CPSC 421: Introduction to Theory of Computing
Assignment #8, due Friday November 30th by 3:00pm, via Gradescope

[1] 1. **LATEX BONUS!** You get 1 bonus mark if the homework is typeset using Latex.

[10] 2. Define

$$\begin{aligned} ODD &= \{ x \in \{0,1\}^n : x \text{ has an odd number of ones } \} \\ EVEN &= \{ x \in \{0,1\}^n : x \text{ has an even number of ones } \} \end{aligned}$$

Alice is given a vector $x \in ODD$ and Bob is given a vector $y \in EVEN$. Their goal is to output *any single* index $i \in \{1, \dots, n\}$ for which $x_i \neq y_i$. (Such a coordinate must exist since obviously $x \neq y$.) Design a deterministic communication protocol for this that uses $O(\log n)$ bits of communication. Briefly explain why your protocol uses only $O(\log n)$ bits.

[10] 3. Let n, k be integers with $1 \leq k \leq n/2$. (For simplicity, you may assume that k divides n.) Let

$$\mathcal{X} = \mathcal{Y} = \{ X : X \subseteq \{1, \dots, n\} \text{ and } |X| = k \}.$$

Let $f : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$ be the disjointness function. That is, for $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, let

$$f(X,Y) = \begin{cases} 1 & (\text{if } X \cap Y = \emptyset) \\ 0 & (\text{otherwise}) \end{cases}$$

Using a fooling set argument to prove that $D(f) \geq \lceil \log(n/k) \rceil$. (You must prove that your fooling set is correct.)

Hints:

- Drawing the communication matrix is probably not going to be helpful — it is too big!
- Find a (very natural) set $\mathcal{Q} = (Q_1, \dots, Q_{n/k}) \subseteq \mathcal{X}$ such that $Q_i \cap Q_j = \emptyset$ for all $i \neq j$. Define $S = \{ (Q_i, Q_i) : Q_i \in \mathcal{Q} \}$.
- Note that $|\mathcal{X}| = \binom{n}{k} \gg n/k$, so you should not take $\mathcal{Q} = \mathcal{X}$. That is much too big to work!

[10] 4. Recall the AUCTION problem. The items for auction are $U = \{1, \dots, n\}$. There are two bidders, Alice and Bob, who run some protocol to decide which items to give to Alice, and which to give to Bob. (There is no auctioneer). Alice and Bob respectively have valuation functions $v_a, v_b : 2^U \to \{0,1\}$. The goal is to collaboratively compute

$$\max_{S \subseteq U} (v_a(S) + v_b(U \setminus S)). \tag{1}$$

(Of course, Alice does not know v_b and Bob does not know v_a.)

In this question, we will prove that Alice and Bob must exchange at least 2^n bits in order to solve the combinatorial auction problem. To do so, we will perform a reduction from $DISJ$ to AUCTION (somewhat like Lecture 31).

Specifically, let $\ell = 2^n$, let $U' = \{1, \dots, \ell\}$, and let $\pi : 2^U \to U'$ be an arbitrary bijection. Perform a reduction from $DISJ_\ell$ to the AUCTION problem. (Recall that in $DISJ_\ell$, Alice receives $A \subseteq U'$, Bob receives $B \subseteq U'$, and they must decide whether $A \cap B = \emptyset$.) To do so, you must design v_a for Alice using A and π, and v_b for Bob using B and π. The aim is that the value $\max_{S \subseteq U} \big(v_a(S) + v_b(U \setminus S)\big)$ determines the value of $DISJ_\ell(A, B)$.

Hints.

- Since π is a bijection from 2^U to U', π^{-1} is a bijection from U' to 2^U.
- Note that v_a and v_b appear slightly differently in (1). So maybe the way that Bob prepares his input for the auction problem should be slightly different than the way that Alice prepares her input.
- What does it mean for $DISJ_\ell(A, B) = 0$? What does it mean for the value of AUCTION to be 2?

[21] 5. Let x and y be Boolean arrays of size $a \times b$. (That is, $x_{i,j} \in \{0, 1\}$ for all $i \in \{1, \ldots, a\}$ and $j \in \{1, \ldots, b\}$.) Define

$$f(x, y) \;=\; \bigwedge_{i=1}^{a} \bigvee_{j=1}^{b} (x_{i,j} \wedge y_{i,j}).$$

Here $\wedge$ denotes Boolean AND, and $\vee$ denotes Boolean OR.

[1] a. Write an English sentence describing what this function computes.

[5] b. Prove that $N(f) = O(a \log b)$.

[5] c. Prove that $N(\neg f) = O(\log(a) + b)$.

[5] d. Prove that $N(\neg f) \geq b$. (So parts (c) & (d) give nearly-tight bounds on $N(\neg f)$.) Your solution should find a fooling set S for f consisting of 0-entries with $|S| \geq 2^b$. then use the following claim that I mentioned in Lecture 32 (taking $g = \neg f$).

Claim 1. Let $g : X \times Y \to \{0, 1\}$ be any function. Suppose S is a fooling set for g consisting of 1-entries (i.e., $g(x, y) = 1 \; \forall (x, y) \in S$). Then $N(g) \geq \lceil \log_2 |S| \rceil$.

Hint: This question does not depend on the value of a, so you might as well first solve it in the case $a = 1$. Try to write an English sentence explaining what the function f is in the case $a = 1$. Is this similar to a function we've seen before??

[5] e. Prove that $N(f) \geq a \log_2 b$. (So parts (b) & (e) give nearly-tight bounds on $N(f)$.) Your solution should find a fooling set S for f consisting of 1-entries with $|S| \geq b^a$, then apply Claim 1.

Hint:

- Find a collection $\mathcal{Q}$ of Boolean arrays such that $f(x, x) = 1 \; \forall x \in \mathcal{Q}$ and $f(x, x') = 0 \; \forall x, x' \in \mathcal{Q}$ but $x \neq x'$.
- Note that the total number of Boolean arrays of size $a \times b$ is $2^{ab} \gg b^a$, so you should not let $\mathcal{Q}$ be the collection of *all* Boolean arrays. That is much too big to work!
- Suppose we determined b ways of assigning 0s and 1s to each row. Then suppose we constructed $\mathcal{Q}$ by independently choosing one of these possibilities for each row. Then we would have $|\mathcal{Q}| = b^a$. What is a natural set of b possibilities for each row?

[2] 6. **OPTIONAL BONUS QUESTION** Given sets $x, y \subseteq [n]$, $MED(x, y)$ is defined to be the median of the multiset $x \cup y$. (If $x \cup y$ contains an even number $t = 2k$ of (not necessarily distinct) elements, then the median is defined as the kth smallest element). Show that $D(MED) = O(\log n)$.

Hint: First, reduce to the case where x and y have an equal number of elements, and that number is a power of two. Then, show that with constant communication, one can either reduce the size of both the sets by a constant fraction or reduce the potential range where the median lies by a constant fraction.