1. Show that the following problems are decidable by describing a single-tape TM that decides it.

   (a) \( L_1 = \{ 0^n1^m : n, m \geq 1 \} \)
   
   (b) \( L_2 = \{ 0^n1^n : n \geq 0 \} \)

2. Informally and briefly describe a TM to decide the following languages. You may use multiple tapes if you wish.

   (a) \( L_1 = \{ a^n b^n c^n : n \geq 0 \} \)
   
   (b) \( L_2 = \{ x#y#z : x + y = z \} \)
   
   (c) \( L_3 = \{ w \in \{0, 1\}^* : w \) contains twice as many 0s as 1s \}

3. Show that the collection of decidable languages is closed under the operation of

   (a) union
   
   (b) concatenation
   
   (c) star
   
   (d) complementation
   
   (e) intersection

4. Show that the collection of Turing-recognizable languages is closed under the operation of

   (a) union
   
   (b) concatenation
   
   (c) star
   
   (d) intersection

Give an example to show that the collection of Turing-recognizable languages is not closed under complementation.

5. Is the collection of decidable languages closed under countable unions? If so, give a proof. If not, give a counterexample.

6. (Another TM variant) A Turing machine with left reset is similar to an ordinary Turing machine, but the transition function has the form

   \[ \delta: Q \times \Gamma \leftarrow Q \times \Gamma \times \{ R, \text{RESET} \}. \]

   If \( \delta(q, a) = (r, b, \text{RESET}) \), when the machine is in state \( q \) reading an \( a \), the machine’s head jumps to the left-hand end of the tape after it writes \( b \) on the tape and enters state \( r \). Note that these machines do not have the usual ability to move the head one symbol left. Show that Turing machines with left reset recognize the class of Turing-recognizable languages.

7. Let \( \text{INFINITE}_{DFA} = \{ \langle A \rangle : A \) is a DFA and \( L(A) \) is an infinite language \}. Show that \( \text{INFINITE}_{DFA} \) is decidable.
8. Let \( S = \{ \langle M \rangle : M \text{ is a DFA that accepts } w^R \text{ whenever it accepts } w \} \). Show that \( S \) is decidable. \textit{Hint:} Use the fact that \( EQ_{DFA} = \{ \langle A, B \rangle : A \text{ and } B \text{ are DFAs and } L(A) = L(B) \} \) is decidable.

9. Let \( BLANKHALT = \{ \langle M \rangle : M \text{ halts on the empty input} \} \). Show that \( BLANKHALT \) is undecidable.

10. Consider the problem of determining whether a single-tape Turing machine ever writes a blank symbol over a nonblank symbol during the course of its computation on any input string. Formulate this problem as a language, and show that it is undecidable.

11. Consider the problem of determining whether a Turing machine \( M \) on an input \( w \) ever attempts to move its head left when its head is on the left-most tape cell. Formulate this problem as a language and show that it is undecidable.

12. Consider the problem of determining whether a Turing machine \( M \) on an input \( w \) ever attempts to move its head at any point during its computation of \( w \). Formulate this problem as a language and show that it is \textit{decidable}.

13. When we are designing a Turing machine, it can often be desirable to use as few states as possible. A seemingly innocent heuristic to do this is to determine if a state will ever be used and, if not, to remove that state (and all its transitions). In a real program (say written using C or Python), this may refer to eliminating dead code and, thus, reduce the program size. Given a Turing machine \( M \) and a state \( q \), show that it is undecidable to determine if \( M \) ever enters state \( q \).

14. Let \( f : \mathbb{N} \to \mathbb{N} \) be defined as

\[
 f(x) = \begin{cases} 
 3x + 1 & \text{if } x \text{ is odd} \\
 x/2 & \text{if } x \text{ is even}
\end{cases}
\]

Starting with a positive integer \( x \) and iterate \( f \), you obtain the sequence, \( x, f(x), f(f(x)), \) etc. The sequence stops whenever it hits 1. For example, if \( x = 3 \), we obtain the sequence 3, 10, 5, 16, 8, 4, 2, 1. The Collatz conjecture asks if for any starting \( x \), the sequence will eventually stop at 1.

Suppose that \( H \) is a TM that solves the halting problem. Use this to design a TM that decides if the sequence starting at \( x \) will eventually hit 1 and stop.

15. Prove that \( \{1\}^* \) contains an undecidable subset.

16. Describe two different Turing machines, \( M \) and \( N \), that, when started on any input, \( M \) outputs \( \langle N \rangle \) and \( N \) outputs \( \langle M \rangle \).

17. Consider the following statement.

“The smallest positive integer that cannot be described by less than fifteen words”.

Does such a number exist?

18. (*) Let \( \Sigma = \{1\} \) and \( A = \{ 1^n : \pi \text{ contains the string } 1^n \text{ in its decimal expansion} \} \). Is \( A \) decidable?
19. (*) Let $\Gamma = \{0, 1, \bot\}$ be the tape alphabet for all TMs. Define the busy beaver function $f : \mathbb{N} \to \mathbb{N}$ as follows. For each value of $k$, consider all $k$-state TMs that halt when started with a blank tape. Let $f(k)$ be the maximum number of 1s that remain on the tape among all of these machines. Show that $f$ is not a computable function.

20. (*) We say that $P$ is a nontrivial property of the language of a Turing machine if there exists TMs $M_1$ and $M_2$ such that $L(M_1)$ satisfies property $P$ and $L(M_2)$ does not satisfy property $P$. For example, the property that a TM accepts no string is nontrivial (why?).

**Theorem 1** (Rice’s Theorem). Let $P$ be any nontrivial property. The language $L_P = \{ \langle M \rangle : L(M) \text{ satisfies } P \}$ is undecidable.

(a) Prove Rice’s Theorem.

(b) Use Rice’s Theorem to show that the language

$$L = \{ \langle M \rangle : |L(M')| \text{ is prime} \}$$

is undecidable.

21. (*) An oracle Turing machine is a Turing machine $M$ with a magical read/write oracle tape along with three additional states $q_{\text{yes}}, q_{\text{no}}, q_{\text{ask}}$. The oracle tape is specified with a language $O$. Whenever $M$ enters the state $q_{\text{ask}}$, $M$ moves to state $q_{\text{yes}}$ if $w \in O$ and $q_{\text{no}}$ if $w \notin O$. Here, $w$ denotes the contents of the oracle tape when $M$ enters $q_{\text{ask}}$. Note that an oracle call counts only as a single computational step.

We will denote a TM with oracle $O$ by $M^O$. If we omit the superscript then the TM has no oracle.

(a) Let $HALT = \{ \langle M, w \rangle : M \text{ is a TM and halts on input } w \}$. Show that $HALT$ is decidable by a TM with oracle $HALT$.

(b) Let $HALT^{HALT} = \{ \langle M^{HALT}, w \rangle : M^{HALT} \text{ is a TM with oracle } HALT \text{ and halts on input } w \}$. Show that $HALT^{HALT}$ is undecidable by a TM with oracle $HALT$.

**Remark.** Although oracle Turing machines seem fairly contrived, they have been used to prove that certain techniques for attacking the $P$ vs $NP$ problem will not work. This is known as the Baker-Gill-Solovay Theorem. We will cover $P$ and $NP$ later in the course, but not this particular theorem.