1. Suppose that
- $A \subseteq \Sigma^*$ is $NP$-complete,
- $B \subseteq \Sigma^*$ is in $P$,
- $A \cap B = \emptyset$, and
- $A \cup B \neq \Sigma^*$

Prove that $A \cup B$ is $NP$-complete.

2. Some questions about Sipser’s proof of Theorem 7.37 (the Cook-Levin theorem). (Beware: other proofs that you might find in other books or online resources might be different. In particular, they might not use the notion of a “configuration”.)

   a. Each row of the tableau is supposed to be a “configuration” (defined on Sipser, page 168). How does the formula $\phi$ ensure that each row (i) contains at least one state $q_i$, and (ii) does not contain two or more states $q_i$.

   b. Give an upper bound on the number of “legal windows” in Sipser proof, in terms of $|\Gamma|$ and $|Q|$, using Big-O notation.

3. The “Interval Depth” problem is as follows. Given a set of $n$ intervals on the real line, we would like to determine the largest subset of these intervals that contain a common point. (Each interval is of the form $[x, y]$ where $x, y \in \mathbb{R}$ and $x < y$.)

   We may write the Interval Depth problem as a language $INTDEPTH$, which contains strings of the form $\langle k, x_1, y_1, \ldots, x_m, y_m \rangle$, where $x_i < y_i$, and there exist $k$ intervals containing a common point.

   a. Describe a polynomial-time reduction from $INTDEPTH$ to $CLIQUE$.

   b. Describe and analyze a polynomial-time algorithm for $INTDEPTH$.

   c. Why don’t these two results imply that $P = NP$?

4. Recall the problem

   $$VC = \{ \langle G, k \rangle : G \text{ has a vertex cover of size } \leq k \}.$$  

   We showed in Lecture 24 that $VC$ is $NP$-complete. (See also Sipser Theorem 7.44.) So, if $P = NP$ then $VC \in P$.

   Consider instead the problem

   $$MINVC = \{ \langle G, k \rangle : \text{the smallest vertex cover in } G \text{ has exactly } k \text{ vertices} \}.$$  

   This problem is not believed to be in $NP$.


   b. Prove that if $P = NP$ then $MINVC \in P$. 


Let us say that a boolean formula is a “four-occurrence CNF formula” if it is in conjunctive normal form and every variable appears at most four times. Define

\[ CNF_4 = \{ \langle \phi \rangle : \phi \text{ is a satisfiable, four-occurrence CNF formula} \}. \]

It is known that \( CNF_4 \) is NP-complete.

Let us say that a boolean formula is a “four-occurrence 4CNF formula” if it is in conjunctive normal form, every variable appears at most four times, and every clause contains exactly four literals (no repetitions). Define

\[ 4CNF_4 = \{ \langle \phi \rangle : \phi \text{ is a satisfiable, four-occurrence 4CNF formula} \}. \]

Prove that \( 4CNF_4 \) is in P.