CPSC 421: Introduction to Theory of Computing
Assignment #1, due Tuesday September 13th

Notes.
• It is **mandatory** that all homework solutions be typeset in Latex. Other word processing software, such as Microsoft Word, is **not** acceptable.

• These questions are intended as review and self-assessment. The intention is that you should be able to solve all questions in about an hour (ignoring any time to get familiar with Latex). If it takes you many hours/days to solve these questions, the rest of the course will likely be a big struggle.

[10] 1. Let $f : A \to B$ be a function where A and B are finite sets. We say f is:

- **Injective (or "one-to-one"):** if $f(a) = f(a')$ then $a = a'$,
- **Surjective (or "onto"):** for every $b \in B$ there exists $a \in A$ with $f(a) = b$.

If f is both injective and surjective, we say that f is a bijection. Let $A = \{a, b, c\}$ and $B = \{d, e\}$.

How many different functions are there from A to B? How many of these are injections? How many are surjections? How many are bijections?

[6] 2. Suppose A and B are finite sets and assume $x \in A \cap B$. For each question either provide a proof that the statement is true or show via counterexample that the statement is false.

[2] a. $x \in \{A\} \cap \{B\}$.
[2] b. $x \in \{A \cup B\}$.

[5] 3. Consider the following algorithmic problem. The input is a graph with n nodes (and no multi-edges). A *rectangle* in the graph is a sequence of four distinct vertices (v_1, v_2, v_3, v_4) such that the edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$ and $\{v_1, v_4\}$ are all present in the graph. (We do not care whether $\{v_1, v_3\}$ or $\{v_2, v_4\}$ are edges.)

Give an algorithm to decide if the given graph contains a rectangle. The running time of your algorithm should be polynomial in n. You may assume whatever computational model you like, and assume that the graph is represented however you like.

Guidelines.

- You should specify what format your input is given in.
- You should briefly justify what the runtime of your algorithm is.
- A very complicated algorithm will require a long explanation, so it's best not to have a very complicated algorithm. My algorithm is explained in one sentence.
- My whole solution is 5 sentences.

[10] 4. Let $S(n) = 1 + 2 + \cdots + n$ be the sum of the first n natural numbers and let $C(n) = 1^3 + 2^3 + \cdots + n^3$ be the sum of the first n cubes. Prove the following equalities by induction on n, to arrive at the curious conclusion that $C(n) = \big(S(n)\big)^2$ for every $n \geq 1$.

Guidelines.

- In this problem, we are checking whether you know how to do a formal induction proof.
- You should clearly write down all steps of a formal induction proof, and show all work.

[5] a. $S(n) = n(n+1)/2$.

[5] b. $C(n) = (n^4 + 2n^3 + n^2)/4 = n^2(n+1)^2/4$.

[10] 5. Let

$$A = \begin{bmatrix} 3 & 6 & 0 \\ 0 & 0 & 0 \\ 2 & 4 & 3 \end{bmatrix}.$$

[5] a. Find a non-singular sub-matrix of A that has maximum size.

[5] b. Let I be the 2×2 identity matrix. Find a matrix B for which the rank of $B - I$ is strictly less than the rank of B. Find a matrix C for which the rank of $C - I$ is strictly greater than the rank of C.

[6] 6. Your friend has an interview with a large tech company and they will be quizzing her on some basic graph theory. He has lost his notes and doesn't have any cellular data or internet connection. Therefore, you are his last hope. He needs to know whether the following are true and false and needs a satisfying explanation. Can you help him out?

If the statement is true, explain in one or two sentences. If the statement is false, give a counterexample in one or two sentences.

[2] a. If G is connected, the the complement of G is disconnected.

[2] b. Removing an edge from a tree creates two trees.

[2] c. The edges of every cycle can be colored using two colors so that no two edges sharing a vertex are colored by the same color.