**Surname (print):** ____________________

**First Name:** ____________________

**Signature:** ____________________

**ID #:** ____________________

University of British Columbia

**CPSC 421/501 Introduction to Theory of Computation**

Final Exam

December 8, 2012

3:30 p.m. - 6:00 p.m.

**INSTRUCTIONS:**

1: Write your name and ID# in the blanks above.

2: There are 13 pages in this exam, including the cover page and three blank pages at the back. Make sure that you have all the pages.

3: No textbooks, notes, cheatsheets or electronic devices are permitted.

| Question | Value | Mark Awarded |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| Total | 100 | |

**Question 1:** True/False Questions. You do not need to justify your answer. When relevant, you may assume that the alphabet is $\{a, b\}$.

1: Every finite language is regular.

2: The language $\{ a^n b a^n : n \geq 0 \}$ is regular.

3: The language $\left\{ a^n b^{2n} a^n : n \geq 0 \right\}$ is context-free.

4: There exists a non-recognizable language that can be recognized by a nondeterministic Turing machine.

5: The complexity class EXP is closed under complement.

6: There exists an NP-complete language $L$ such that $\overline{L}$ is undecidable.

7: There is a language known to be in $\mathrm{NP} \cap \mathrm{coNP}$ that is not known to be in P.

8: Every 3CNF formula has an assignment that satisfies at least half of the clauses.

9: In the the theory of context free languages, the acronym "CFL" actually stands for Calcutta Football League.

10: In the theory of probabilistically checkable proofs, the acronym "PCP" actually stands for Peruvian Communist Party.

**Question 2:** Recall the undecidable language $A_{TM}$ defined in class

$$A_{TM} = \{ \langle M, w \rangle : \text{the Turing machine } M \text{ accepts the string } w \} .$$

**(a):** Is $A_{TM}$ recognizeable? Prove your answer in a few sentences.

**(b):** Is $A_{TM}$ NP-hard? Prove your answer in a few sentences.

**(c):** Is $\overline{A_{TM}}$ recognizeable? Prove your answer in a few sentences. (Here $\overline{A_{TM}}$ denotes the complement of $A_{TM}$.) You may use any theorem proven in class (except for any theorems specifically about recognizeability of $\overline{A_{TM}}$).

**Question 3:** Let $G = (V, E)$ be an undirected graph. Recall that a clique $G$ is a subset $U \subseteq V$ such that $G$ contains an edge between every pair of distinct vertices in $U$. Recall that

$$\mathit{CLIQUE} \;=\; \{\, \langle G, k \rangle \;:\; G \text{ is a graph containing a clique of size at least } k \;\}\,.$$

Define

$$\mathit{HALFCLIQUE} \;=\; \{\, \langle H \rangle \;:\; H \text{ is a graph with } n \text{ nodes containing a clique of size at least } n/2 \;\}\,.$$

**(a):** Prove that $\mathit{HALFCLIQUE} \in NP$.

**(b):** Prove that $\mathit{CLIQUE} \leq_P \mathit{HALFCLIQUE}$.

**(c):** Conclude that *HALFCLIQUE* is $NP$-complete. You may use theorems proven in class.

**Question 4:**

**(a):** Define the complexity class coNP.

**(b):** A language $L \in \text{coNP}$ is said to be coNP-complete if, for every language $A \in \text{coNP}$, we have $A \leq_P L$. Prove that $L$ is coNP-complete if and only if $\overline{L}$ is NP-complete.

**Question 5:** Let $\Sigma_n$ be the class of functions of the form

$$f(x_1, ..., x_n) \;=\; \sum_{i=1}^{n} \sum_{j=1}^{n} c_{i,j} x_i x_j$$

where $x_1, ..., x_n$ are variables and each of the coefficients $c_{i,j}$ is a real number. (If you are bothered by real numbers, you can think of each coefficient as being an integer that can each be represented in $n$ bits.) Let $\Sigma = \bigcup_{n \geq 1} \Sigma_n$.

We say $f, g \in \Sigma_n$ are **equivalent** if there exists a permutation $\pi$ of the $n$ variables such that

$$f(x_1, ..., x_n) - g(x_{\pi(1)}, ..., x_{\pi(n)}) \;=\; 0.$$

(Here 0 represents the polynomial for which all coefficients are equal to zero, i.e., evaluating this polynomial at any point produces the value zero. Here $\pi$ is a one-to-one and onto map from $\{1, ..., n\}$ to itself, i.e., it just reorders the variables.)

**(a):** Let $L$ be the language

$$L \;=\; \{ \, \langle f, g \rangle \;:\; f, g \in \Sigma \text{ are equivalent} \, \} \, .$$

Prove that $L \in \mathrm{NP}$.

**(b):** Let $L'$ be the language

$$L' \;=\; \{\, \langle f, g \rangle \;:\;\; f, g \in \Sigma \text{ are } \textbf{not} \text{ equivalent} \,\}\,.$$

Prove that $L' \in \mathrm{IP}$. In other words, give an interactive proof for $L'$, where each string $x \in L'$ is accepted with probability 1, and each string $x \notin L'$ is accepted with probability at most 1/3.

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