}} + \frac{1}{\sqrt{\sqrt{r}}} \ }{1 - \frac{1}{\sqrt{r}}} \right)^2.$$ 

This right-hand side is at most $d/2r$ for all $r \geq 9$. It follows that $\|U_{d/r}\|_F^2 \leq d/2r$ with probability at least $1 - 2^{-\Omega(d^2/r)}$, completing the proof.

E.2 Proof of Theorem 5.6

This proof builds on the lower bound construction for learning a single Gaussian (Theorem 5.3), and extends it to a lower bound for learning a mixture of Gaussians. The high-level idea is simple: create a family of distributions in $k$-mix$(G^d)$ such that each Gaussian uses a covariance matrix as constructed in Theorem 5.3. As we will use Lemma 5.1 again to obtain the sample complexity lower bound, it suffices to construct $2^{\Omega(kd^2)}$ distributions in $k$-mix$(G^d)$ with pairwise KL divergence $O(\varepsilon^2)$ and pairwise TV distance $\Omega(\varepsilon)$. Some care is required to ensure that the TV distance is large, and we will adopt some ideas used in earlier work for mixtures of spherical Gaussians [10] Appendix C.2. In more detail, the construction proceeds as follows.

First, we construct a family of covariance matrices. The proof of Theorem 5.3 shows that there exists a family of symmetric, positive definite matrices $\Sigma_1, \ldots, \Sigma_T$ with $T = 2^{\Omega(d^3)}$ satisfying

$$\text{KL}(N(0, \Sigma_i) \parallel N(0, \Sigma_j)) \leq O(\varepsilon^2) \quad \forall i \neq j \quad (23a)$$

$$\text{TV}(N(0, \Sigma_i), N(0, \Sigma_j)) \geq \Omega(\varepsilon) \quad \forall i \neq j \quad (23b)$$

$$\Sigma_i \prec 2I \quad \forall i. \quad (23c)$$

Next we will create a family of distributions in $k$-mix$(G^d)$ for which each Gaussian in the mixture uses one of these $\Sigma_i$ matrices as its covariance matrix. However, there is a tension. On the one hand, we’d like any two of these mixture distributions to use disjoint sets of covariance matrices, so that the TV distance between the mixtures is large. On the other hand, that constraint would greatly reduce the number of mixture distributions we can create, and we want many distributions in order to maximize the lower bound. This tension is resolved by a compromise obtained via error-correcting codes.

The formal construction proceeds as follows. First, we pick $\mu_1, \ldots, \mu_k \in \mathbb{R}^d$, which will serve as the means for the Gaussians. The only constraint is that they should be far apart: for some $\Delta$, to be chosen later, we have $\|\mu_i - \mu_j\|_2 \geq \Delta$ for all $i \neq j$. Each mixture distribution will be a uniform mixture of $k$ Gaussians, for which the $i$th Gaussian has mean $\mu_i$. The choice of covariance matrices is determined using the error-correcting code. Specifically, let $X \subset [T]^k$ be a set as in Lemma E.1 below. The family of mixture distributions is

$$\mathcal{F} := \{ f_x : x \in X \} \quad \text{where} \quad f_x := \frac{1}{k} \left( N(\mu_1, \Sigma_{x_1}) + \cdots + N(\mu_k, \Sigma_{x_k}) \right).$$

As desired, we have $|\mathcal{F}| = T^{\Omega(k)} = 2^{\Omega(kd^2)}$.

This lemma constructs the desired code. It is proven at the end of the section.

Lemma E.1. Let $T \geq 4$ and $k \in \mathbb{N}$. There exists a set of tuples $X \subseteq [T]^k$ such that $|X| \geq T^{\Omega(k)}$ and every pair of distinct $x, y \in X$ have Hamming distance $|\{ i \in [k] : x_i \neq y_i \}| \geq k/4$. 

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To analyze $\mathcal{F}$, the first task is to prove the pairwise $KL$ divergence upper bound. This is straightforward. Fix distinct $x, y \in \mathcal{X}$. For each $i$, (23a) shows that
\[
KL(\mathcal{N}(\mu_i, \Sigma_{x_i}) \| \mathcal{N}(\mu_i, \Sigma_{y_i})) = KL(\mathcal{N}(0, \Sigma_{x_i}) \| \mathcal{N}(0, \Sigma_{y_i})) \leq O(\varepsilon^2).
\]
Convexity of $KL$ divergence [12, Theorem 2.7.2] then shows that $KL(f_x \| f_y) \leq O(\varepsilon^2)$.

The remaining task is to prove $TV(f_x, f_y) \geq \Omega(\varepsilon)$ for all distinct $f_x, f_y \in \mathcal{F}$. The intuition is as follows. Say that index $i \in [k]$ disagrees if $x_i \neq y_i$. Whenever $i$ disagrees, the $i$th Gaussian in $f_x$ and $i$th Gaussian in $f_y$ have TV distance $\Omega(\varepsilon)$. Proving this formally requires somewhat more care because each Gaussian is supported on all of $\mathbb{R}^d$, so there is interaction between all Gaussians involved in the mixtures. However, the parameter $\Delta$ ensures that the means are far apart, so the interaction is negligible.

More formally, let $A_j' \subseteq \mathbb{R}^d$ be such that
\[
Pr_{g \sim N(\mu_j, \Sigma_{x_j})}[g \notin B_i] = Pr_{g \sim N(\mu_j, \Sigma_{x_j})}[g \in A_j'] - Pr_{g \sim N(\mu_j, \Sigma_{y_j})}[g \in A_j'] = TV(N(\mu_j, \Sigma_{x_j}), N(\mu_j, \Sigma_{y_j})).
\]
Define
\[
A_j = A_j' \cap B_j \quad \text{where} \quad B_j = \{ x \in \mathbb{R}^d : \|x - \mu_j\|_2 < \Delta/2 \}.
\]
Note that the separation of $\mu_1, \ldots, \mu_k$ implies that the balls $B_1, \ldots, B_k$ are pairwise disjoint. Consequently, the sets $A_1, \ldots, A_k$ are also pairwise disjoint.

Several preliminary inequalities are required concerning these events. First,
\[
Pr_{g \sim N(\mu_i, \Sigma_{x_i})}[g \notin B_i] = Pr_{g \sim N(\mu_i, \Sigma_{x_i})}[\|g - \mu_i\|_2^2 \geq (\Delta/2)^2] = Pr_{g \sim N(0, \Sigma_{x_i})}[\|g\|_2^2 \geq (\Delta/2)^2] \quad \text{(translating to zero-mean)}
\]
\[
\leq Pr_{g \sim N(0, I_d)}[\|g\|_2^2 \geq \Delta^2/8] \quad \text{(by (23d))}
\]
\[
\leq \varepsilon^2/\Delta^2,
\]
by applying Lemma A.9 with $t = 2 \ln(k/\varepsilon)$ and choosing $\Delta$ to satisfy $\Delta^2/8 = d + 2\sqrt{d}t + 2t$. Inequality (25) also holds replacing $x_i$ with $y_i$. Since $A_i' \setminus A_i \subseteq B_i^c$, (25) shows that
\[
\left| Pr_{g \sim N(\mu_i, \Sigma_{x_i})}[g \in A_i] - Pr_{g \sim N(\mu_i, \Sigma_{x_i})}[g \in A_j'] \right| \leq Pr_{g \sim N(\mu_i, \Sigma_{x_i})}[g \notin B_i] \leq \varepsilon^2/\Delta^2.
\]
This inequality also holds using $y_i$ instead of $x_i$. For $i \neq j$, we have $A_j \subseteq B_i^c$, so
\[
Pr_{g \sim N(\mu_i, \Sigma_{x_i})}[g \notin B_i] = Pr_{g \sim N(\mu_j, \Sigma_{y_j})}[g \not\in B_i] \leq \varepsilon^2/\Delta^2.
\]
Finally, by (24), (26) and the triangle inequality,
\[
Pr_{g \sim N(\mu_j, \Sigma_{x_j})}[g \in A_j] - Pr_{g \sim N(\mu_j, \Sigma_{y_j})}[g \in A_j] \geq TV(N(\mu_j, \Sigma_{x_j}), N(\mu_j, \Sigma_{y_j})) - 2\varepsilon^2/\Delta^2.
\]
The total variation distance is lower bounded as follows. Let $A := A_1 \cup \cdots \cup A_k$. Then
\[
TV(f_x, f_y) \geq Pr_{g \sim f_x}[g \in A] - Pr_{g \sim f_y}[g \in A]
\]
\[
= \sum_{j=1}^k \left( Pr_{g \sim f_x}[g \in A_j] - Pr_{g \sim f_y}[g \in A_j] \right) \quad \text{(by disjointness of the $A_j$)}
\]
\[
= \frac{1}{k} \sum_{j=1}^k \sum_{i=1}^k \left( Pr_{g \sim N(\mu_i, \Sigma_{x_i})}[g \in A_j] - Pr_{g \sim N(\mu_j, \Sigma_{y_j})}[g \in A_j] \right) \quad \text{(expanding $f_x$ and $f_y$ as $k$-mixtures)}
\]
\[
= \frac{1}{k} \sum_{j=1}^k \left( Pr_{g \sim N(\mu_j, \Sigma_{x_j})}[g \in A_j] - Pr_{g \sim N(\mu_j, \Sigma_{y_j})}[g \in A_j] \right) \quad \text{(summands with $i = j$)}
\]
\[
+ \frac{1}{k} \sum_{j=1}^k \sum_{i \neq j} \left( Pr_{g \sim N(\mu_i, \Sigma_{x_i})}[g \in A_j] - Pr_{g \sim N(\mu_j, \Sigma_{y_j})}[g \in A_j] \right) \geq 0 \quad \text{(summands with $i \neq j$)}
\]
\[
\leq \varepsilon^2/\Delta^2 \quad \text{by (27)}
\]

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\[ \geq \frac{1}{k} \sum_{j=1}^{k} \left( \Pr_{g \sim N(\mu_j, \Sigma_{xj})}[g \in A_j] - \Pr_{g \sim N(\mu_j, \Sigma_{yj})}[g \in A_j] \right) - \varepsilon^2 \]

\[ \geq \frac{1}{k} \sum_{j=1}^{k} \left( TV\left( N(\mu_j, \Sigma_{xj}), N(\mu_j, \Sigma_{yj}) \right) - 2\varepsilon^2/k^2 \right) - \varepsilon^2 \quad \text{(by (28))} \]

\[ \geq \frac{1}{k} (k/4) \Omega(\varepsilon) - 3\varepsilon^2 = \Omega(\varepsilon), \]

where the last inequality is because \( TV\left( N(\mu_j, \Sigma_{xj}), N(\mu_j, \Sigma_{yj}) \right) \geq \Omega(\varepsilon) \) whenever \( x_j \neq y_j \), which is the case for at least \( k/4 \) of the indices \( j \).

**Proof (of Lemma E.1).** This can be proven several different ways. The conclusion of the lemma states that \( \chi \) is a code over the alphabet \([T]\) of rate \( \Omega(1) \) and relative distance at least \( 1/4 \). By standard results \([24, Proposition 3.3.2]\), the \( T \)-ary entropy function \( H_T \) satisfies \( 1 - H_T(1/4) \geq 1/4 \) as \( T \geq 4 \). By the Gilbert-Varshamov bound \([24, Theorem 4.2.1]\), there exists such a code of rate \( 1/8 \). A self-contained proof can be found in our technical report \([7]\). \( \square \)

**References**


