1. Let $R$ and $S$ be regular expressions over some alphabet $\Sigma$. Formally prove or disprove each of the following statements. (Here we use “=” to mean that the two regular expressions describe the same language.)

\[ a. \quad (R^*S^*)^* = (R \cup S)^*. \]

\[ b. \quad R^* \cup S^* = (R \cup S)^*. \]

\[ c. \quad (R \cup S)^*S = (R^*S)^*. \]

2. Prove that the following languages over $\Sigma = \{0, 1\}$ are not regular. You may use the fact that the class of regular languages is closed under unions, concatenations and complements.

\[ a. \quad L = \{ 0^n1^m : n \leq m \}. \]

\[ b. \quad L = \{ w : \text{\textit{\textbar w\textbar is a perfect square}} \}. \text{\textit{(That is, \textbar w\textbar = n^2 for some integer n \geq 0.)}} \]

\[ c. \quad L = \{ w : w \text{ is not a palindrome} \}. \text{\textit{(A palindrome is a string that reads the same forward and backward.)}} \]

3. Consider the language $F = \{ a^ib^jc^k : i,j,k \geq 0 \text{ and if } i = 1 \text{ then } j = k \}$.

\[ a. \quad \text{Show that } F \text{ is not regular.} \]

\[ b. \quad \text{Show that } F \text{ satisfies the condition of the pumping lemma. In other words, give an integer } p \geq 1 \text{ and demonstrate that for all strings } w \in F \text{ with } \textbar w\textbar \geq p \text{ we can find } x,y,z \in \{a,b,c\}^* \text{ such that} \]

\[ \bullet \quad \textbar y\textbar > 0, \]

\[ \bullet \quad \textbar xy\textbar \leq p, \text{ and} \]

\[ \bullet \quad xy^nz \in F \text{ for all } n \geq 0. \]

\[ c. \quad \text{Explain why this does not contradict the pumping lemma.} \]

4. Let $\Sigma = \{0, 1\}$ and let $B$ be the collection of strings that contain at least one 1 in their second half. In other words, let

\[ B = \{ wv : u \in \Sigma^*, v \in \Sigma^*1\Sigma^* \text{ and } \textbar u\textbar \geq \textbar v\textbar \}. \]

\[ a. \quad \text{Give a PDA that recognizes } B. \]

\[ b. \quad \text{Give a CFG that generates } B. \]

5. **OPTIONAL BONUS QUESTION**

Professor Dumas thinks he understands the pumping lemma. His interpretation is that “a finite automaton has a limited amount of complexity, so if it accepts a very long string, that string must have some repeated structure inside”. To formalize his interpretation, he proposes the following conjecture.

**Conjecture 1.** Let $L$ be regular language over $\Sigma$. Then there exists an integer $p \geq 1$ such that, for every string $w \in L$ with $\textbar w\textbar \geq p$,

\[ \exists x, y, z \in \Sigma^* \text{ and } i \geq 2 \text{ such that } w = xy^iz \text{ and } y \neq \epsilon. \quad (1) \]
Professor Dumas’ conjecture is false. Let’s try to understand why.

[10] a. Find a DFA $M = (Q, \Sigma, \delta, q_0, F)$ with $|Q| \leq 3$ and $|\Sigma| \geq 3$ and a string $w$ that is accepted by $M$, and has $|w| > |\Sigma|^2$, but does not satisfy (1).

[10] b. Can you find an even longer string $w$ that is accepted by your $M$ but does not satisfy (1)? You can use computer simulations if you like. If your solution involves any ideas found in the literature or online, be sure to give a citation.