CPSC 421: Introduction to Theory of Computing
Assignment #9, due Friday December 2nd by 3:00pm, via Gradescope

[6] 1. Does nondeterminism help in computing a function, or its complement? Not necessarily!

Assume $n \geq 2$, and let $X = Y = \{1, \ldots, 2^n\}$. Let GTE be the function

$$GTE(x,y) \;=\; \begin{cases} 1 & \text{(if } x \geq y\text{)} \\ 0 & \text{(otherwise)} \end{cases}$$

In class we saw that $D(GTE) = n + 1$. In this question we will show that nondeterminism does not help compute GTE or its negation!

[3] a. Use a fooling set to prove that $N(GTE) \geq n$.

[3] b. Use a fooling set to prove that $N(\neg GTE) \geq n$.

[10] 2. Let x and y be matrices of size $a \times b$ with entries in $\{0,1\}$. (That is, $x_{i,j} \in \{0,1\}$ for all $i \in \{1, \ldots, a\}$ and $j \in \{1, \ldots, b\}$.) Define

$$f(x,y) \;=\; \bigwedge_{i=1}^{a} \bigvee_{j=1}^{b} (x_{i,j} \wedge y_{i,j}).$$

(Here $\wedge$ denotes Boolean AND, and $\vee$ denotes Boolean OR.)

[5] a. Prove that $N(f) = O(a \log b)$.

[5] b. Prove that $N(\neg f) = O(\log(a) + b)$.

[1] c. **OPTIONAL BONUS QUESTION** (but not difficult). Use fooling sets to prove that $N(\neg f) \geq b$ and $N(f) \geq a \log_2 b$.

[10] 3. Let us recall the definition of randomized communication complexity of f (for public-coin protocols with 1-sided error). We defined

$$R_{pub}^{one}(f) \;=\; \min_{\text{randomized protocol } P} \;\; \max_{\text{inputs } (x,y)} \;(\text{\# bits communicated by } P \text{ on input } (x,y))$$

where P must satisfy:

- for all (x,y) with $f(x,y) = 1$, $\Pr[\,\text{Alice and Bob both accept}\,] \geq 1/2$,
- for all (x,y) with $f(x,y) = 0$, $\Pr[\,\text{Alice and Bob both reject}\,] = 1$.

Here the probability is over the public random string $r \in \{0,1\}^*$ that is provided to both Alice and Bob.

Sometimes a failure probability of 1/2 is not good enough. With that in mind, for any $0 < \epsilon < 1$, define

$$R_{pub}^{one,\epsilon}(f) \;=\; \min_{\text{randomized protocol } P} \;\; \max_{\text{inputs } (x,y)} \;(\text{\# bits communicated by } P \text{ on input } (x,y))$$

where P must satisfy:

- for all (x, y) with $f(x, y) = 1$, $\Pr[\,\text{Alice and Bob both accept}\,] \geq 1 - \epsilon$,
- for all (x, y) with $f(x, y) = 0$, $\Pr[\,\text{Alice and Bob both reject}\,] = 1$.

[10] a. Prove that $R^{one,\epsilon}_{pub}(f) \leq \lceil \log_2(1/\epsilon) \rceil \cdot R^{one}_{pub}(f)$.
Hint: See Practice Problem Set 6.

[1] b. **OPTIONAL BONUS QUESTION** (but not hard). We showed in lecture that $R^{one}_{pub}(\neg EQ) \leq 2$. Combining with part (a) shows that $R^{one,\epsilon}_{pub}(\neg EQ) \leq 2 \cdot \lceil \log_2(1/\epsilon) \rceil$. This can be slightly improved.
Prove that $R^{one,\epsilon}_{pub}(\neg EQ) \leq \lceil \log_2(1/\epsilon) \rceil + 1$.

[10] 4. In class we showed that $R^{two}_{priv}(GTE) \leq O(\log^2 n)$. The "public coin" setting is always easier than the "private coin" setting, so let's try to improve that result by using public coins. Modify the protocol discussed in class to obtain a public coin protocol showing that $R^{two}_{pub}(GTE) \leq O(\log(n) \cdot \log\log(n))$.

[3] 5. **OPTIONAL BONUS QUESTION** (rather involved). The result of the previous question is not quite optimal. Show the optimal result, which is $R^{two}_{pub}(GTE) \leq O(\log n)$.

Hint: Allow the equality tests to make mistakes, and use results from this research paper: http://cs.brown.edu/~eli/papers/SICOMP23FRPU.pdf