CPSC 421: Introduction to Theory of Computing
Assignment #6, due Friday November 4th by 12pm (noon), via Gradescope

[10] 1. Suppose that

- $A \subseteq \Sigma^*$ is NP-complete,
- $B \subseteq \Sigma^*$ is in P,
- $A \cap B = \emptyset$, and
- $A \cup B \neq \Sigma^*$

Prove that $A \cup B$ is NP-complete.

[15] 2. Some questions about Sipser's proof of Theorem 7.37 (the Cook-Levin theorem). (Beware: other proofs that you might find in other books or online resources might be different. In particular, they might not use the notion of a "configuration".)

[7] a. Each row of the tableau is supposed to be a "configuration" (defined on Sipser, page 168). How does the formula ϕ ensure that each row (i) contains *at least one* state q_i, and (ii) does not contain *two or more* states q_i.

[8] b. Give an upper bound on the number of "legal windows" in Sipser proof, in terms of $|\Gamma|$ and $|Q|$, using Big-O notation.

[15] 3. The "Interval Depth" problem is as follows. Given a set of n intervals on the real line, we would like to determine the largest subset of these intervals that contain a common point. (Each interval is of the form $[x, y]$ where $x, y \in \mathbb{R}$ and $x < y$.)

We may write the Interval Depth problem as a language $INTDEPTH$, which contains strings of the form $\langle k, x_1, y_1, \ldots, x_m, y_m \rangle$, where $x_i < y_i$, and *there exist* k intervals containing a common point.

[7] a. Describe a polynomial-time reduction from $INTDEPTH$ to $CLIQUE$.

[5] b. Describe and analyze a polynomial-time algorithm for $INTDEPTH$.

[3] c. Why don't these two results imply that $P = NP$?

[15] 4. Recall the problem

$$VC \;=\; \{\, \langle G, k \rangle \,:\, G \text{ has a vertex cover of size} \leq k \,\}.$$

We showed in Lecture 24 that VC is NP-complete. (See also Sipser Theorem 7.44.) So, if $P = NP$ then $VC \in P$.

Consider instead the problem

$$MINVC \;=\; \{\, \langle G, k \rangle \,:\, \text{the smallest vertex cover in } G \text{ has exactly } k \text{ vertices} \,\}.$$

This problem is *not believed* to be in NP.

[8] a. Prove that $MINVC$ is NP-hard. *Hint:* See notes to Lectures 23 & 24.

[7] b. Prove that if $P = NP$ then $MINVC \in P$.

[2] 5. **OPTIONAL BONUS QUESTION:**

Let us say that a boolean formula is a "four-occurrence CNF formula" if it is in conjunctive normal form and every variable appears at most four times. Define

$$CNF_4 \;=\; \{\, \langle\phi\rangle \,:\, \phi \text{ is a satisfiable, four-occurrence CNF formula} \,\}\,.$$

It is known that CNF_4 is NP-complete.

Let us say that a boolean formula is a "four-occurrence 4CNF formula" if it is in conjunctive normal form, every variable appears at most four times, and every clause contains exactly four literals (no repetitions). Define

$$4CNF_4 \;=\; \{\, \langle\phi\rangle \,:\, \phi \text{ is a satisfiable, four-occurrence 4CNF formula} \,\}\,.$$

Prove that $4CNF_4$ is in P.