CPSC 421: Introduction to Theory of Computing
Assignment #7, due Tuesday November 15th by 11:59pm (midnight), via Gradescope

[10] 1. Let $DUMBPP$ be the complexity class for with $L \in DUMBPP$ if and only if there is a polynomial-time TM M for which

$$x \in L \implies \Pr[\, M \text{ accepts } x\,] < 1/3$$
$$x \notin L \implies \Pr[\, M \text{ rejects } x\,] < 1/3$$

What complexity class does $DUMBPP$ equal? Briefly explain your answer.

[10] 2. Consider the following purported proof that $BPP \cap NP = RP$.

Proof. We already saw in class that $RP \subseteq BPP$ and $RP \subseteq NP$, so $RP \subseteq BPP \cap NP$.

Now consider any $L \in BPP \cap NP$. Every string $x \in L$ can be accepted with probability at least $2/3$ since $L \in BPP$. Every string $x \notin L$ can be rejected with probability 1 since $L \in NP$. These acceptance/rejection probabilities are the same as in definition of RP, so $L \in RP$. So $BPP \cap NP \subseteq RP$. □

Is this a valid proof? If so, explain how it can be made precise. If not, explain what the flaw is. In either case, ensure that your answer is explained carefully.

Hint: Similar logic is used in a problem near the end of Tutorial 1.

[10] 3. For this problem, you may consider either of these problems.

$$\begin{aligned} MINVC &= \{\, \langle G, k \rangle \,:\, \text{the smallest vertex cover in } G \text{ has exactly } k \text{ vertices} \,\}. \\ MAXCLIQUE &= \{\, \langle G, k \rangle \,:\, \text{the largest clique in } G \text{ has exactly } k \text{ vertices} \,\}. \end{aligned}$$

(Probably it is easier to consider $MAXCLIQUE$.)

In Assignment 6, we argued that $MINVC$ is NP-hard. (Looking at the solutions, we see that the proof also shows that $MAXCLIQUE$ is NP-hard.) We also remarked that $MINVC$ is *not believed to be in NP*. In this question, we will see why: if $MINVC \in NP$, then we will reach the unlikely conclusion that $NP = coNP$.

(If you have a valid solution that does not follow this breakdown into (a) and (b), that is fine.)

[8] a. Show that $\overline{MINVC}$ is NP-hard. (Or, if you prefer, show that $\overline{MAXCLIQUE}$ is NP-hard.)

Hints: In Lecture 23, we showed a reduction from $3SAT$ to $CLIQUE$ such that:

- a satisfiable formula maps to a graph whose maximum clique size is $= k$,
- an unsatisfiable formula maps to a graph whose maximum clique size is $< k$.

We want to tweak the reduction so that

- a satisfiable formula maps to a graph whose maximum clique size is $\neq \ell$,
- an unsatisfiable formula maps to a graph whose maximum clique size is $= \ell$.

Unfortunately, for the graphs coming from *un*satisfiable formulas, we don't know what the maximum clique size is. Is there some way to fix that?

[2] b. Suppose that $MINVC \in NP$. (Or, if you prefer, suppose that $MAXCLIQUE \in NP$.) Conclude that $NP = coNP$.

Hints: There are useful results in Practice Problem Set 5, which you may use.

[15] 4. Let us define the "Safe Marriage" problem. There are n people. Each pair of people u and v either like or dislike each other. A "Safe Marriage of size k" is a set of pairs

$$\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_k, v_k\}$$

such that:

- u_i and v_i like each other,
- u_i is the *only* person that v_i likes amongst $\{u_1, v_1, \ldots, u_k, v_k\}$,
- v_i is the *only* person that u_i likes amongst $\{u_1, v_1, \ldots, u_k, v_k\}$.

The objective is to decide if there is a Safe Marriage of size k. (There is no notion of gender in this problem.)

We can model this problem using an undirected graph G, where the vertices correspond to people, and the edges correspond to pairs who like each other. Let us define the decision problem

$$SAFEMARRIAGE = \{ \langle G, k \rangle : G \text{ has a safe marriage of size } k \}.$$

Prove that $SAFEMARRIAGE$ is NP-complete.

Hint: Try a reduction from Independent Set, which you may assume is known to be NP-complete. (There are very simple connections between Independent Set and Clique.)

[3] 5. **OPTIONAL BONUS QUESTION:**

Let G be a social network (i.e., a graph) whose vertices correspond to people and whose (undirected) edges correspond to friend relationships between pairs of people.

Suppose that some pairs of people (who are friends) get married. We allow the possibility of polygamy: person a could simultaneously be married to person b and to person c. A friend relationship between person x and person y is said to be **defunct** if x did not marry y, but either x or y married someone else say, person z. (For example, if x and y are friends, but x marries z, then x and y are unlikely to hang out anymore.) The entire social network is said to be **dreary** if every relationship is either a married couple, or a defunct friendship.

Let $DREARY$ the the language consisting of strings $\langle G, k \rangle$ such that G is a social network in which k marriages can make the network become dreary. Prove that $DREARY$ is NP-complete.