CPSC 421: Introduction to Theory of Computing
Assignment #5, due Friday November 2nd by 3pm, via Gradescope

[1] 1. **LATEX BONUS!** You get 1 bonus mark if the homework is typeset using Latex.

[10] 2. [5] a. In class we claimed that, if a polynomial-time algorithm is discovered for some problem, it is usually possible to discover a reasonably efficient algorithm, say one running in $O(n^5)$ time.

We could try to formalize this by saying that $P \subseteq TIME(n^5)$. Is this a true statement? Explain why or why not. (You may refer to theorems from class or the textbook.)

[5] b. Another standard complexity class is $E = \bigcup_{c>0} TIME(2^{cn})$. Is it true that $E = EXP$? Explain why or why not. (You may refer to theorems from class or the textbook.)

[10] 3. In class we claimed that P is a nice complexity class because polynomial-time computations are closed under composition. Let's check whether polynomial-time computations are closed under a polynomial number of compositions.

Let M be a program with two inputs: $i \in \mathbb{N}$ and $w \in \Sigma^*$. Let $c > 0$ be a fixed constant (which depends on M but not on i or w) and let $n = |w|$. Suppose that

- On any inputs, M makes $\Theta(n^c)$ basic computational steps (each of which takes constant time).
- $M(i, w)$ makes $\Theta(n^c)$ calls to $M(i + 1, w)$ when $i < n$.
- $M(n, w)$ only does basic computational steps, and does not calls M again as a subroutine.

Does $M(0, w)$ run in time polynomial in n?

[12] 4. [2] a. Give an example of an infinite, decidable language L (with, say, $\Sigma = \{0, 1\}$) satisfying the following property. For *every* Turing Machine M that decides L, and *every* $x \in L$, the machine M performs at least 100 steps before accepting x.

My solution is two sentences long.

[12] b. Suppose there is a language L and a TM M such that, for every string x, M halts on input x after at most $\sqrt{|x|}$ steps. Prove that M must actually halt on input x after a *constant* number of steps.

Hints.

- How does the last symbol of the input string affect the output?
- Use strong induction.

(Here $|x|$ denotes the length of string x. To avoid pedantic issues, let's ignore the case $x = \epsilon$.)

[1] c. **OPTIONAL BONUS QUESTION:** Let $t(n)$ be any increasing function that grows asymptotically slower than n, i.e., $t(n) \leq t(n + 1)$ for all n, and $t(n) = o(n)$. (For example, $t(n) = \sqrt{n}$ or $t(n) = \log n$.) Prove that $TIME(t(n)) = TIME(1)$.

(Pedantic detail: We assume that $t(n) \geq 1$ for all n.)

[15] 5. Let us consider a decision problem about a generalized form of Sudoku. (The case $n = 3$ corresponds to ordinary Sudoku.) A problem instance consists of an integer $n \geq 3$ and a two-dimensional grid of cells, with n^2 rows and n^2 columns. In the initial problem instance, each cell is either blank or contains a number in $\{1, \ldots, n^2\}$.

The goal is to place a number into every blank cell such that:

(1) Each column contain every number in $\{1, \ldots, n^2\}$ exactly once.

(2) Each row contain every number in $\{1, \ldots, n^2\}$ exactly once.

(3) For every $i, j \in \{0, \ldots, n-1\}$, the square at the intersection of rows $\{ni+1, \ldots, n(i+1)\}$ and columns $\{nj+1, \ldots, n(j+1)\}$ contains every number in $\{1, \ldots, n^2\}$ exactly once.

[6] a. The decision problem $SUDOKU$ is: given an initial problem instance (in which each cell could be blank or contain a number), decide whether the blanks can be filled in such that conditions (1)-(3) are satisfied. Show that $SUDOKU$ is in NP.

[9] b. Suppose that someone proves that $SUDOKU$ (the decision problem) is in P. Give a polynomial-time algorithm with the following behavior: given an initial problem instance (in which each cell could be blank or contain a number), output either:

- A value for each cell such that conditions (1)-(3) are satisfied, or
- "Reject" if there is no way to satisfy conditions (1)-(3).

[2] 6. **OPTIONAL BONUS QUESTION:** This question relates to section 6.1 of the textbook, which discusses the Recursion Theorem (Theorem 6.3).

In class we used a reduction from A_{TM} to prove that $REGULAR_{TM}$ is undecidable (see Theorem 5.3 in the text).

In this question, you must use the Recursion Theorem to prove to prove that $REGULAR_{TM}$ is undecidable.