CPSC 421: Introduction to Theory of Computing
Assignment #8, due Thursday November 24th by 11:59pm, via Gradescope

[10] 1. (This relates to Lecture 19.) Let $DUMBPP$ be the complexity class for with $L \in DUMBPP$ if and only if there is a polynomial-time TM M for which

$$\begin{aligned} x \in L &\implies \Pr[\, M \text{ accepts } x \,] \;\leq\; 1/3 \\ x \notin L &\implies \Pr[\, M \text{ rejects } x \,] \;\leq\; 1/3 \end{aligned}$$

What complexity class does $DUMBPP$ equal? Briefly explain your answer.

Hint: My solution has two sentences.

[15] 2. Consider the problem

$$MAXCLIQUE \;=\; \{\, \langle G, k \rangle \;:\; \text{the largest clique in } G \text{ has exactly } k \text{ vertices} \,\}.$$

In Assignment 7, we argued that $MAXCLIQUE$ is NP-hard. We also remarked that $MAXCLIQUE$ is *not believed to be in NP*. In this question, we will see why: if $MAXCLIQUE \in NP$, then we will reach the unlikely conclusion that $NP = coNP$.

- [2] a. Suppose that $NP \setminus coNP \neq \emptyset$. Prove that $coNP \setminus NP \neq \emptyset$.
- [2] b. Suppose that $coNP \subseteq NP$. Prove that $NP = coNP$.
  Hint: My solution has two sentences.
- [6] c. Show that $\overline{MAXCLIQUE}$ is NP-hard.
  Note: You must use our standard definition of reductions, i.e., "polynomial time mapping reductions", as defined in Sipser Definition 7.29.
  Hint: My solution has two (long-ish) sentences.
- [2] d. Show that $MAXCLIQUE$ is coNP-hard.
  Hint: My solution has two sentences.
- [3] e. Suppose that $MAXCLIQUE \in NP$. Conclude that $coNP = NP$.

[15] 3. Let us define the "Safe Marriage" problem. There are n people (but there is no notion of gender in this problem). Each pair of people u and v either like or dislike each other. A "Safe Marriage of size k" is a set of pairs

$$\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_k, v_k\}$$

such that:

- u_i and v_i like each other,
- u_i is the *only* person that v_i likes amongst $\{u_1, v_1, \ldots, u_k, v_k\}$,
- v_i is the *only* person that u_i likes amongst $\{u_1, v_1, \ldots, u_k, v_k\}$.

The objective is to decide if there is a Safe Marriage of size k.

We can model this problem using an undirected graph G, where the vertices correspond to people, and the edges correspond to pairs who like each other. The decision problem is

$$SAFEMARRIAGE = \{\, \langle G, k \rangle \;:\; G \text{ has a safe marriage of size } k \,\}.$$

Prove that $SAFEMARRIAGE$ is NP-hard.

Hint: Try a reduction from Independent Set, which is known to be NP-complete (see Assignment 7).

[3] 4. **OPTIONAL BONUS QUESTION:**

Let $[m]$ denote $\{1, \ldots, m\}$. Define the language $TWOTREES$ to be the set of strings of the form $\langle G, H \rangle$, where

- G and H are both graphs with n vertices and m edges.
- Each edge of G has a unique label in $[m]$. (In other words, there is a bijection from $[m]$ to the edges of G.)
- Each edge of H has a unique label in $[m]$. (In other words, there is a bijection from $[m]$ to the edges of H.)
- There exists a set $T \subseteq [m]$ (where $|T| = n - 1$) such that
  - the edges of G whose labels are in T form a spanning tree of G, and
  - the edges of H whose labels are in T form a spanning tree of H.

(We allow the possibility that some edges may be self-loops, which cannot belong to any spanning tree.)

It is obvious that $TWOTREES \in NP$. In this problem, we will prove that $TWOTREES \in coNP$, so actually $TWOTREES \in NP \cap coNP$.

[1] a. Define two functions $g, h : [m] \to \mathbb{Z}$ as follows:

- $g(S)$ is the size (number of edges) of the largest forest in G that uses only the edges with labels in S.
- $h(S)$ is the size (number of edges) of the largest forest in H that uses only the edges with labels in S.

Prove that

$$g(A \cup B) + g(A \cap B) \leq g(A) + g(B) \qquad \forall A, B \subseteq [m]. \tag{1}$$

(So the analogous inequality holds for h too.)

Hint: It can be proven using the https://en.wikipedia.org/wiki/Kirchhoff%27s_theorem theorem. It can also be proven using properties of Kruskal's algorithm.

[1] b. Prove that, if $\min_{S \subseteq [m]} \big(g(S) + h([m] \setminus S) \big) < n - 1$ then $\langle G, H \rangle \notin TWOTREES$.

[1] c. Prove, by induction on m, that if $\min_{S \subseteq [m]} \big(g(S) + h([m] \setminus S) \big) = n - 1$ then $\langle G, H \rangle \in TWOTREES$.

Conclude that $TWOTREES \in coNP$.

Hint: Find a label so that the corresponding edges are not self-loops in either graph. Consider what happens when you delete this edge, and when you contract this edge. In both cases, the graph has fewer edges so one may apply the inductive hypothesis.