CPSC 421/501 Intro to Theory of Computing (Term 1, 2013-14)
Assignment 0

Due: Wednesday September 11th, in class.

Question 1: Let L_1 and L_2 be finite sets. Suppose that C is a finite set with $L_1 \in C$ and $L_2 \in C$. Suppose that $x \in L_1 \cap L_2$. Is $x \in C$? If so, explain why; if not, give a counterexample.

Question 2: Let A and B be finite sets. A *bijection* from A to B is a function $f : A \to B$ such that $f(a) = f(a')$ implies $a = a'$, and for every $b \in B$ there exists $a \in A$ with $f(a) = b$.

- **(a):** Suppose that $|A| = |B| = n \geq 1$. How many bijections from A to B are there?
- **(b):** Suppose that $|A| = n$ and $|B| = 2n$ for $n \geq 1$. How many bijections from A to B are there?

Question 3: Let $f(x)$, $g(x)$ and $h(x)$ be univariate polynomials of degree d. What is the degree of $f(g(h(x)))$?

Question 4:

- **(a):** Suppose we flip a biased coin that comes up heads with probability p and tails with probability $1 - p$. Suppose we perform k independent flips of this coin. What is the probability that we see heads at least once during these k flips?
- **(b):** There are two decks of cards. One is complete, but the other is missing the Ace of spades. Suppose you pick one of the two decks with equal probability and then select a card from that deck uniformly at random. What is the probability that you picked the complete deck, given that you selected the eight of hearts?

Question 5: Consider the following algorithmic problem. The input is a graph with n nodes (and no multi-edges). A *rectangle* in the graph is a sequence of four vertices (v_1, v_2, v_3, v_4) such that the edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$ and $\{v_1, v_4\}$ are all present in the graph. (We do not care whether $\{v_1, v_3\}$ or $\{v_2, v_4\}$ are edges.)

Give an algorithm to decide if the given graph contains a rectangle. The running time of your algorithm should be polynomial in n. You may assume whatever computational model you like, and assume that the graph is represented however you like.

Question 6: Problem 0.11 (3rd edition of Sipser).

Let $S(n) = 1 + 2 + \cdots + n$ be the sum of the first n natural numbers and let $C(n) = 1^3 + 2^3 + \cdots + n^3$ be the sum of the first n cubes. Prove the following equalities by induction on n, to arrive at the curious conclusion that $C(n) = \big(S(n)\big)^2$ for every $n \geq 1$.

- **(a):** $S(n) = n(n + 1)/2$.
- **(b):** $C(n) = (n^4 + 2n^3 + n^2)/4 = n^2(n + 1)^2/4$.

Question 7: Problem 0.13 (3rd edition of Sipser), which is Problem 0.12 (2nd edition of Sipser).

Let G be a graph with two or more nodes. We assume that G has no multiedges or self-loops, but it might be disconnected. Show that G must contain (at least) two nodes whose degrees are equal.