CPSC 421: Introduction to Theory of Computing
Assignment #4, due Friday October 26th by 3pm, via Gradescope

[1] 1. **LATEX BONUS!** You get 1 bonus mark if the homework is typeset using Latex.

[5] 2. Fix the alphabet $\Sigma = \{0, 1\}$. Let $\mathcal{A}$ be the class of languages whose cardinality is a power of two. That is,

$$\mathcal{A} \;=\; \{\; L \subseteq \Sigma^* \;:\; \exists n \in \mathbb{N} \text{ s.t. } |L| = 2^n \;\}\,.$$

Is $\mathcal{A}$ countable or not? Justify your answer.

[8] 3. Recall that $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers. Let $\mathcal{P} = \{2, 3, 5, 7, 11, \ldots,\}$ be the set of all prime numbers. Let

$$\mathcal{F} = \{ \text{ function } f \;:\; f : \mathbb{Z} \to \mathcal{P} \;\}$$

be the set of all functions whose domain is $\mathbb{Z}$ and whose co-domain is $\mathcal{P}$. **Without** using a diagonalization argument, prove that $\mathcal{F}$ is uncountable.

[10] 4. Fix an alphabet Σ. Prove, using a diagonalization argument, that there exists a language L such that neither L nor $\bar{L}$ are Turing-Recognizable.

[7] 5. Define the language

$$B_{TM} \;=\; \{\; \langle M, w \rangle \;:\; M \text{ is a Turing machine that rejects input } w \;\}\,.$$

Describe a reduction $B_{TM} \leq_T HALT_{TM}$. Argue briefly that this reduction is correct.

[10] 6. Let C be any language $C \neq \Sigma^*$. Let

$$B(C) \;=\; \{\; \langle N \rangle \;:\; N \text{ is a TM and } L(N) \not\subseteq C \;\}\,.$$

Show that $B(C)$ is undecidable, for any such C.

Hint: Let z be any string in $\Sigma^* \setminus C$. Design a reduction that uses z in an important way.

[2] 7. **OPTIONAL BONUS QUESTION**: Fix any finite alphabet Σ. Let

$$\mathcal{A} \;=\; \{\; A \subseteq \Sigma^* \;:\; A \text{ is regular and } A \text{ has infinite cardinality} \;\}\,.$$

Prove that there is a language $C \subseteq \Sigma^*$ such that $C \cap A$ is not recognizable for all $A \in \mathcal{A}$.