[4] 1.  [2] a. Prove that if $L_1$ and $L_2$ are regular, then so is

$$L_1 \setminus L_2 = \{ x \in \Sigma^* : x \in L_1 \text{ and } x \not\in L_2 \}.$$ 

You may wish to use results from the textbook (e.g., pages 46 and 85). (My solution is one sentence long.)

[2] b. Let $k$ be a fixed positive integer. Define $L_k = \{ 0^{kn}1^m : n, m \geq 0 \}$. Is $L_k$ regular? Briefly explain why or why not.

[6] 2. Let $R$ be the regular expression \(((1^*01) \cup (00))^*10^*((0^*01) \cup \epsilon))$. Let $L(R)$ be the language that it generates.

The following are true/false questions, and you do not need to justify your answers.

[2] a. Is $010 \in L(R)$?
[2] b. Is $10100 \in L(R)$?
[2] c. Is $001001 \in L(R)$?

[8] 3. Let $\Sigma = \{a, b\}$ and $L = \{ w \in \Sigma^* : w \text{ is a palindrome } \}$. (Recall that a palindrome is any string $x$ that equals the reverse of itself.) In this question we will prove that $L$ is non-regular using the Pumping Lemma.

[3] b. For every integer $p$, define a string $w$ (of length at least $p$) that is a palindrome over $\Sigma$ and for which the first $p$ characters are identical. (But $w$ should not be of the form $a^p$ or $b^p$).
[3] c. Show that for any decomposition of your palindrome $w$ into $xyz$ where $|y| > 0$ and $|xy| \leq p$, that $xy^2z$ is no longer a palindrome.

[8] 4. Let $\Sigma = \{a, b, c, d\}$ and $L = \{ a^7b^nc^3d^5n : n \geq 0 \}$. Prove that $L$ is non-regular using the Pumping Lemma.

[15] 5. Let $\Sigma = \{0, 1, 2\}$. Consider the language

$$F = \{ 0^n1^n2^n : n \geq 1 \} \cup \left\{ 0^k w : k \neq 1, w \in \{1, 2\}^* \right\}.$$ 

The purpose of this question is to prove that $F$ is non-regular, but it satisfies the pumping condition. In other words, the situation looks like this figure:
[7] a. Show that $F$ is not regular. You may use the pumping lemma and the fact that the class of regular languages is closed under unions, concatenations and complements. You may assume that
\[
\{ 01^n2^n : n \geq 0 \}
\]
is non-regular since we have shown something very similar in class.

[7] b. Show that $F$ satisfies the pumping condition. In other words, find an appropriate integer $p$ and demonstrate that for all strings $w \in F$ with $|w| \geq p$, we can decompose $w = xyz$ (where $x, y, z \in \{0, 1, 2\}^*$) such that
\begin{itemize}
  \item $|y| > 0$,
  \item $|xy| \leq p$, and
  \item $xy^n z \in F$ for all $n \geq 0$.
\end{itemize}

[1] c. Explain why this does not contradict the pumping lemma.

[10] 6. Let $\Sigma = \{a, b\}$. Let
\[
L = \{ x \in \Sigma^* : x \text{ is a palindrome and } |x| \text{ is a multiple of 3} \}.
\]


Professor Dumas thinks he understands the pumping lemma. His interpretation is that “a finite automaton has a limited amount of complexity, so if it accepts a very long string, that string must have some repeated structure inside”. To formalize his interpretation, he proposes the following conjecture.

**Conjecture 1.** Let $L$ be regular language. Suppose there is a DFA with $p$ states that accepts $L$. Then for every string $w \in L$ with $|w| \geq p$,
\[
\exists x, y, z \in \Sigma^*, i \geq 2 \text{ such that } w = xy^iz \text{ and } y \neq \epsilon.
\] (1)

Professor Dumas’ conjecture is false. Let’s try to understand why.

Let $\Sigma = \{a, b, c\}$. Find a DFA $M = (Q, \Sigma, \delta, q_0, F)$ and a string $w$ that is accepted by $M$, and has $|w| > 20|Q|$, but does not satisfy (1).