CPSC 421: Introduction to Theory of Computing
Practice Problem Set #0, Not to be handed in

1. Let $A = \{0, 2, 3\}$, $B = \{2, 3\}$, and $C = \{1, 5, 9\}$ be subsets of some universal set $U = \{0, 1, 2, ..., 9\}$. Determine:
   (a) $A \cap B$
   (b) $A \cup B$
   (c) $A \cup C$
   (d) $A \cap C$
   (e) $A - B$
   (f) $B - A$
   (g) $\overline{A}$
   (h) $\overline{C}$

2. Let $A = \{1, 2, 3\}$, $B = \{2, 3\}$, $C = \{1, 4\}$, and let the universal set be $U = \{0, 1, 2, 3, 4\}$. List the elements in:
   (a) $A \times B$
   (b) $B \times A$
   (c) $A \times B \times C$
   (d) $A \times \overline{A}$
   (e) 2^A
   (f) $A \times \emptyset$
   (g) $B \times 2^B$

3. Let A, B, and C be as in question 1. and let $D = \{3, 2\}$ and $E = \{2, 3, 2\}$. Determine which of the following are true. Give reasons for your decisions.
   (a) $A = B$
   (b) $B = D$
   (c) $B = E$
   (d) $A - B = B - A$

4. For sets A, B and C, prove the following:
   (a) If $A \subseteq B$ then $A \cap B = A$
   (b) If $A \subseteq B$ then $A \cup B = B$
   (c) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

5. Prove that if $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$

6. Given that U = all university students, D = day students, M = mathematics majors, and G = graduate students. Draw Venn diagrams illustrating this situation and shade in the following sets.

(a) evening (i.e. non-day) students
(b) undergraduate mathematics majors
(c) non-math graduate students
(d) non-math undergraduate students
(e) graduate students or math majors who take day classes

7. Let A be a set with $|A| = n$. How many distinct two-element subsets are there in the set 2^A?

8. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d, e, f\}$, and $C = \{+, -\}$. Define the functions $f : A \to B$ such that $f(k) =$ the k^{th} letter in the alphabet, and $g : B \to C$ such that $g(\alpha) = +$ if α is a vowel and $g(\alpha) =$ - if α is a consonant.

   (a) Find $g \circ f$
   (b) Does it make sense to discuss $f \circ g$? If not ,why not?
   (c) Does f^{-1} exist? Why or why not?
   (d) Does g^{-1} exist? Why or why not?

9. (*) For each of the following pairs of sets X and Y, give a function $f : X \to Y$ that is:

   (i) Injective but not surjective
   (ii) Surjective but not injective
   (iii) Bijective

   Or else explain why this is not possible.

   (a) $X = \{a, b\}$, $Y = \{1, 2, 3\}$
   (b) $X = \{a, b, c\}$, $Y = \{1, 2\}$
   (c) $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$

10. (*) There is a theorem stating that two sets have the same cardinality (i.e., number of elements) if and only if there exists a bijection between them. This is rather trivial if the sets are finite, but can be a powerful tool for analyzing the sizes of infinite sets. Show that there exists a bijection from $\mathbb{N}$, the set of natural numbers ($\{0, 1, 2, ...\}$), to $\mathbb{Z}$, the set of integers ($\{... - 2, -1, 0, 1, 2, ...\}$). Conclude that these two sets in fact have the same number of elements.

11. Let A be an arbitrary set, and let $B = \{x \in A : x \notin x\}$. That is, B contains all sets in A that do not contain themselves: For all y,

    $(*)\quad y \in B$ if and only if $(y \in A$ and $y \notin y)$.

    Can it be that $B \in A$?

12. Let $\mathcal{A}$ be a family of sets. The union of the sets in $\mathcal{A}$ is

    $$\bigcup_{A \in \mathcal{A}} A = \{a : a \in A \text{ for some } A \in \mathcal{A}\},$$

    and the intersection of the sets in $\mathcal{A}$ is

    $$\bigcap_{A \in \mathcal{A}} A = \{a : a \in A \text{ for all } A \in \mathcal{A}\}.$$

(a) What is $\bigcup_{A\in\emptyset} A$?

(b) What is $\bigcap_{A\in\emptyset} A$? (Which *a*s do *not* satisfy the definition of intersection ?)

(c) Show that $(\bigcup_{A\in\mathcal{A}} A)^c = \bigcap_{A\in\mathcal{A}} A^c$. Note that $\mathcal{A}$ needn't be countable; it is arbitrary.

13. Show that for any sets A and B,

$$A = (A \cap B) \cup (A \cap B^c) = ((A \cup B) \cap B^c) \cup (A \cap B).$$

14. Let a and b be any constants. Prove that $a^n \cdot n^b = O((a+1)^n)$.