Computing the independence polynomial in Shearer’s region for the LLL

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Abstract

The independence polynomial has been widely studied in algebraic graph theory, in statistical physics, and in algorithms for counting and sampling problems. Seminal results of Weitz (2006) and Sly (2010) have shown that in bounded-degree graphs the independence polynomial can be efficiently approximated if the argument is positive and below a certain threshold, whereas above that threshold the polynomial is hard to approximate. Furthermore, this threshold exactly corresponds to a phase transition in physics, which demarcates the region within which the Gibbs measure has correlation decay.

Evaluating the independence polynomial with negative or complex arguments may not have a counting interpretation, but it does have strong connections to combinatorics and to statistical physics. The independence polynomial with negative arguments determines the maximal region of probabilities to which the Lovász Local Lemma (LLL) can be extended, and also gives a lower bound on the probability in the LLL’s conclusion (Shearer 1985). In statistical physics, complex roots of the independence polynomial relate to existence of phase transitions, and there is a relation between negative and complex roots (Penrose 1963).

We study algorithms for approximating the independence polynomial with negative and complex arguments. Whereas many algorithms for computing combinatorial polynomials are restricted to the univariate setting, we consider the multivariate independence polynomial, since there is a natural multivariate region of interest — Shearer’s region for the LLL. Our main result is: for any \(n\)-vertex graph of degree at most \(d\), any \(\alpha \in (0, 1]\), and any complex vector \(p\) such that \((1 + \alpha) \cdot p\) lies in Shearer’s region, there is a deterministic algorithm to approximate the independence polynomial at \(p\) within \((1 + \varepsilon)\) multiplicative error and with runtime \(\left(\frac{n}{\varepsilon \alpha}\right)^{O(\log(d)/\alpha)}\). Our results also extend to graphs of unbounded degree that have a bounded connective constant. Our analysis uses a novel multivariate form of the correlation decay technique.

On the hardness side, we prove that every algorithm must have some dependence on \(\alpha\), as it is \#P-hard to evaluate the polynomial within any \(\text{poly}(n)\) factor at an arbitrary given point in Shearer’s region. Similarly, deciding if a given point lies in Shearer’s region is also \#P-hard. Finally, it is NP-hard to approximate the independence polynomial in a graph of degree at most \(d\) with a fixed negative argument that is only a constant factor outside Shearer’s region.

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1 Introduction

The independence polynomial is the generating function of independent sets of a graph. Formally, given a graph \( G = (V, E) \), and a vector \( \mathbf{x} = (x_v)_{v \in V} \) of vertex activities, it is the multi-linear polynomial

\[
Z_G(\mathbf{x}) = \sum_{I \text{ indep. in } G} \prod_{v \in I} x_v.
\]

Aside from its natural importance in combinatorics as a generating function, the independence polynomial has also been studied extensively in statistical physics where it arises as the partition function of the hard core lattice gas, which has been used as a model of adsorption. In both settings, the partition function and its derivatives encode important properties of the model. For example, in the combinatorial setting, \( Z_G \) encodes a weighted count of the independent sets, while the derivatives of \( \log Z_G \) encode relevant average quantities, such as the mean size of an independent set. As such, much effort has gone into understanding the complexity of computing \( Z_G \). The exact evaluation of the independence polynomial at non-trivial evaluation points turns out to be \#P-hard [42]. As for approximate computation, the problem is well studied in the setting where the activities are positive and real valued. In this setting, the problem has served to highlight some of the tightest known connections between phase transitions and computational complexity: we will discuss this line of work in more detail below.

In this paper, we are concerned instead with the problem of approximately computing the independence polynomial at possibly negative and even complex valued vertex activities. The interest in studying partition functions at complex values of the activities originally comes from statistical mechanics, where there is a paradigm of studying phase transitions in terms of the analyticity of \( \log Z_G \). This paradigm has led to the question of characterizing regions of the complex plane where the partition function is non-zero [22, 44]. However, one of our main motivations for studying the independence polynomial at complex activities comes from another delightful connection between combinatorics and statistical mechanics that arose in the work of Shearer [34] and Scott and Sokal [33] on the Lovász Local Lemma.

The Lovász Local Lemma (LLL) is a fundamental tool used in combinatorics to argue that the probability that none of a set of suitably constrained bad events occurs is positive. In abstract terms, the lemma is formulated in terms of \( n \) events \( \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n \) and a probability distribution \( \mu \) on the events. However, only two pieces of data about the distribution \( \mu \) are used in the formulation of lemma:

- The marginal probabilities \( p_i := \mu(\mathcal{E}_i) \) of the events, and
- A dependency graph \( G = (V, E) \) associated with \( \mu \). The vertices \( V \) are identified with the events \( \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n \), and the graph is interpreted as stipulating that under the distribution \( \mu \), the event \( \mathcal{E}_i \) is independent of all the other events conditioned on its immediate neighbors in the graph \( G \).

Various versions of the LLL provide sufficient conditions on the \( p_i \) and the dependency graph \( G \) that ensure that the probability \( \mu(\bigwedge_{i=1}^n \neg \mathcal{E}_i) \) is positive. Under suitable restrictions on the structure of the the events and the distribution \( \mu \), several algorithmic versions of the LLL have also been derived: here, given the sample space \( \Omega \) of the events \( \mathcal{E}_i \) and an implicit description of the distribution \( \mu \), the objective is to find, in randomized polynomial time, a point in \( \Omega \) where none of the events occur; see, e.g., [1, 2, 17, 18, 21, 28]. In this paper, our concern is with a different algorithmic question about the event \( \bigwedge_{i=1}^n \neg \mathcal{E}_i \):

**Question 1.** Given the marginal probabilities \( \mu(\mathcal{E}_i) \) and the dependency graph \( G \), compute the minimum value of \( \mu(\bigwedge_{i=1}^n \neg \mathcal{E}_i) \) over all probability distributions \( \mu \) that respect the dependency graph \( G \).

This is a natural question in probabilistic combinatorics, and indeed it previously arose in work of Holroyd and Liggett [20, Section 9], where concrete examples were analyzed numerically by an inefficient algorithm.

The seminal work of Shearer [34] provided necessary and sufficient conditions for the conclusion of the Lovász Local Lemma (i.e., \( \mu(\bigwedge_{i=1}^n \neg \mathcal{E}_i) > 0 \)) to hold for a given dependency graph \( G = (V, E) \) and probabilities \( (p_v)_{v \in V} \) for the vertices of \( G \). Scott and Sokal [33] showed that Shearer’s conditions can be
expressed very succinctly in the language of partition functions. Specifically, \( \mu(\bigwedge_{i=1}^n -E_i) > 0 \) holds for a given graph \( G \) and probabilities \( p = (p_v)_{v \in V} \in (0, 1)^V \) if and only if \( p \) lies in the set
\[
S := \left\{ p \in (0, 1)^V : Z_G(z) \neq 0 \ \forall z \in \mathbb{C}^V \ \text{s.t.} \ |z| \leq p \right\}.
\]
where \(|z|\) means coordinate-wise magnitude, and \(\leq\) also applies coordinate-wise. We shall refer to the set \( S \) as the Shearer region; to emphasize that \( S \) depends on \( G \) we sometimes write \( S_G \).

In other words, when \( p \in S \), the conclusion \( \mu(\bigwedge_{i=1}^n -E_i) > 0 \) holds for all events \( E_i \) satisfying \( \mu(E_i) \leq p_i \) and satisfying the constraints imposed by the dependency graph. On the other hand, when \( p \in (0, 1)^V \setminus S \), then there exist events \( E_i \) with \( \mu(E_i) \leq p_i \) satisfying the constraints imposed by the dependency graph for which \( \mu(\bigwedge_{i=1}^n -E_i) = 0 \). The results of Shearer [34] and Scott and Sokal [33] are in fact sharper, and show that for any set of events \( E_i \) as above and any \( p \in S \)
\[
\mu(\bigwedge_{i=1}^n -E_i) \geq Z_G(-p),
\]
which is positive by continuity since \( Z_G(0) = 1 \). Furthermore, there exist events with dependency graph \( G \) and probabilities \( p \) which achieve equality in eq. (1.2). In view of eq. (1.2), our Question 1 reduces to the question of approximating the partition function \( Z_G(-p) \) when \( p \in S \).

### 1.1 Our results

We solve Question 1 in a more general context by giving a fully polynomial time approximation scheme (FPTAS) for \( Z_G(z) \) when \( z \) is a vector of possibly complex activities with the property that the vector \(|z|\) of their magnitudes lies in the Shearer region. The algorithm runs in polynomial time for bounded-degree graphs, and quasi-polynomial time for general graphs.

**Theorem 1.1 (FPTAS for \( Z_G \)).** Let \( G \) be an \( n \)-vertex graph with maximum degree \( d \). Suppose that \( \alpha, \epsilon \in (0, 1] \), and that \( z \in \mathbb{C}^V \) satisfies \( (1 + \alpha) \cdot |z| \in S \). Then a \((1 + \epsilon)\)-approximation to \( Z_G(z) \) can be computed in time \( (n/\epsilon)^{O((\log(d))/\alpha)} \).

**Remark 1.1.** Theorem 1.1 extends to graphs of unbounded maximum degree that have a bounded connective constant [16, 27, 36, 37]. We refer to Appendix D for the details of this extension.

An approximation scheme is necessary in Theorem 1.1 since computing \( Z_G(-p) \) exactly is \#P-hard (see also Section 4). Restricting to a univariate partition function, we have the following corollary.

**Corollary 1.2 (FPTAS for the univariate case).** Let \( G \) be an \( n \)-vertex graph with maximum degree \( d \). Define \( \rho_G = \min\{ |z| : z \in \mathbb{C}, Z_G(z1) = 0 \} \). It is known that \( \rho_G > (d-1)^{d-1} \). Let \( \alpha, \epsilon \in (0, 1] \) and \( z \in \mathbb{C} \) satisfy \( (1 + \alpha) \cdot |z| \leq \rho_G \). Then a \((1 + \epsilon)\)-approximation to \( Z_G(z1) \) can be computed in time \( (n/\epsilon)^{O((\log(d))/\alpha)} \).

In contrast, approximating the univariate partition function at negative activities outside the Shearer region can be NP-hard. See Theorem 4.4 for a hardness result in this direction. Note that our FPTAS needs the “slack parameter” \( \alpha \), which measures how far \( p \) is to the boundary of the Shearer region. Some dependence on the slack parameter is necessary, as we show in the following result.

**Theorem 1.3 (Necessity of slack).** If there is an algorithm to estimate \( Z_G(-p) \), assuming \((1 + \alpha) p \in S \), within a poly \( n \) multiplicative factor in running time \( n \log \frac{1}{\alpha} \cdot O(\log n) \) then \#P \subseteq DTIME\((n^{O(\log n)})\).

### 1.2 Related work

As discussed above, the exact computation of the independence polynomial turns out to be \#P-hard. This is a fate shared by the partition functions of several other “spin systems” (e.g., the Ising model) in statistical physics, and by now there is extensive work on the complexity theoretic classification of partition functions in terms of dichotomy theorems (see e.g., [8]).
The approximation problem for a univariate partition function with a positive real argument is also well studied and has strong connections with phase transitions in statistical mechanics. In two seminal papers, Weitz [43] and Sly [38] (see also [13, 14, 39]) showed that there exists a critical value \( \lambda_c(d) \) such when \( \lambda < \lambda_c(d) \), there is an FPTAS for the partition function \( Z_G(\lambda) \) on graphs of maximum degree \( d \), while when \( \lambda > \lambda_c(d) \), approximating \( Z_G(\lambda) \) on \( d \)-regular graphs is NP-hard under randomized reductions.

The approach for our FPTAS builds upon the correlation decay technique pioneered by Weitz, which has since inspired several results in approximate counting (see, e.g., [7, 11, 15, 23–25, 35, 36]). Unlike previous work, where the partition function has positive activities and induces a probability distribution on the underlying structures, our emphasis is on negative and complex activities. It turns out that Weitz’s proof can be easily modified to handle a univariate independence polynomial with a negative (and indeed, complex) parameter \( z \) satisfying \( |z| < \frac{(d-1)^{d-1}}{d^d} \), analogous to the \( \lambda < \lambda_c(d) \) condition mentioned above; this observation appears in [41]. Our work considers a much more general scenario: the multivariate independence polynomial under a global condition incorporating all vertex activities (i.e., the set \( S \)). This yields a result for the univariate case stronger than [41], as our threshold \( \rho_G \) in Corollary 1.2 depends on \( G \) not just on \( d \).

Starting with a paper of Barvinok [4], a different approach to approximating partition functions in their zero-free regions has emerged. Here, the analyticity of \( \log Z \) in the zero-free region of \( Z \) is used to provide an additive approximation to \( \log Z \) (which translates to a multiplicative approximation for \( Z \)) via a Taylor expansion truncated at an appropriate degree. While this method has by now been applied to several classes of partition functions [3, 5, 6, 31], the resulting algorithms have so far turned out to be quasi-polynomial because of the lack of a method to efficiently compute coefficients of terms of degree \( \Omega(\log n) \) in the Taylor expansion of \( \log Z \) (which, in the case of the hard core model, correspond to \( \Omega(\log n) \)-wise correlations among vertices in a random independent set).

1.3 Techniques

As in previous work, our starting point is the standard self-reducibility argument showing that the problem of designing an FPTAS for \( Z_G(\lambda) \) is equivalent to the problem of designing an FPTAS for computing the occupation ratio \( r_v \) of a given vertex \( v \) in any given graph \( G \) (i.e., the ratio of the total weights of the independent sets containing \( v \) to the total weight of those that do not). In previous work, these occupation ratios are actual likelihood ratios that can be translated to the probability that the vertex \( v \) is occupied, but because of complex weights, we do not have the luxury of this interpretation. However, as in earlier work, we can still write formal recurrences for these occupation ratios. As Weitz showed [43], this recursive computation is naturally structured as a tree which has the same structure as the the tree \( T_{SAW}(v, G) \) whose nodes correspond to self-avoiding walks in \( G \) starting at \( v \). However the tree \( T_{SAW}(v, G) \) has size exponential in \( |V| \), so this reduction does not immediately give a polynomial time algorithm.

The crucial step in correlation decay algorithms is to show that this tree can be truncated to polynomial size without incurring a large error in the value computed at the root. In earlier work on positive activities, especially since Restrepo et al. [32], the standard method for doing this has been to consider instead a recurrence for an appropriately chosen function \( \phi(r_v) \), known as the message, that is chosen so that when correlation decay holds on the \( d \)-regular tree, each step of the recurrence on the truncated \( T_{SAW} \) contracts the error introduced by the truncation by a constant factor. Thus, by expanding the tree to \( \ell = O(\log \frac{1}{\epsilon}) \) levels (so \( d^\ell = \text{poly}(n, \frac{1}{\epsilon}) \) nodes), one obtains a \( (1 + O(\frac{\epsilon}{\ell})) \)-approximation to the value at the root.

Our approach also involves truncating the computation tree at an appropriate depth and then controlling the errors introduced due to truncation. However, in part because of the lack of a uniform bound on the vertex

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1 Very recently we have learned of simultaneous, independent work of Patel and Regts [29], giving an FPTAS for some partition functions, including the univariate independence polynomial, in bounded-degree graphs. Their result for the univariate independence polynomial seems to be implied by the observations in [41] and by our Corollary 1.2, although their technique is different.

2 Interpreting the computation tree as the \( T_{SAW} \) tree will have less prominence in our analysis than in previous work.
activities, we are not able to recreate a message-based approach. Instead, we perform a direct amortization 
argument, where we define recursively for each node in the computation tree an \textit{error sensitivity parameter}, 
and then measure errors at that node as a fraction of the local error sensitivity parameter. We then establish 
two facts: (1) that the errors, when measured as a fraction of the error sensitivity parameter, do indeed decay 
by a constant fraction at each step of the recurrence (even though the absolute errors may not), and (2) that 
the error sensitivity parameter of the root node is not too large, so that the absolute error of the final answer 
can be appropriately bounded. The detailed argument appears in Section 3.2.

\section{Overview of the correlation decay method}
In this section we summarize the basic concepts and facts relating to Weitz’s correlation decay method. 
Since all the claims are simple or known, the proofs are omitted or appear in the appendix.

\textbf{Partition functions and occupation ratios.} Since we are primarily interested in the hard-core partition 
function (i.e., independence polynomial) with negative activities, it will be convenient to introduce the following 
notation. Let $G = (V,E)$ be a fixed graph, and let $p$ be a fixed vector of (possibly complex) 
parameters on the vertices of $V$. For $S \subseteq V$, let $\text{Ind}(S) = \text{Ind}_G(S) = \{ I \subseteq S : I \text{ independent in } G \}$. 
Following the notation of [19, 21], we define the \textit{alternating-sign independence polynomial} for any subset 
$S$ of $V$ to be

$$ \tilde{q}_S = \tilde{q}_S(p) := \sum_{I \in \text{Ind}(S)} (-1)^{|I|} \prod_{v \in I} p_v. $$

Note that $\tilde{q}_V(p) = Z_G(-p)$. The computation of $\tilde{q}_V$ will be reduced to the computation of \textit{occupation ratios} defined as follows. For a pair $(S,u)$, where $S \subseteq V$ and $u \in S$, the occupation ratio $r_{S,u}$ is

$$ r_{S,u} = r_{S,u}(p) := \frac{\sum_{I \in \text{Ind}(S), u \in I} (-1)^{|I|} \prod_{v \in I} p_v}{\sum_{I \in \text{Ind}(S), u \notin I} (-1)^{|I|} \prod_{v \in I} p_v}. $$

For readers familiar with the notation of Weitz [43], we note that $r_{S,u}$ agrees with his definition of occupation ratios except for the negative signs used in the definition here. Using the definition of $\tilde{q}_S$, and the notation 
$\Gamma'(u) = \{ v : u \text{ is a neighbor of } v \text{ in } G \}$, $\Gamma'(u) = \Gamma(u) \cup \{ u \}$, we can also rewrite this quantity as

$$ r_{S,u} = \frac{p_u \tilde{q}_{S \setminus \Gamma'(u)} + 1}{\tilde{q}_{S \setminus \{ u \}}} = 1 - \frac{\tilde{q}_S - \tilde{q}_{S \setminus \{ u \}}}{\tilde{q}_{S \setminus \{ u \}}}. $$

A standard self-reducibility argument now reduces the computation of $\tilde{q}_V$ to that of the $r_{S,u}$.

\textbf{Claim 2.1.} Fix an arbitrary ordering $(v_1, v_2, \ldots, v_n)$ of $V$, and let $S_i = \{ v_i, v_{i+1}, \ldots, v_n \}$. We then have

$$ \tilde{q}_V = \prod_{i=1}^n \frac{\tilde{q}_{S_i}}{\tilde{q}_{S_{i+1}}} = \prod_{i=1}^n (1 - r_{S_i,v_i}). $$

\textbf{Recurrences for the occupation ratios.} An important observation in Weitz’s work [43] was that the computation of occupation ratios similar to the $r_{S,u}$ can be carried out over a tree-like recursive structure. We follow a similar strategy, although we find it convenient to work with a somewhat different notation.

\textbf{Definition 2.2 (Child subproblems).} Given a pair $(S,u)$ with $S \subseteq V$ and $u \in S$, and an arbitrary ordering 
$(v_1, v_2, \ldots, v_k)$ of $\Gamma(u) \cap S$, we define the set of \textit{child} sub-problems $C(S,u) = C(v_1,v_2,\ldots,v_k)(S,u)$ to be

$$ C(S,u) = \{ (S \setminus \{ u \}, v_1), (S \setminus \{ u, v_1 \}, v_2), \ldots, (S \setminus \{ u, v_1, \ldots, v_{k-1} \}, v_k) \}. $$

Note that the ordering of neighbors used in the definition of $C(S,u)$ is completely arbitrary and orderings 
between neighbors of different vertices do not share any consistency constraints. The recurrence relation for 
the computation of the $r_{S,u}$, analogous to Weitz’s computation tree, is then the following:
Lemma 2.3 (Computational recurrence). Fix a graph $G = (V, E)$ and a vector $p \in \mathbb{C}^V$ of complex parameters. Let $(S, u)$ be such that $u \in S$ and $S \subseteq V$. Fix an arbitrary ordering $(v_1, v_2, \ldots, v_k)$ of $\Gamma(u) \cap S$ and define the corresponding set $C(S, u)$ of child subproblems. We then have

$$r_{S,u} = p_u \cdot \prod_{(S', u') \in C(S, u)} \frac{1}{1 - r_{S', u'}}. \quad (2.4)$$

Remark 2.1. As observed by Weitz [43], each node in the computation tree of $r_{V,u}$ at depth $\ell$ corresponds to a unique self-avoiding walk of length $\ell$ starting at $v$.

The Shearer region. The Shearer region was defined (implicitly) by Shearer [34] as follows.

Definition 2.4 (Shearer region). Given a graph $G = (V, E)$, the Shearer region $S$ is the set of vectors $p \in (0, 1)^V$ such that $\hat{q}_S(p) > 0$ for all $S \subseteq V$.

Shearer proved that this is the maximal region of probability vectors to which the Lovász Local Lemma can be extended. The equivalence with our earlier definition (1.1) is due to Scott and Sokal [33, Theorem 2.10]: the Shearer region can be equivalently defined by the absence of roots in a certain polydisc, as follows.

Theorem 2.5. A probability vector $p \in (0, 1)^V$ is in the Shearer region if and only if for all vectors $z = (z_v)_{v \in V}$ of complex activities such that $|z_v| \leq p_v$, it holds that $Z_G(z) \neq 0$.

Considering this, it is natural to extend the definition of the Shearer region to complex parameters. In the following, we consider primarily this complex extension of the Shearer region.

Definition 2.6 (Complex Shearer region). Given a graph $G = (V, E)$, the complex Shearer region is

$$S = S_G := \{ p \in \mathbb{C}^V : Z_G(z) \neq 0 \ \forall z \in \mathbb{C}^V, |z_v| \leq |p_v| \}. $$

We now state some important properties of the occupation ratios and $\hat{q}_S$ in the setting of real, positive parameters $p$. These results are essentially translations of the results of [33, 34] into our notation.

Lemma 2.7 (Monotonicity and positivity of $\hat{q}$). Let $G = (V, E)$ be any graph and let $p \in (0, 1)^V$ be such that $p \in S$. Then for any subsets $A$ and $B$ of $V$ such that $A \subseteq B$, we have $\hat{q}_A(p) \geq \hat{q}_B(p) > 0$.

Lemma 2.8 (Occupation ratios are bounded). Let $G = (V, E)$ be any graph and let $p \in (0, 1)^V$ be such that $p \in S$. Then, for any subset $S$ of $V$ and any vertex $u \in S$, we have $p_u \leq r_{S,u} < 1$.

The correlation decay algorithm. Weitz’s high-level approach to compute the independence polynomial is to compute the partition function via a telescoping product analogous to (2.3), and to compute each occupation ratio via a recurrence analogous to (2.4). As discussed in Section 1.3, the recursion is truncated to $\ell$ levels, and the analysis shows that the occupation ratio at the root is not affected heavily by the occupation ratios where the truncation occurred.

We follow that same high-level approach here, although the details of the analysis are quite different. Algorithm 1 presents pseudocode giving a compact description of the full algorithm. The main procedure, \textsc{ComputeIndependencePolynomial}(G, p, $\ell$) implements (2.3) to estimate $\hat{q}_V(p)$ for a graph $G = (V, E)$, a parameter vector $p$, and a desired recursion depth $\ell$. (The required value of $\ell$ depends upon the accuracy parameter $\epsilon$; see Theorem 3.8). The recursive procedure \textsc{OccupationRatio}(G, p, $\ell$, $S$, $u$) implements (2.4) to estimate the occupation ratio $r_{S,u}$ by executing $\ell$ levels of recursion.

3 The analysis

Let us turn to the analysis of the correlation decay method in our setting. The notion of correlation decay in the hard core model refers to the decaying dependence of the occupation probability at a given vertex $v$ on the conditioning on a set of vertices at a certain distance from $v$. In the setting of positive activities $z$ [43],
Algorithm 1 Our algorithm to compute a \((1 + \epsilon)\)-approximation to \(\bar{q}_V(p)\).

1: \textbf{procedure} \textsc{ComputeIndependencePolynomial}(\(G = (V, E), p, \ell\))
2: \hspace{1cm} Fix an ordering \(V = (v_1, v_2, \ldots, v_n)\)
3: \hspace{1cm} \(\bar{q} \leftarrow 1\)
4: \hspace{1cm} \textbf{for} \(i \leftarrow 1, \ldots, n\) \textbf{do}
5: \hspace{2cm} \(\bar{q} \leftarrow \bar{q} \cdot (1 - \textsc{OccupationRatio}(G, \ell, \{v_i, \ldots, v_n\}, v_i))\)
6: \hspace{1cm} \textbf{return} \(\bar{q}\)
7: \textbf{procedure} \textsc{OccupationRatio}(\(G, p, \ell, S, u\))
8: \hspace{1cm} \textbf{if} \(\ell = 0\) \textbf{then} \textbf{return} 0
9: \hspace{1cm} \textbf{Let} \((w_1, w_2, \ldots, w_k)\) denote a fixed ordering of \(S \cap \Gamma(u)\)
10: \hspace{1cm} \(r \leftarrow p_u\)
11: \hspace{1cm} \textbf{for} \(i \leftarrow 1, \ldots, k\) \textbf{do}
12: \hspace{2cm} \(r \leftarrow r / (1 - \textsc{OccupationRatio}(G, p, \ell - 1, S \setminus \{u, w_1, \ldots, w_{i-1}\}, w_i))\)
13: \hspace{1cm} \textbf{return} \(r\)

3.1 The error sensitivity parameter

For simplicity of notation, we fix the input graph \(G = (V, E)\) and an ordering on vertices \(V = \{v_1, \ldots, v_n\}\) for the rest of this section. Recall that \(p = (p_1, p_2, \ldots, p_n)\) denotes a vector of vertex parameters in the complex plane. (We have \(p = -z\) where \(z\) are the usual activities in the hard core model.) We use \(|p|\) to denote the vector \((|p_1|, |p_2|, \ldots, |p_n|)\). Note that \(p\) is in the Shearer region \(S\) if and only if \(|p|\) is in \(S\).

The error sensitivity parameter for \(p\) is defined by considering the point \(|p|\) since this is, in certain ways, the worst case. The reader who wishes to understand the main ideas while avoiding some mild technical details may henceforth assume that \(p\) is a real positive vector, and therefore \(r_{S,u}^i(p) = r_{S,u}(p)\).

Definition 3.1 (Error sensitivity parameter). For \(u \in S\) and \(p \in \mathbb{C}^V\), define

\[ r_{S,u}^i(p) := r_{S,u}(|p|). \]

The error sensitivity parameter \(\beta_{S,u} := \beta_{S,u}(p)\) is defined as

\[ \beta_{S,u}(p) := \left. \frac{dr_{S,u}^i(1 + t)p}{dt} \right|_{t=0}. \]  \hspace{1cm} (3.1)

We will now prove several properties of the error sensitivity parameter. We begin by establishing a recursive formula for the \(\beta_{S,u}\) (Claim 3.2). We note that this formula is very similar to the total differential of \(r_{S,u}^i\) as a function of the variables \(r_{S,v}^j\) in (2.4), with \(\beta_{S,u}\) playing the role of \(dr_{S,u}^i\). The total differential describes how infinitesimal errors would propagate in the computation tree. This is the intuitive basis for
We induct on the consideration of $\beta_{S,u}$ as an error sensitivity parameter.

**Claim 3.2.** Let $(S, u)$ be node in the computation tree. Then

$$\beta_{S,u} = r_{S,u}^i \cdot \left( 1 + \sum_{c \in C(S,u)} \frac{\beta_c}{1 - r_c^i} \right).$$

**Proof.** By a direct calculation using the definition of $\beta_{S,u}$,

$$\beta_{S,u} := \left. \frac{\frac{d r_{S,u}^i((1 + t)p)}{dt}}{(1 + t) |p_u| \cdot \prod_{c \in C(S,u)} \frac{1}{1 - r_c^i((1 + t)p)}} \right|_{t=0} = \left. \frac{\frac{d}{dt} \left( 1 + \sum_{c \in C(S,u)} \frac{1}{1 - r_c^i((1 + t)p)} \right) \cdot \prod_{c \in C(S,u)} \frac{1}{1 - r_c^i((1 + t)p)}}{(1 + t) |p_u| \cdot \prod_{c \in C(S,u)} \frac{1}{1 - r_c^i((1 + t)p)}} \right|_{t=0}$$

$$= \prod_{c \in C(S,u)} \frac{1}{1 - r_c^i((1 + t)p)} \cdot \left( |p_u| + (1 + t) |p_u| \sum_{c \in C(S,u)} \frac{\beta_c}{1 - r_c^i((1 + t)p)} \right)$$

$$= r_{S,u}^i(p) \cdot \left( 1 + \sum_{c \in C(S,u)} \frac{\beta_c}{1 - r_c^i} \right). \quad \square$$

**Lemma 3.3.** Fix a parameter vector $p \in S$. Let $t_0$ be such that for $0 \leq t \leq t_0$, $(1 + t)p$ is also in $S$. Define $\beta_{S,u}(p, t) = \frac{d}{dt} r_{S,u}^i((1 + t)p)$. (Note that Definition 3.1 is consistent with $\beta_{S,u}(p) = \beta_{S,u}(p, 0)$.) Then, for all nodes $(S, u)$ in the computation tree, $\beta_{S,u}(p, t)$ is a non-negative, non-decreasing function of $t$ for $t \in [0, t_0]$. Thus, the map $t \mapsto r_{S,u}^i((1 + t)p)$ is non-decreasing and convex over the same domain.

**Proof.** We induct on $|S|$. The base case is $S = \{u\}$, so $r_{\{u\},u}^i((1 + t)p) = (1 + t) |p_u|$, We therefore have

$$\beta_{\{u\},u}(p, t) = \frac{d}{dt} r_{\{u\},u}^i((1 + t)p) = |p_u|,$$

which is a constant (and hence non-decreasing), non-negative function of $t$.

For the inductive case, we use a recursive formula for $\beta_{S,u}(p, t)$ as in the proof of Claim 3.2. We have

$$\beta_{S,u}(p, t) = \left( |p_u| \cdot \prod_{(S', u') \in C(S,u)} \frac{1}{1 - r_{S', u'}((1 + t)p)} + \frac{\beta_{S', u'}(p, t)}{1 - r_{S', u'}((1 + t)p)} \right).$$

By the induction hypothesis, $\beta_{S', u'}(p, t) \geq 0$ for each $|S'| < |S|, u' \in S'$. Since $(1 + t)p \in S$, Lemma 2.8 implies $0 \leq r_{S', u'}((1 + t)p) < 1$. Therefore $\beta_{S,u}(p, t) \geq 0$ as well. Moreover, the inductive hypothesis implies that both $r_{S', u'}((1 + t)p)$ and $\beta_{S', u'}(p, t)$ are non-decreasing in $t$. Since the whole expression is monotone in $r_{S', u'}$ and $\beta_{S', u'}(p, t)$, the left-hand side $\beta_{S,u}(p, t)$ is also a non-decreasing function of $t$. \quad \square

We can now prove the following relations between the $\beta_{S,u}$ and the $r_{S,u}^i$.

**Lemma 3.4.** Let $p \in C^V$ and $\alpha > 0$ satisfy $(1 + \alpha)p \in S$. We then have the following inequalities for all nodes $(S, u)$ in the computation tree. (We use the shorthand notation $r_{S,u}^i = r_{S,u}^i(p)$ and $\beta_{S,u} = \beta_{S,u}(p)$.)

1. $\beta_{S,u} < (1 - r_{S,u}^i)/\alpha$.
2. $r_{S,u}^i \leq \beta_{S,u} \leq (1 + d_u/\alpha) \cdot r_{S,u}^i$, where $d_u$ is the degree of the vertex $u$ in $G$.
3. $r_{S,u}^i < \frac{1}{1+\alpha} r_{S,u}^i((1 + \alpha)p) < \frac{1}{1+\alpha}$.  

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Proof. Since \((1 + \alpha)p \in S\), Lemma 2.8 implies \(r_{S,u}^i((1 + \alpha)p) < 1\). Further, from Lemma 3.3, we know that \(t \mapsto r_{S,u}^i((1 + t)p)\) is convex for \(t \in [0,\alpha]\). Item 1 of the lemma then follows from the inequalities

\[
1 > r_{S,u}^i((1 + \alpha)p) \geq r_{S,u}^i + \alpha \frac{dr_{S,u}^i}{dt} \bigg|_{t=0} \geq r_{S,u}^i + \alpha \beta_{S,u}.
\]

(3.2)

Item 2 follows from Claim 3.2, using \(\beta_{S,u} \geq 0\), \(|C(S,u)| \leq d_u\), and \(\beta_{S,u} < (1 - r_{S,u}^i)/\alpha\) from item 1.

To prove the first inequality in item 3, we again use eq. (3.2), and substitute \(\beta_{S,u} \geq r_{S,u}^i\) from item 2. The second inequality follows from the fact that \((1 + \alpha)p \in S\), so that \(r_{S,u}^i((1 + \alpha)p) < 1\).

Finally, we relate the quantities \(r_{S,u}^i\) to the quantities \(r_{S,u}\) that we actually want to approximate.

Claim 3.5. Let \(p\) lie in the complex Shearer region. For any node \((S, u)\) in the computation tree, we have

\[
|r_{S,u}(p)| \leq r_{S,u}^i(p).
\]

Proof: The proof is by induction on \(|S|\). When \(S\) is a singleton, we have \(r_{\{u\},u}(p) = |p_u| = r_{\{u\},u}^i(p)\).

For the inductive case, we use the recursion for \(r_{S,u}(p)\) to obtain

\[
|r_{S,u}(p)| = \left| p_u \prod_{c \in C(S,u)} \frac{1}{1 - r_c(p)} \right| \leq |p_u| \prod_{c \in C(S,u)} \frac{1}{1 - |r_c(p)|} \leq |p_u| \prod_{c \in C(S,u)} \frac{1}{1 - r_c^i(p)} = r_{S,u}^i.
\]

Here, the first inequality follows from Fact B.1 since \(p \in S\) implies that \(|r_c(p)|, r_c^i(p) < 1\), while the second inequality follows from the induction hypothesis.

3.2 Correlation decay with complex activities

We now use the error sensitivity parameters to establish the correlation decay results needed for our FPTAS.

Let \(G = (V,E)\) be a graph on \(n\) vertices, and let \(p \in \mathbb{C}^V\) be such that \((1 + \alpha)p \in S\). The root of the recursion is a pair \((A,a)\) where \(A \subseteq V\) and \(a \in A\). Let \(\ell \geq 0\) be arbitrary. Recall that Algorithm 1 recursively computes an estimate \(R_{A,a}\) of \(r_{A,a}\), where for every pair \((S, u)\) encountered, we have

\[
R_{S,u} = \begin{cases} 
0 & \text{(if } (S, u) \text{ is at depth } \ell \text{ in the computation tree)} \\
 p_u \prod_{c \in C(S,u)} (1 - R_c)^{-1} & \text{(otherwise)}
\end{cases}
\]

The depth \(\delta\) of a node is defined as follows: the root has \(\delta(A,a) = 0\), its children have \(\delta(S,u) = 1\), etc.

Theorem 3.6. For notational simplicity, let \(r_{S,u}^i = r_{S,v}^i(p)\) and \(r_{S,v}' = r_{S,v}'((1 + \alpha)p)\). Similarly, let \(\beta_{S,u} = \beta_{S,u}(p)\) and \(\beta_{S,u}' = \beta_{S,u}'((1 + \alpha)p)\). For a node \((S, u)\) in a computation tree of depth \(\ell\),

\[
|r_{S,u} - R_{S,u}| \leq \frac{\beta_{S,u}(p)}{1 + \alpha^{\ell - \delta(S,u)}}.
\]

Proof. The proof is by induction on \(\ell - \delta(S,u)\). The base case is \(\delta(S,u) = \ell\), so \(R_{S,u} = 0\). Then

\[
|r_{S,u} - R_{S,u}| = |r_{S,u}^i| \leq r_{S,u}^i \leq r_{S,u}' \leq \beta_{S,u}',
\]

by Claim 3.5, Lemma 3.3, and item 2 of Lemma 3.4.

For the inductive step, we apply the mean value theorem (Theorem B.3) with \(\gamma_c := r_c\), obtaining

\[
|r_{S,u} - R_{S,u}| \leq |p_u| \prod_{c \in C(S,u)} \frac{1}{1 - r_c} \sum_{c' \in C(S,u)} \frac{|r_{c'} - R_{c'}|}{1 - r_{c'}} r_{S,u}' \sum_{c \in C(S,u)} \frac{|r_{c} - R_{c}|}{1 - r_{c}}.
\]

(3.3)

By definition, \(\delta(c) = \delta(S,u) + 1\) for all \(c \in C(S,u)\). By the induction hypothesis, we therefore have

\[
|r_{S,u} - R_{S,u}| \leq r_{S,u}' \sum_{c \in C(S,u)} \frac{\beta_{c}'}{(1 + \alpha)^{\ell - \delta(S,u) - 1}}.
\]
From item 3 of Lemma 3.4, we have $r_{T,v}^i \leq \frac{1}{1+\alpha} r_{T,v}' \leq r_{T,v}'$ for all $T \subseteq V$ and $v \in T$, so that we get

$$|r_{S,u} - R_{S,u}| \leq \frac{r_{S,u}'}{1+\alpha} \sum_{c \in C(S,u)} \frac{\beta'_c}{(1+\alpha)^{\ell-\delta(S,u)-1}(1-r_{e}')} \leq \frac{r_{S,u}'}{(1+\alpha)^{\ell-\delta(S,u)}} \sum_{c \in C(S,u)} \frac{\beta'_c}{1-r_{e}'} \leq \frac{\beta_{S,u}'}{(1+\alpha)^{\ell-\delta(S,u)}},$$

where the last inequality uses the recurrence for $\beta_{S,u}$ derived in Claim 3.2 (evaluated at $(1+\alpha)p$), and the fact that $r_{S,u}'$ is non-negative. This completes the induction. \(\square\)

**Corollary 3.7.** Given a graph $G = (V,E)$, let $p$ be a complex parameter vector such that $(1+\alpha)^2 p \in S$. Let $(A,a)$ be the root of the recursive computation of Theorem 3.6, where $a$ is a vertex of degree $d_a$ in $G$. Then, we have

$$|r_{A,a} - R_{A,a}| \leq \frac{1+d_a/\alpha}{(1+\alpha)^{\ell}}.$$

**Proof.** Recall from Lemma 2.8 that $r_{A,a} \in [0,1)$ when $p \in S$. Then item 2 of Lemma 3.4 implies that $\beta_{A,a}((1+\alpha)p) \leq (1+d_a/\alpha) \cdot r_{A,a}((1+\alpha)p) \leq 1+d_a/\alpha$.

We can apply Lemma 3.4 to $\beta_{A,a}((1+\alpha)p)$ (as opposed to $\beta_{A,a}(p)$) because of the assumption that $(1+\alpha)^2 p \in S$. The claim now follows from Theorem 3.6 since $(A,a)$ is at depth 0 in the computation tree. \(\square\)

**Remark 3.1.** We note here that an inductive argument identical to that made in Claim 3.5 shows that when $p \in S$ we have $|R_{S,u}(p)| \leq |r_{S,u}(p)| \leq r_{S,u}'(p)$.

We can now prove that our algorithm indeed provides an FPTAS for the quantity $\hat{q}_V(p)$. We remark that Theorem 1.1 follows from here by a simple reformulation.

**Theorem 3.8 (FPTAS for $\hat{q}$).** Given a graph $G = (V,E)$ on $n$ vertices with maximum degree $d$, a parameter vector $p$ such that $(1+\alpha)^2 p \in S$, and a positive $\epsilon \leq 1/n$, define $\ell = \left\lceil \log_{1+\alpha} \left( \frac{(1+\alpha)(1+d/\alpha)}{\epsilon} \right) \right\rceil$. Then a $(1+O(\epsilon n))$-approximation to $\hat{q}_V(p)$ can be computed in time $O(nd^\ell)$.

**Proof.** Order the vertices of $G$ arbitrarily as $v_1, v_2, \ldots, v_n$. Recall that

$$\hat{q} := \hat{q}_V = \prod_{i=1}^n (1 - r_{S,v_i}),$$

where $S_i := \{v_i, v_{i+1}, \ldots, v_n\}$. Let us denote by capital letters the estimates computed by Algorithm 1. $R_{S_i,v_i}$ is computed using $\ell$ levels of the recurrence in Theorem 3.6, where $\ell$ is chosen as in the statement of the theorem. Note that the number of nodes of the computation tree explored in the computation of each $R_{S_i,v_i}$ is $O(d^\ell)$ since the graph is assumed to be of degree at most $d$. The algorithm outputs the quantity

$$\hat{Q} := \prod_{i=1}^n (1 - R_{S_i,v_i}).$$

We now prove that this is indeed a $(1+O(\epsilon n))$-approximation for $\hat{q}_V(p)$. For ease of notation, we define $\xi_i := 1 - r_{S_i,v_i}$ and $\Xi_i := 1 - R_{S_i,v_i}$. From Corollary 3.7 we have, for each $i$,

$$\left| \frac{\Xi_i}{\xi_i} - 1 \right| = \frac{|r_{S_i,v_i} - R_{S_i,v_i}|}{1 - r_{S_i,v_i}} \leq \frac{1+\alpha}{\alpha} \cdot \frac{1+d/\alpha}{(1+\alpha)^\ell}. \quad (3.4)$$

Here, the last inequality uses Claim 3.5 (which implies that $|r_{S_i,v_i}| \leq r_{S_i,v_i}'$) and item 3 from Lemma 3.4 (which implies that when $p \in S$, $r_{S_i,v_i}' \leq \frac{1}{1+\alpha} < 1$). Together, with Fact B.1, these two inequalities
imply that \( \frac{1}{1-r^2_{n,i}} \leq \frac{1+\alpha}{\alpha} \). Since \( \ell = \left\lceil \log_{1+\alpha} \left( \frac{(1+\alpha)(1+d/\alpha)}{\epsilon} \right) \right\rceil \), we therefore have \( \left\lceil \frac{\alpha}{\xi_i} - 1 \right\rceil \leq \epsilon \), for all \( i \). Combining this with Fact B.2, and recalling that \( \bar{q} = \prod_{i=1}^n \xi_i \) and that \( \bar{Q} = \prod_{i=1}^n \bar{\xi}_i \), we obtain \( \bar{Q} - \bar{q} \leq 2n\epsilon \bar{q} \), which proves the theorem.

\[ \square \]

4 Hardness of evaluation and deciding membership

In this section we complement our algorithmic results with some hardness results. Due to space restrictions, only a brief statement of the results appears here. The full discussion appears in Appendix C.

To begin, we show the hardness of evaluating \( \bar{q}_V \) up to exponentially small error.

**Theorem 4.1.** For a 4-regular graph \( G = (V,E) \) and \( |V| < k < |V|^2 \) given on the input, it is \#P-hard to compute \( \bar{q}_V(1/k, \ldots, 1/k) \) within an additive error of \( 1/(2k|V|^3) \).

A small modification of this argument also allows us to show \#P-hardness of testing the sign of \( \bar{q}_V \). Since the Shearer region is defined using positivity of \( \bar{q}_S \) (see Definition 2.4), this leads to hardness of testing membership in the Shearer region, even with a small gap.

**Theorem 4.2.** For a graph \( G = (V,E) \) and rational \( (p_1, \ldots, p_n) \in [0,1]^V \) given on the input, it is \#P-hard to distinguish between \( (p_1 + \epsilon, \ldots, p_n + \epsilon) \in \mathcal{S}_G \) and \( (p_1, \ldots, p_n) \notin \mathcal{S}_G \), for \( \epsilon = 1/|V|^{|V|} \).

Since \( \mathcal{S}_G \) is the maximal region to which the LLL applies, it is unfortunate that testing membership in \( \mathcal{S}_G \) is hard. However, the original statement of the LLL [12, 40] required that \( p \) belong to a certain set \( \mathcal{L}_G \subset \mathcal{S}_G \), and interestingly it turns out that deciding membership in \( \mathcal{L}_G \) is easy. For details, see Section C.3.

Our main algorithm, described in Theorem 1.1, requires the slack parameter \( \alpha \). Next we show that every algorithm must have some dependence on the slack: for a point \( p \) with exponentially small slack, approximating \( \bar{q}_V(p) \) is \#P-hard. This directly implies Theorem 1.3.

**Theorem 4.3.** For a graph \( G = (V,E) \), \( |V| = n \), and rational \( p \in [0,1]^V \) given on the input, such that \( (1 + \frac{1}{n^{2n}})p \in \mathcal{S}_G \), it is \#P-hard to approximate \( \bar{q}_V(p) \) with any poly \( n \) factor.

The previous result shows hardness of approximation for points that are \textbf{inside} the Shearer region and exponentially close to the boundary. Our next result shows hardness of approximation for points that are \textbf{outside} the Shearer region and a constant factor away from the boundary, even in the univariate case.

**Theorem 4.4.** Fix \( d \geq 62 \) and a rational number \( \lambda > 39/d \). Suppose there exists a PTAS for computing \( \bar{q}_V(\lambda 1) \) for input graphs \( G \) of degree at most \( d \) (under the promise that \( \bar{q}_V(\lambda 1) > 0 \)). Then, \( \text{NP} = \text{RP} \).

5 Conclusions and open questions

There are two main questions left open by our work. The first asks whether dependence on the slack in Theorem 1.1 can be improved from exponential to polynomial.

**Question 2.** Is there an algorithm to estimate \( \bar{q}_V(p) \) up to a \( (1 + \epsilon) \)-multiplicative factor in \( n \)-vertex graphs of maximum degree \( d \), assuming that \( (1 + \alpha)p \in \mathcal{S} \), in running time \( \left( \frac{n}{\alpha} \right)^O(\log d) \)?

The second question asks whether the Shearer threshold, \( \frac{(d-1)^{d-1}}{d^d} \), delineates the boundary of computational efficiency for negative activities, analogous to the threshold \( \lambda_c(d) = \frac{(d-1)^{d-1}}{(d-2)^d} \) for positive activities. We conjecture that the boundary is even more striking for negative activities: beyond the threshold, it is \#P-hard even to decide the \textit{sign} of the polynomial. This conjecture forms a natural counterpart to Corollary 1.2.

**Conjecture 5.1.** Let \( d \geq 3 \), and suppose that \( \lambda > \frac{(d-1)^{d-1}}{d^d} \). Then, for graphs of maximum degree \( d \), it is \#P-hard to decide whether \( \bar{q}_V(\lambda 1) > 0 \).
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A Proofs from Section 2

Proof of Claim 2.1. From eq. (2.2), we have, for every \(1 \leq i \leq n\),
\[
1 - r_{S_i,v_i} = \frac{\tilde{q}_{S_i}}{\tilde{q}_{S_{i+1}}}. \tag{A.1}
\]
Multiplying these equations, we get
\[
\prod_{i=1}^{n}(1 - r_{S_i,v_i}) = \frac{\tilde{q}_{S_1}}{\tilde{q}_{S_{n+1}}} = \frac{\tilde{q}_V}{\tilde{q}_\emptyset}.
\]
Since \(\tilde{q}_\emptyset = 1\), this yields the claim. \(\Box\)

Proof of Lemma 2.3. Define \(S_i = S \setminus \{u,v_1,v_2,\ldots,v_{i-1}\}\) for \(1 \leq i \leq k\). From eq. (2.2), we then have, for \(1 \leq i \leq k\),
\[
1 - r_{S_i,v_i} = \frac{\tilde{q}_{S_{i+1}}}{\tilde{q}_{S_i}}. \tag{A.2}
\]
Multiplying these equations, we get
\[
\prod_{i=1}^{k} \frac{1}{1 - r_{S_i,v_i}} = \frac{\tilde{q}_{S_{k+1}}}{\tilde{q}_{S_1}} = \frac{\tilde{q}_{S \setminus \Gamma^+(u)}}{\tilde{q}_{S \setminus \{u\}}},
\]
where in the last equation we use \(S_{k+1} = S \setminus \Gamma^+(u)\) and \(S_1 = S \setminus \{u\}\). (Recall that \(\Gamma^+(u)\) is the set containing \(u\) and all its neighbors in \(G\)). The claim of the lemma now follows since \(r_{S,u} = p_u \cdot \frac{\tilde{q}_{S \setminus \Gamma^+(u)}}{\tilde{q}_{S \setminus \{u\}}}\) (see eq. (2.2)). \(\Box\)

Proof of Lemma 2.7. This is a special case of Corollary 2.27 (b) of Scott and Sokal [33]. See also [19, Section 5.3]. \(\Box\)

Proof of Lemma 2.8. From eq. (2.2) we have \(r_{S,u} = p_u \cdot \frac{\tilde{q}_{S \setminus \Gamma^+(u)}}{\tilde{q}_{S \setminus \{u\}}} = 1 - \frac{\tilde{q}_S}{\tilde{q}_{S \setminus \{u\}}}\). From Lemma 2.7, we have \(0 < \tilde{q}_V \leq \tilde{q}_S \leq \tilde{q}_{S \setminus \{u\}} \leq \tilde{q}_{S \setminus \Gamma^+(u)}\), which yields the claim. \(\Box\)

B Inequalities

Finally, we enumerate some simple inequalities that will be needed in our proofs.

Fact B.1. Let \(z\) be a complex number such that \(|z| \leq \tau < 1\). Then \(\frac{1}{1 - z} \leq \frac{1}{1 - \tau}\).

Proof. \(|1 - z| \geq 1 - |z| \geq 1 - \tau\), which implies the claim since \(\tau < 1\). \(\Box\)

Fact B.2. Let \((x_i)_{i=1}^{n}\) and \((y_i)_{i=1}^{n}\) be two sequences of complex numbers with the \(y_i\) non-zero such that
\[
\left| \frac{x_i}{y_i} - 1 \right| \leq \epsilon, \tag{B.1}
\]
where \(\epsilon \leq 1/n\). Then, we have
\[
\left| \prod_{i=1}^{n} x_i - \prod_{i=1}^{n} y_i \right| \leq 2n \epsilon \cdot \prod_{i=1}^{n} |y_i|. \tag{B.2}
\]
Proof. For each $i$, define $z_i$ so that $x_i = y_i (1 + z_i)$. Note that $|z_i| \leq \epsilon$ for each $i$. We therefore have
\[
\left| \prod_{i=1}^{n} x_i - \prod_{i=1}^{n} y_i \right| = \left| \prod_{i=1}^{n} y_i \right| \left| \prod_{i=1}^{n} (1 + z_i) - 1 \right| \leq \sum_{i=1}^{n} \binom{n}{i} \epsilon^i \cdot \prod_{i=1}^{n} |y_i|
\]
\[
\leq n \epsilon \sum_{i=1}^{n} \frac{(n \epsilon)^{i-1}}{i!} \cdot \prod_{i=1}^{n} |y_i|, \text{ using } \binom{n}{i} \leq \frac{n^i}{i!},
\]
\[
\leq \prod_{i=1}^{n} |y_i| \cdot n \epsilon \sum_{i=1}^{n} \frac{1}{i!} \leq n \epsilon \cdot (e - 1) \cdot \prod_{i=1}^{n} |y_i| \leq 2n \epsilon \cdot \prod_{i=1}^{n} |y_i|,
\]
where the last line uses the fact that $n \epsilon < 1$. □

We will also need the following consequence of the mean value theorem. Fix a complex number $\lambda$ and a positive integer $d > 0$ and let $f(x) = f(x_1, x_2, \ldots, x_d)$ be defined as
\[
f(x_1, x_2, \ldots, x_d) := \lambda \prod_{i=1}^{d} \frac{1}{1 - x_i}
\]
defined when $|x_i| < 1$ for all $i$.

Theorem B.3 (Mean value theorem). Let $x = (x_1, x_2, \ldots, x_d)$ and $y = (y_1, y_2, \ldots, y_d)$ be two sequences of complex numbers and let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d)$ be such that $|x_i|, |y_i| \leq \gamma_i < 1$ for $1 \leq i \leq d$. Then
\[
|f(x) - f(y)| \leq |f(\gamma)| \sum_{i=1}^{d} \frac{|x_i - y_i|}{1 - \gamma_i}.
\]

Proof. Let $g(t) := f(tx + (1 - t)y)$ for $t \in [0, 1]$. Note that $g$ is continuously differentiable on its domain (since $|x_i|, |y_i| < 1$). Hence $|g'(t)|$ attains its maximum at some point $t_0 \in [0, 1]$. Let $z = t_0x + (1 - t_0)y$. Note that $|z_i| \leq \gamma_i$ for all $i$. We now have
\[
|f(x) - f(y)| = |g(1) - g(0)| \leq \int_{0}^{1} |g'(t)| \, dt \leq |g'(t_0)|
\]
\[
= |f(x)| \left| \sum_{i=1}^{d} \frac{y_i - x_i}{1 - z_i} \right|
\]
\[
\leq |f(\gamma)| \sum_{i=1}^{d} \frac{|x_i - y_i|}{1 - \gamma_i},
\]
where the last line uses the form of $f$ and Fact B.1. □

C Hardness of evaluation and deciding membership

In this section we complement our positive results with some negative ones. First, in Section C.1 we show that exactly evaluating $\tilde{q}_V$ and deciding membership is $\#P$-hard. Then, in Section C.2 we show similar results with exponentially small error, and show that algorithms for approximating $\tilde{q}_V$ must have runtime that depends on the slack. In Section C.2.1 we show that it is hard to approximate $\tilde{q}_V$ at points that lie outside the Shearer region by a constant factor. Finally, in Section C.3 we show a positive result: that one can efficiently decide membership in the region for the original LLL, which is a strict subset of Shearer’s region.
C.1 Exact evaluation and membership

Our starting point is the following known hardness result.

**Theorem C.1** ([9]). For a given 3-regular bipartite graph, it is \#P-hard to compute the number of perfect matchings.

From here, we obtain by a standard reduction the hardness of computing the alternating-sign independence polynomial. In the following, to emphasize the graph under consideration, we deviate from our previous notation slightly by letting \( \tilde{q}_G(p) = \sum_{I \in \text{Ind}(V)} (-1)^{|I|} p^I \), where \( G = (V,E) \).

**Theorem C.2.** For a 4-regular graph \( G = (V,E) \) and \( |V| < k < |V|^2 \) given on the input, it is \#P-hard to compute \( \tilde{q}_G(1/k, \ldots, 1/k) \).

We note that for a 4-regular graph, the Shearer region is known to contain at least the line segment between \((0, \ldots, 0)\) and \((\frac{1}{4k}, \ldots, \frac{1}{4k})\). So we claim that it is \#P-hard to evaluate Shearer’s polynomial even on points that are inside the Shearer region with a large slack.

**Proof.** Let \( H \) be a given 3-regular bipartite graph on \( n + n \) vertices. We define \( G = (V,E) \) to be the line graph of \( H \), which is 4-regular. We have \( |V| = 3n \), the number of edges of \( H \). Independent sets in \( G \) correspond to matchings in \( H \). We have

\[
\tilde{q}_G(1/k, \ldots, 1/k) = \sum_{I \in \text{Ind}(G)} (-1)^{|I|} k^{-|I|} = \sum_{\text{matching } M \subset H} \left( \frac{-1}{k} \right)^{|M|} = \frac{1}{k^n} \sum_{\text{matching } M \subset H} (-1)^{|M|} k^{-|M|}.
\]

Let us denote \( b_k = \sum_{\text{matching } M \subset H} (-1)^{|M|} k^{-|M|} \), which is an integer. Perfect matchings in \( H \) have cardinality \( n \). Therefore, each non-perfect matching contributes a multiple of \( k \) here, only perfect matchings contribute \( 1 \) (with a sign depending on the parity on \( n \); assume wlog that \( n \) is even). Hence we have \( b_k = \# \text{ perfect matchings } \pmod{k} \). If we could compute \( b_k \), say for any \( |V| < k < |V|^2 \), then we could recover the number of perfect matchings in \( H \) by the Chinese remainder theorem with \( |V| \) choices of prime numbers \( k \), \( |V| < k < |V|^2 \) (which exist for large enough \( |V| \) by the prime number theorem), since the number of perfect matchings is upper-bounded by \( n! < |V|^{|V|} \). This proves that computing \( \tilde{q}_G(1/k, \ldots, 1/k) = b_k/k^n \) for \( |V| < k < |V|^2 \) is \#P-hard.

Next, we show that for an unrestricted point \( p \), it is \#P-hard even to compute the sign of \( \tilde{q}_G(p) \).

**Theorem C.3.** For a graph \( G = (V,E) \) and rational \( p \in [0, 1]^V \) given on the input, it is \#P-hard to decide whether \( \tilde{q}_G(p) > 0 \).

**Proof.** Let \( G \) be a given graph as in the proof of Theorem C.2, \( |V| = 3n \) and \( |V| < k < |V|^2 \). Define \( G' \) to be a graph obtained from \( G \) by adding a new vertex \( z \) and adding edges between \( z \) and all the vertices of \( G \). We have

\[
\tilde{q}_{G'}(1/k, \ldots, 1/k, p_z) = \tilde{q}_G(1/k, \ldots, 1/k) - p_z
\]

because the only independent set in \( G' \) containing \( z \) is \{z\}. We can also assume that \( \tilde{q}_G(1/k, \ldots, 1/k) = b_k/k^n \) for some integer \( b_k \), as in the proof of Theorem C.2. The possible range for \( b_k \) is \([-8k^n, 8k^n]\), since in the proof of Theorem C.2, \( |b_k| \leq k^n |\text{Ind}(G)| \leq k^{n+2}n \).

Suppose that we can decide whether \( \tilde{q}_{G'}(1/k, \ldots, 1/k, p_z) > 0 \) for a given \( p_z = b'/k^n \). That is, we can decide whether \( \tilde{q}_G(1/k, \ldots, 1/k) > p_z \). Then we can compute \( \tilde{q}_G(1/k, \ldots, 1/k) = b_k/k^n \) by a binary search on \( p_z = b'/k^n \). Since we have \( 2(8k)^n \) possible values for \( b' \), the binary search takes \( 1 + n \log_2(8k) \leq 1 + n \log_2(72n^2) \) steps. Therefore we could compute the value of \( \tilde{q}_G(1/k, \ldots, 1/k) \), which is \#P-hard.

\[\square\]
Theorem C.4. For a graph \( G = (V, E) \) and rational \( p \in [0, 1]^V \) given on the input, it is \#P-hard to decide whether \( p \) is in the Shearer region.

Proof. Let \( S_G \) denote the Shearer region for a graph \( G \). Let \( G' = G + z \) be a graph as in the proof of Theorem C.3, with all edges between \( z \) and \( G \). For a given \( p_z \in [0, 1] \) and \( |V| < k < |V|^2 \), we would like to decide whether \( \tilde{q}_{G'}(1/k, \ldots, 1/k, p_z) = \tilde{q}_G(1/k, \ldots, 1/k) - p_z > 0 \). As above, we know that \((1/k, \ldots, 1/k) \in S_G\), which means that \( 0 < \tilde{q}_G(1/k, \ldots, 1/k) < 1 \). Therefore, as \( p_z \) varies from 0 to 1, \( \tilde{q}_{G'}(1/k, \ldots, 1/k, p_z) = \tilde{q}_G(1/k, \ldots, 1/k) - p_z \) decreases from a positive value to a negative one.

We use the following characterization: \( p \in S_{G'} \) if and only if there is a continuous path from the origin to \( p \) such that \( \tilde{q}_{G'}(x) > 0 \) for each point \( x \) on the path [33, Theorem 2.10]. Here, we know that \((1/k, \ldots, 1/k, 0) \in S_{G'} \) and \( \tilde{q}_{G'}(1/k, \ldots, 1/k, p_z) \) is decreasing in \( p_z \); therefore checking whether \((1/k, \ldots, 1/k, p_z) \in S_{G'} \) is equivalent to checking whether \( \tilde{q}_{G'}(1/k, \ldots, 1/k, p_z) > 0 \), which is \#P-hard.

C.2 Approximate evaluation and membership

Perhaps a more interesting question is how accurately we can evaluate \( \tilde{q}_G(p) \) or check membership in the Shearer region, when errors are allowed. As our main positive result shows, \( \tilde{q}_G(p) \) for \( p \) well inside the Shearer region (with constant slack) can indeed be evaluated approximately, within polynomially small error. Our hardness reductions here show that certain exponentially small errors are not achievable. We obtain the following results automatically, from the fact that the possible values of \( \tilde{q}_G(p) \) in our reduction are integer multiples of \( 1/k^n \), where \(|V| = 3n \) and \( k < |V|^2 \), so \( 1/k^n > 1/(2k|V|^3) \).

Theorem C.5 (restatement of Theorem 4.1). For a 4-regular graph \( G = (V, E) \) and \(|V| < k < |V|^2 \) given on the input, it is \#P-hard to compute \( \tilde{q}(1/k, \ldots, 1/k) \) within an additive error of \( 1/(2k|V|^3) \).

Theorem C.6. For a graph \( G = (V, E) \) and rational \( p \in [0, 1]^V \) given on the input, it is \#P-hard to distinguish whether \( \tilde{q}_G(p) > 1/|V|^{|V|} \) or \( \tilde{q}_G(p) \leq 0 \).

With a slight extension of the above proof for membership hardness, we get the following.

Theorem C.7 (restatement of Theorem 4.2). For a graph \( G = (V, E) \) and rational \((p_1, \ldots, p_n) \in [0, 1]^V \) given on the input, it is \#P-hard to distinguish between \((p_1 + \epsilon, \ldots, p_n + \epsilon) \in S_G \) and \((p_1, \ldots, p_n) \notin S_G \), for \( \epsilon = 1/|V|^{|V|} \).

Proof. Consider the reduction we used in the proof of Theorem C.3. It shows that it is \#P-hard to distinguish whether \( \tilde{q}_G(1/k, \ldots, 1/k) \geq b'/k^n \) or \( \tilde{q}_G(1/k, \ldots, 1/k) \leq (b' - 1)/k^n \), for given \( b' > 0 \); here, \(|V| = 3n\). We let \( p_z = (b' - 1)/k^n \) and consider the graph \( G' = G + z \). In the first case, \((1/k, \ldots, 1/k, p_z) \) is in the Shearer region of \( G' \) while in the second case it is not.

In the first case, when \( \tilde{q}_G(1/k, \ldots, 1/k) \geq b'/k^n \), we consider a modified point \((1/k + \epsilon, \ldots, 1/k + \epsilon, p_z + \epsilon) \) where \( \epsilon = 1/|V|^{|V|} \). By the convexity of \( \tilde{q}_G(\lambda p) \) in \( \lambda \) (see [19]), we have

\[
\tilde{q}_G(1/k + \epsilon, \ldots, 1/k + \epsilon) \geq 1 - \frac{1/k + \epsilon}{1/k} (1 - \tilde{q}_G(1/k, \ldots, 1/k)) \geq \frac{b'}{k^n} - k\epsilon.
\]

Furthermore,

\[
\tilde{q}_{G'}(1/k + \epsilon, \ldots, 1/k + \epsilon, p_z + \epsilon) = \tilde{q}_G(1/k + \epsilon, \ldots, 1/k + \epsilon) - p_z - \epsilon \geq \frac{b'}{k^n} - k\epsilon - \frac{b' - 1}{k^n} - \epsilon > 0
\]

since \( \epsilon = 1/|V|^{|V|} < 1/k^{3n/2} \). Moreover, there is a path from the origin to \((1/k + \epsilon, \ldots, 1/k + \epsilon, p_z + \epsilon) \) where \( \tilde{q}_{G'} \) is positive, which means that \((1/k + \epsilon, \ldots, 1/k + \epsilon, p_z + \epsilon) \in S_{G'} \).
In the second case, when \( \tilde{q}_{G'}(1/k, \ldots, 1/k) \leq (b' - 1)/k^n \), we have
\[
\tilde{q}_{G'}(1/k, \ldots, 1/k, p_z) = \tilde{q}_{G}(1/k, \ldots, 1/k) - p_z \leq 0.
\]
Here, \((1/k, \ldots, 1/k, p_z) \not\in S_{G'}\). Therefore, distinguishing between these two cases would allow us to solve a \#P-hard problem.

It remains open whether membership in the Shearer region is polynomially checkable within polynomially small error.

Next, we use Theorem C.7 to prove that it is in fact \#P-hard to approximate the independence polynomial even within polynomially large factors, when \( p \) is inside but close to the boundary of the Shearer region.

**Theorem C.8** (restatement of Theorem 4.3). For a graph \( G = (V, E) \), \(|V| = n\), and rational \( z \in [0, 1]^V \) given on the input, such that \( (1 + \frac{1}{n^{2c}})z \in S_G \), it is \#P-hard to approximate \( \tilde{q}_V(z) \) with any poly \((n)\) factor.

**Proof.** Suppose that given \( G, z \) as above, we can compute a number \( \tilde{Q}_V \) such that \( \tilde{q}_V(z) \leq \tilde{Q}_V \leq n^c \tilde{q}_V(z) \), for some absolute constant \( c > 0 \). Then clearly we can also do this for \( \tilde{q}_S(z) \), \( S \subseteq V \), by considering the subgraph induced by \( S \). Suppose also that \( n \) is sufficiently large, say \( n \geq 2c + 2 \). We claim that then by a polynomial number of calls to such an algorithm, we can distinguish for a given point \( p \in [0, 1]^V \) whether \( p + \frac{1}{n^{2c}}1 \in S_G \) or \( p \not\in S_G \), which is a \#P-hard problem by Theorem C.7.

Let \( \phi(t) = \tilde{q}_V(tp) \). Clearly \( \phi(0) = 1 \), and it was shown in [19] that \( \phi \) is convex and decreasing. We aim to find the minimum \( t > 0 \) such that \( \phi(t) = 0 \), which defines the nearest point on the boundary of \( S_G \) in the direction of \( p \). We can assume that \( \sum_{i=1}^n p_i \geq 1 \), otherwise \( p \in S_G \) trivially. We use the following algorithm: We start with \( t = 0 \). Given \( t \), we estimate \( \phi(t) \) and \( \phi'(t) \) (within polynomial factors) using the assumed algorithm. This can be done, since \( \phi(t) = \tilde{q}_V(tp) \), and
\[
\phi'(t) = \frac{d}{dt} \tilde{q}_V(tp) = \sum_{i=1}^n p_i \frac{\partial}{\partial z_i} \tilde{q}_V(z)|_{tp} = - \sum_{i=1}^n p_i \tilde{q}_V \Gamma^{+}(i)(tp).
\]

We will show that we only apply this computation to points \( t \) such that \( (1 + \frac{1}{n^{2c}})tp \in S \). For such points \( \phi(t) > 0 \), \( \phi'(t) < 0 \) and we can also estimate \( \frac{\phi(t)}{\phi'(t)} \). Let \( D(t) \) be our estimate, such that \( n^{-2c} \frac{\phi(t)}{\phi'(t)} \leq D(t) \leq \frac{\phi(t)}{\phi'(t)} \). Given this estimate, we replace \( t \) by \( t' = t + \frac{1}{2}D(t) \). We repeat this process as long as \( D(t) \geq 1/n^{n+1+2c} \) and \( t < 1 \). If we reach \( t \geq 1 \), we answer YES; else if \( D(t) \) drops below \( 1/n^{n+1+2c} \), we answer NO.

We note the following: Assuming that the minimum positive root of \( \phi \) is \( \xi_0 \) and \( 0 \leq t < \xi_0 \), we have \( t + D(t) \leq t + \frac{\phi(t)}{\phi'(t)} \leq \xi_0 \) by convexity of \( \phi \). Therefore, the additive slack at any point \( t \) is at least \( D(t) \). Since we update the point to \( t' = t + \frac{1}{2}D(t) \), we always retain slack at least \( \frac{1}{2}D(t) \), which is guaranteed to be at least \( \frac{1}{2n^{n+1+2c}} \) (for \( n \geq 2c + 2 \)), otherwise we terminate. This proves the above claim that we only evaluate at points \( t \) such that \( (1 + \frac{1}{n^{2c}})tp \in S \).

On the other hand, if \( \delta := \xi_0 - t \), we have \( \tilde{q}_V(tp + \delta p_i e_i) \geq 0 \) for \( 1 \leq i \leq n \) since \( (t + \delta)p \) is at the boundary of \( S_G \). We then have
\[
\phi(t + \delta) - \phi(t) \leq \min_{1 \leq i \leq n} \tilde{q}_V(tp + \delta p_i e_i) - \tilde{q}_V(tp) = - \max_{1 \leq i \leq n} \delta p_i \frac{\partial \tilde{q}_V}{\partial z_i}|_{z=tp} \leq - \frac{\delta}{n} \sum_{i=1}^n p_i \frac{\partial \tilde{q}_V}{\partial z_i}|_{z=tp} = \frac{\delta}{n} \phi'(t).
\]
Therefore, since \( \phi(t + \delta) = 0 \), we get \( \frac{\phi(t)}{\phi'(t)} \geq \frac{\delta}{n} = \frac{\xi_0 - t}{n} \). By our approximation guarantee, \( D(t) \geq n^{-2c} \frac{\phi(t)}{\phi'(t)} \geq n^{-2c-1}(\xi_0 - t) \). (Note that this also means that \( (t + n^{1+2c} D(t))p \not\in S \).) So when we replace


\begin{equation}
\left(1 - \frac{1}{2n^{2c+1}}\right)^{2n^{2c+1}(n+2+2c)\log n} \leq \frac{1}{n^{n+2+2c}}.
\end{equation}

Initially, we have \( \xi_0 \leq n \) because \( p_i \geq 1/n \) for some \( i \in [n] \). Hence, the quantity \( \xi_0 - t \) as well as \( D(t) \) must shrink below \( 1/n^{n+1+2c} \) in a polynomial number of steps.

If we terminate because \( t \geq 1 \), we have certified that \( p \in S \) and we can answer \text{YES}. If we terminate because \( D(t) < 1/n^{n+1+2c} \) then it is the case that \( t < 1 \), and we know that \( (t + n^{1+2c}D(t))p \notin S \). Hence \((1 + 1/n^t)p \notin S \) and we can answer \text{NO}.

\textbf{Corollary C.9} (restatement of Theorem 1.3). If there is an algorithm to estimate \( \tilde{q}_V(p) \) to within a \( n \)-multiplicative factor, assuming that \( (1 + \alpha)p \in S \), and running in time \( (n \log \frac{1}{\alpha})^{O(\log n)} \), then \#P \( \subseteq \) DTIME\( (n^{O(\log n)}) \).

\textit{Proof.} Suppose we have such an algorithm. Then we can run it for \( \alpha = \frac{1}{\pi c} \), and solve a \#P-hard problem (from Theorem C.8) in running time \( (n \log \frac{1}{\alpha})^{O(\log n)} = n^{O(\log n)} \). \hfill \Box

In contrast, the running time of our algorithm (for a constant-factor approximation) is \( n^O(\frac{1}{\alpha} \log d) \). Again, there is an open question here, whether there is an approximation algorithm (possibly even an FPTAS) under these conditions with running time at most quasi-polynomial in \( n, 1/\alpha \).

\subsection{C.2.1 Hardness of approximation at a fixed point}

Finally, we observe that there is no PTAS for the univariate alternating-sign independence polynomial on bounded degree graphs, even for a fixed value of the parameter \( \lambda \) when that parameter is a constant factor away from being in the Shearer region of the input graph. Further, the hardness result holds even with the added promise that the alternating sign independence polynomial at the fixed parameter \( \lambda \) is positive on the given input graph. The result below is not optimal, and one might expect that a similar hardness result holds also for \( \lambda \) just outside the boundary of the Shearer region. Our reduction is similar in spirit to that of Luby and Vigoda [26].

\textbf{Theorem C.10} (Restatement of Theorem 4.4). Fix a positive integer \( d \geq 62 \) and a rational number \( \lambda > 39/d \). Suppose there exists a PTAS for computing \( \tilde{q}_G(\lambda 1) = Z_G(-\lambda) \) for input graphs \( G \) of degree at most \( d \) when it is promised that \( \tilde{q}_G(\lambda 1) > 0 \). Then, \text{NP} = \text{RP}.

\textit{Proof.} We reduce from the inapproximability of \( Z_G(x) \) for positive \( x \). In this case, the results of Galanis et al. [13, 14] and Sly and Sun [39] show that for any fixed rational \( \lambda_0 \geq 1 \), there is no PTAS for computing \( Z_G(\lambda_0) \) on graphs \( G \) of maximum degree 6 unless \text{NP} = \text{RP}.

Now let \( d \) and \( \lambda \) be as in the statement of the lemma and suppose there exists and PTAS for \( Z_G(-\lambda) \) when \( G \) is a graph of degree at most \( d \). Set \( k = \left|\frac{d\lambda_1}{12}\right| \geq 4 \). Let \( \lambda_1 := (1 - k\lambda)^2 - 1 \); a simple calculation shows that when \( d \geq 62 \) and \( \lambda > 39/d, \lambda_1 \geq 1 \). Now, given an input graph \( H \) of degree at most 6, we show how to construct in polynomial time a graph \( G \) of degree at most \( d \) such that

\begin{equation}
Z_H(\lambda_1, \lambda_1) = Z_G(\lambda_1),
\end{equation}

which implies an PTAS for \( Z_H(\lambda_1) \) starting from the assumed PTAS for \( \tilde{q}_G(\lambda 1) = Z_G(-\lambda) \). (Note that the latter PTAS only needs to be valid under the additional promise that \( Z_G(-\lambda) > 0 \), since we know that \( Z_H(\lambda_1) > 0 \).) Since \( \lambda_1 \geq 1 \), this will prove the claim in combination with the hardness results for \( Z_H \) cited above.

We now provide the construction of \( G \). Let \( H = (V_H, E_H) \) and consider the graph \( B = (V_B, E_B) \) consisting of two disjoint copies of the \( k \)-clique. The graph \( G \) is a product graph of \( H \) and \( B \) obtained
by replacing each vertex of $H$ by a copy of $B$ and each edge of $H$ by a complete bipartite graph. More
formally, $G = (V,E)$ where
\[
V = \{ (u,v) : u \in V_H \text{ and } v \in V_B \}, \quad \text{and}
\]
\[
E = \big\{ \{ (u,v), (u,v') \} : u \in V_H \text{ and } \{ v, v' \} \in E_B \big\} 
\cup \big\{ (u,v), (u',v') \big\} : (u,u') \in E_H \text{ and } v, v' \in V_B \big\}.
\]
With this definition, we have $Z_G(x) = Z_H(Z_B(x) - 1)$. To see why this is true, note that
\[
Z_G(x) = \sum_{I \text{ indep. in } G} x^{\|I\|} = \sum_{I \text{ indep. in } H} \prod_{u \in I} \left( \sum_{J \notin \emptyset} x^{\|J\|} \right)
= \sum_{I \text{ indep. in } H} \prod_{u \in I} (Z_B(x) - 1) = Z_H(Z_B(x) - 1).
\]
Observing that $Z_B(x) = (1+kx)^2$, we obtain eq. (C.1) by setting $x = -\lambda$. \hfill \square

### C.3 Membership in the original LLL region

The original statement of the Lovász Local Lemma [12,40] had a stronger hypothesis than Shearer’s formulation. It stated that $\mu(A_{\leq 1} \cap E_i) > 0$ if $G$ is a dependency graph for events $E_1, \ldots, E_n$ and if $p$ lies in the set
\[
\mathcal{L}_G := \big\{ p \in [0,1]^V : \exists x \in (0,1)^V \text{ s.t. } p_i \leq x_i \cdot \prod_{(i,j) \in E} (1-x_j) \ \forall i \in V \big\}.
\]
Shearer’s results imply that $\mathcal{L}_G \subseteq S_G$ (see also [19,33]). Interestingly, although deciding membership in $S_G$ is #P-hard, membership in $\mathcal{L}_G$ can be decided in polynomial time within exponentially small errors.

**Theorem C.11.** For a given graph $G = (V,E)$, $|V| = n$, rational $(p_1, \ldots, p_n) \in [0,1]^V$ and $\epsilon > 0$, we can distinguish between $(p_1 + \epsilon, \ldots, p_n + \epsilon) \in \mathcal{L}_G$ and $(p_1, \ldots, p_n) \notin \mathcal{L}_G$ , in time $\text{poly}(n, \log \frac{1}{\epsilon})$.

**Proof.** By taking logs, we can write equivalently
\[
\mathcal{L}_G = \big\{ p \in [0,1]^V : \exists x \in (0,1)^V \text{ s.t. } \log p_i \leq \log x_i + \sum_{(i,j) \in E} \log(1-x_j) \ \forall i \in V \big\}.
\]
Thus $p \in \mathcal{L}_G$ is equivalent to the following set being nonempty:
\[
\mathcal{X}_{G,p} = \big\{ x \in (0,1)^V : \log p_i \leq \log x_i + \sum_{(i,j) \in E} \log(1-x_j) \ \forall i \in V \big\}.
\]
Note that this is a convex set: $p_i$ is fixed here, and $\phi_i(x) = \log x_i + \sum_{(i,j) \in E} \log(1-x_j)$ is a concave function of $x \in (0,1)^V$. Also, it is easy to implement a separation oracle for $\mathcal{X}_{G,p}$. Given a point $x$, we can check directly if all the constraints are satisfied, and if not we can compute a separating hyperplane whose normal vector is the gradient of $\phi_i(x)$.

Suppose now that $p + \epsilon = (p_1 + \epsilon, \ldots, p_n + \epsilon) \in \mathcal{L}_G$. Let $x \in (0,1)^V$ be such that $p_i + \epsilon \leq x_i \prod_{(i,j) \in E} (1-x_j)$. Clearly, $x_i \geq \epsilon$. Also, for any $\xi_i \in [0,\epsilon]$, $x_i - \xi_i \prod_{(i,j) \in E} (1-(x_j - \xi_j)) \geq (x_i - \xi_i) \prod_{(i,j) \in E} (1-x_j) \geq x_i \prod_{(i,j) \in E} (1-x_j) - \xi_i \geq p_i$.

This means that the box $[x - \epsilon, x]$ is contained in $\mathcal{X}_{G,p}$. The volume of this box is $\epsilon^n$, while $\mathcal{X}_{G,p}$ is contained in the box $[0,1]^V$, of volume 1. Therefore, by the ellipsoid method, we can find a point in $\mathcal{X}_{G,p}$ in poly$(n, \log \frac{1}{\epsilon})$ iterations, which certifies that $p \in \mathcal{L}_G$ and we can answer YES. If the ellipsoid method fails to find such a point, it must be the case that $p + \epsilon \notin \mathcal{L}_G$, in which case we can answer NO. \hfill \square

In particular, in poly$(n)$ time we can decide about membership in the LLL region within a $1/n^n$ additive error, which is #P-hard for the Shearer region.
D Extension to graphs of bounded connective constant

The connective constant, first studied by Hammersley [16], is a natural notion of the average degree of a graph. The definition is best motivated in the setting of infinite regular lattices (e.g., $\mathbb{Z}^2$), though it extends easily to general graph families. Note that the maximum and average degrees of $\mathbb{Z}^2$ are both 4, and in this respect it is not distinguishable from the infinite 4-regular tree. However, it is clear that $\mathbb{Z}^2$ is very different from the regular tree (in particular due to its small girth), and the connective constant may be seen as a notion of average degree that tries to capture this difference.

For a fixed vertex $v$ in $\mathbb{Z}^2$, consider $N(v, \ell)$, the number of self-avoiding walks in the lattice starting at $v$. (In the special case of the lattice, this number depends only upon $\ell$ and not $v$.) We then have

$$2^\ell < N(v, \ell) < 3^\ell.$$  

The connective constant measures the rate of growth of $N(v, \ell)$ as a function of $\ell$. Formally, the connective constant $\Delta(\mathbb{Z}^2)$ of $\mathbb{Z}^2$ is given by

$$\Delta(\mathbb{Z}^2) = \lim_{\ell \to \infty} N(v, \ell)^{1/\ell}.$$  

(The limit on the right hand side above can be shown to exist in the case of $\mathbb{Z}^2$ and other regular lattices; see, e.g., [27]. However, computing the exact value of the connective constant is an open problem for most regular lattices, with one celebrated exception [10].) For an infinite family of finite graphs, we may similarly define the connective constant as follows:

**Definition D.1 (Connective constant: finite graphs [36]).** Let $\mathcal{G}$ be an infinite family of finite graphs. We say that the connective constant of graphs in $\mathcal{G}$ is at most $\Delta$ if there exist positive constants $a$ and $c$ such that for any $G \in \mathcal{G}$ with at least $n$ vertices, any $\ell \geq a \log n$ and any vertex $v$ in $G$, the number $N(v, \ell)$ of self-avoiding walks in $G$ of length $\ell$ starting at $v$ is at most $c\Delta^\ell$.

Note that the connective constant of graphs of maximum degree $d$ is at most $d - 1$. However, as in the case of lattices, it can be much smaller than this crude bound; in particular, it can be bounded even when the maximum degree is unbounded. An important example is that of graphs sampled from the sparse Erdős-Rényi random graph model $\mathcal{G}(n, d/n)$. When $d$ is a constant, the connective constant of such graphs is at most $d$ w.h.p. On the other hand, the maximum degree of such a graph on $n$ vertices is $\Omega\left(\frac{\log n}{\log \log n}\right)$ w.h.p.

The connective constant turns out to have an important connection with correlation decay based algorithms for the independence polynomial. Recall that Weitz showed that when $0 \leq \lambda < \lambda_c(d)$, there is an FPTAS for $Z_G(\lambda)$ on graphs of degree at most $d$. In [36], this was extended to all graphs of connective constant $d - 1$, without any bound on the maximum degree. (Note that graphs of maximum degree $d$ have connective constant at most $d - 1$, so this is a strict generalization of Weitz’s result even in the bounded degree setting.)

In the setting of complex activities, our main theorem, Theorem 1.1, also generalizes to graphs of bounded connective constant. The proof presented in Section 3.2 is already sufficient to establish this extension with a few small modifications, which we now proceed to describe. In particular, we prove the following modification of Theorem 3.8.

**Theorem D.2 (FPTAS for graphs of bounded connective constant).** Let $\mathcal{G}$ be an infinite family of finite graphs with connective constant at most $\Delta$, and let the constant $a$ be as in the definition of the connective constant. Given a graph $G = (V, E) \in \mathcal{G}$ on $n$ vertices, a parameter vector $p$ such that $(1 + \alpha)^2 p \in S$, and a positive $\epsilon \leq 1/n$, define $\ell = \max\left\{a \log n, \left[\log_{1+\alpha}\left(\frac{(1+\alpha)(1+n/\alpha)}{\epsilon\alpha}\right)\right]\right\}$. Then a $(1 + O(\epsilon n))$-approximation to $\hat{q}_V(p)$ can be computed in time $O(n\Delta^\ell)$.

**Proof (sketch).** The proof is very similar to that of Theorem 3.8 in Section 3.2 so we only describe the steps that need to be modified. The first observation, already alluded to in the remark following Lemma 2.3, is
that the size of the computation tree for computing $r_{V,v}$ up to depth $\ell$ is at most $N(v, \ell)$, the number of self-avoiding walks of length $\ell$ starting at $v$. Since $G$ belongs to a graph family of connective constant at most $\Delta$, we have $N(v, \ell) = O(\Delta^\ell)$ when $\ell \geq a \log n$, so that the cost of expanding the tree to depth $\ell$ is at most $O(\Delta^\ell)$. Thus, the total cost of computing each of the $R_{S_i,v_i}$ as in the proof of Theorem 3.8 is $O(n \Delta^\ell)$.

It remains to show that this achieves a $1 \pm O(\epsilon n)$ factor approximation to $\tilde{q}_V$. The proof is again similar, except that we now apply Corollary 3.7 using $n$ as the bound $d$ on the maximum degree, and using the new definition of $\ell$ in the statement of above theorem (which also replaces the $d$ used in the definition of $\ell$ in Theorem 3.8 by $n$). With these two modification, we again obtain

$$\left| \frac{\Xi}{\xi} - 1 \right| \leq \epsilon \quad \text{for each } i,$$

and can complete the proof exactly as before. \qed