A second course in randomized algorithms

(some notes that are not intended for publication)

Nick Harvey
University of British Columbia

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Contents

Chapter 20

A Warmup: Set Cover

20.1 Definition and Background

Suppose we have a family of sets

$$S_1, \ldots, S_m \qquad \text{where} \qquad \bigcup_{j=1}^{m} S_j = [n].$$

The set $[n]$ is called the ***ground set***; recall that $[n]$ is standard notation for $\{1, \ldots, n\}$. Clearly each set in the family satisfies $S_j \subseteq [n]$.

We would like to choose as few of those sets as possible, while preserving the property that their union equals $[n]$. In mathematical notation, the Set Cover problem is

$$\min \left\{ |C| \,:\, C \subseteq [m] \text{ and } \bigcup_{j \in C} S_j = [n] \right\}.$$

Let OPT to refer to this minimum value.

An algorithm for this problem is called an α-approximation if it always outputs a cover C satisfying $|C| \leq \alpha \cdot \text{OPT}$. Clearly $\alpha \geq 1$. Here α could be a constant or it could depend on other parameters, like n or m.

Some known facts about the Set Cover problem.

- It is NP-hard, as was shown by Karp in 1972. Thus, it is unlikely that any polynomial time algorithm can guarantee to produce an exact solution.
- The natural *greedy algorithm* is a $\ln n$-approximation algorithm.
- For every constant $c < 1$, it is very unlikely that there is a $(c \ln n)$-approximation algorithm. (If such an algorithm existed, this would have similar consequences to P=NP.) Thus, the greedy algorithm is the best possible.

Instead of discussing the greedy algorithm, we will discuss this problem with a viewpoint of mathematical optimization.

Integer program. The first step is to write the problem as an integer program (IP). For each $j \in [m]$, we create a Boolean variable x_j that indicates whether we select the set S_j. Using these variables, we can write the problem as

$$\begin{aligned} \min \quad & \sum_{j=1}^{m} x_j & & \\ \text{s.t.} \quad & \sum_{j \,:\, i \in S_j} x_j & \geq 1 \qquad & \forall i \in [n] \\ & x_j \in \{0,1\} & & \forall j \in [m] \end{aligned}$$

The objective function $\sum_{j=1}^{m} x_j$ counts the number of chosen sets. The i^{th} inequality constraint ensures that at least one of the chosen sets contains element i. Thus, this integer program captures the Set Cover problem exactly. It follows that its minimum value is also OPT, and it is also NP-hard to solve exactly.

Linear program. The next idea is to relax the integer program to a ***linear program*** (LP), by allowing the variables x_j to lie in the interval $[0,1]$ rather than the discrete set $\{0,1\}$. The resulting LP can be written

$$\begin{aligned} \min \quad & \sum_{j=1}^{m} x_j & & \\ \text{s.t.} \quad & \sum_{j \,:\, i \in S_j} x_j & \geq 1 \qquad & \forall i \in [n] \\ & 0 \leq x_j \leq 1 & & \forall j \in [m] \end{aligned}$$

We will use LPOPT to denote the minimum value of this LP.

Question 20.1.1. Does the value of the LP always equal the value of the IP? That is, does LPOPT always equal OPT?

Answer.

No. If it were true, then we'd have proven P=NP, because LPs can be solved in polynomial time. That is not an airtight argument, but we'll see an example below where they differ.

Question 20.1.2. Do we always have LPOPT $\leq$ OPT or LPOPT $\geq$ OPT?

Answer.

We always have LPOPT $\leq$ OPT. To see this, note that every feasible solution x to the IP is also a feasible solution to the LP. Thus, the LP is minimizing over a larger set of feasible solutions, and therefore its value can only be smaller.

Example 20.1.3. Consider the following example with $n = m = 3$.

$$S_1 = \{1,2\} \qquad S_2 = \{2,3\} \qquad S_3 = \{1,3\}\,.$$

It is clear that we must choose at least two sets in order to cover all three elements of the ground set. However, if we set $x_1 = x_2 = x_3 = 0.5$, then one can verify that this is a feasible solution to the LP with objective value 1.5.

Problem to Ponder 20.1.4. In this example, what is LPOPT, the optimal value of the LP?

The term ***integrality gap*** is used to refer to the ratio of the optimum values of an IP and its corresponding LP. Example 20.1.3 shows that our LP for Set Cover has an integrality gap greater than 1. Nevertheless, the LP is still useful. We will show two results.

1. The integrality gap of this LP is at most $\approx \ln n$.
2. Given any feasible solution x to the LP, there is a randomized algorithm that, with constant probability, can produce a valid set cover C whose size is at most $\approx \ln(n) \cdot \sum_j x_j$.

Question 20.1.5. Do you see why the second assertion implies the first?

Answer.

gap.

Consider an optimum solution to the LP. It has objective value at most LPOPT. Applying the randomized algorithm, we obtain a set cover satisfying $|C| \lessapprox \ln(u)\text{LPOPT}$. Since the best Set Cover solution is no larger than C, it follows that $\text{OPT} \lessapprox \ln(u)\text{LPOPT}$, which gives the desired bound on the integrality

20.2 Randomized Rounding

In this section we design an algorithm that will prove the second assertion above. This is called a ***rounding algorithm***, because it converts a fractional solution into an integral solution.

Algorithm 20.1 Rounding the Set Cover LP. Assume that the input x is a feasible solution to the LP.

```
1: function SetCoverRound(x)
2:     Let C ← ∅                                  ▷ The indices of the chosen sets
3:     Let L ← ln(4n)                             ▷ The rounding has L phases
4:     for t = 1, ..., L
5:         for j = 1, ..., m
6:             Add j to C independently with probability x_j
7:     return C
8: end function
```

Lemma 20.2.1. The output C fails to cover all elements in $[n]$ (i.e., $\bigcup_{j \in C} S_j \neq [n]$) with probability at most $1/4$.

Claim 20.2.2. Consider the t^{th} phase of the algorithm. Each element $i \in [n]$ fails to be covered by the sets chosen in this phase with probability at most $1/e$.

Proof. Element i is not covered if every set containing i fails to be chosen. The probability of this is

$$\begin{aligned}
\Pr[\,(j \text{ not chosen}) \ \forall j \text{ s.t. } i \in S_j\,] &= \prod_{j\,:\,i \in S_j} \Pr[\,j \text{ not chosen}\,] \qquad \text{(by independence)} \\
&= \prod_{j\,:\,i \in S_j} (1 - x_j) \\
&< \prod_{j\,:\,i \in S_j} \exp(-x_j) \qquad \text{(by Fact A.2.5)} \\
&= \exp\Big(-\sum_{j\,:\,i \in S_j} x_j\Big).
\end{aligned}$$

This is at most $1/e$ since x is feasible for the LP. □

Proof of Lemma 20.2.1. First let us analyze the probability that a particular element is not covered by C. For any $i \in [n]$,

$$\begin{aligned}
\Pr[\text{element } i \text{ not covered}] &= \prod_{t=1}^{L} \Pr[\text{element } i \text{ not covered in phase } t] \\
&\le \prod_{t=1}^{L} (1/e) \qquad \text{(by Claim 20.2.2)} \\
&= \exp(-L) = \frac{1}{4n} \qquad \text{(since } L = \ln(4n)\text{)}.
\end{aligned}$$

By a union bound (Fact A.3.8), the probability that *any* element fails to be covered by C is

$$\Pr[\textit{any} \text{ element not covered by } C] \le \sum_{i=1}^{n} \Pr[\text{element } i \text{ not covered by } C] \le \sum_{i=1}^{n} \frac{1}{4n} = \frac{1}{4}. \quad \square$$

Lemma 20.2.3. The output C has $|C| > 4L \cdot \sum_j x_j$ with probability at most $1/4$.

Proof. In each phase, set j is added to C with probability x_j, so the expected number of sets added is $\sum_j x_j$. The expected number of edges added in *all* iterations is at most L times larger. That is, $\mathrm{E}[|C|] \le L \cdot \sum_j x_j$. It follows that

$$\begin{aligned}
\Pr\left[|C| \ge 4L \cdot \textstyle\sum_j x_j \right] &\le \frac{\mathrm{E}[|C|]}{4L \cdot \sum_j x_j} \qquad \text{(by Markov's inequality, Fact A.3.22)} \\
&\le \frac{L \cdot \sum_j x_j}{4L \cdot \sum_j x_j} = \frac{1}{4}.
\end{aligned}$$ □

We conclude with the following theorem.

Theorem 20.2.4. The output C covers all elements and has size at most $4L \cdot \sum_j x_j$ with probability at least $1/2$.

References: (Shmoys and Shmoys, 2010, Section 1.7).

Proof. By Lemma 20.2.1, C fails to cover all elements with probability at most $1/4$. By Lemma 20.2.3, C has size exceeding $4L \cdot \sum_j x_j$ with probability at most $1/4$. By a union bound, the probability that either of these occurs is at most $1/2$. □

Corollary 20.2.5. There is a randomized, polynomial-time algorithm that gives a $4\ln(4n)$-approximation to the Set Cover problem.

Proof. Compute an optimum solution x to the linear program; it satisfies $\sum_j x_j = \mathrm{LPOPT}$. This can be done in polynomial time, for example by interior point methods. By Theorem 20.2.4, the rounding algorithm above has probability $1/2$ of producing a Set Cover solution C with

$$|C| \le 4L \cdot \sum_j x_j = 4L \cdot LPOPT \le 4L \cdot OPT. \quad \square$$

Problem to Ponder 20.2.6. By adjusting the parameters, show that the integrality gap is at most $(1 + 2\epsilon)\ln(n/\epsilon)$ for any $\epsilon > 0$.

Chapter 21

Concentration Bounds, with details

21.1 Chernoff bound, in detail

The Chernoff bound was presented in a simplified form in Theorem 9.2.2. Let us now present it in a more elaborate form.

Let $X_1, \dots, X_n$ be independent random variables such that X_i always lies in the interval $[0, 1]$. Define $X = \sum_{i=1}^{n} X_i$. The expectation $\mathrm{E}[X]$ need not be exactly known, but we assume that $\check{\mu} \leq \mathrm{E}[X] \leq \hat{\mu}$. If it is exactly known, we may define $\check{\mu} = \hat{\mu} = \mathrm{E}[X]$.

Theorem 21.1.1. For all $\delta > 0$,

$$\text{Right tail:}\quad \Pr[X \geq (1+\delta)\hat{\mu}] \overset{(a)}{\leq} \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\hat{\mu}} \overset{(b)}{\leq} \begin{cases} e^{-\delta^2\hat{\mu}/3} & (\text{if } \delta \leq 1) \quad \text{“Gaussian tail”} \\ e^{-(1+\delta)\ln(1+\delta)\hat{\mu}/4} & (\text{if } \delta \geq 1) \quad \text{“Poisson tail”} \\ e^{-\delta\hat{\mu}/3} & (\text{if } \delta \geq 1) \quad \text{“Exponential tail”} \end{cases}$$

$$\text{Left tail:}\quad \Pr[X \leq (1-\delta)\check{\mu}] \overset{(c)}{\leq} \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\check{\mu}} \overset{(d)}{\leq} e^{-\delta^2\check{\mu}/2} \quad \text{“Gaussian tail”}.$$

Inequalities (c) and (d) are only valid for $\delta < 1$, but $\Pr[X \leq (1-\delta)\check{\mu}] = 0$ if $\delta > 1$.

References: (McDiarmid, 1998, Theorem 2.3), (Lehman et al., 2018, Theorem 20.5.1), (Motwani and Raghavan, 1995, Section 4.1), (Mitzenmacher and Upfal, 2005, equations (4.2) and (4.5)), (Klenke, 2008, Exercise 5.2.1), Wikipedia.

The tails have a qualitative difference in their dependence on δ.

- The “Gaussian tails” depend on δ^2. This resembles a Gaussian distribution (see Appendix B.4.3), whose tails look like $\exp(-\delta^2/2)$ (see Fact B.4.8).
- The “Poisson tail” depends on $\delta \ln \delta$. This resembles a Poisson distribution, whose mass function function looks like[1] $1/\delta! \approx \exp(-\delta \ln \delta)$.
- The “Exponential tail” depends on δ. This resembles the exponential distribution, whose tail looks like $e^{-\delta}$. This is weaker, but more convenient, than the “Poisson tail”.

[1] See, e.g., (Vershynin, 2018, Theorem 1.3.4) or these lecture notes.

21.2 Proofs for Chernoff Bound

21.2.1 Proof of inequality (a)

The Chernoff bounds would not be true without the assumption that $X_1, \ldots, X_n$ are independent. What special properties do independent random variables have? One basic property is that

$$\mathrm{E}[A \cdot B] \;=\; \mathrm{E}[A] \cdot \mathrm{E}[B] \tag{21.2.1}$$

for any independent random variables A and B.

References: (Lehman et al., 2018, Theorem 19.5.6), (Anderson et al., 2017, Fact 8.10), (Feller, 1968, Theorem IX.2.3), (Cormen et al., 2001, Exercise C.3-5), (Motwani and Raghavan, 1995, Proposition C.6), (Mitzenmacher and Upfal, 2005, Theorem 3.3), (Grimmett and Stirzaker, 2001, Lemma 3.3.9), (Durrett, 2019, Theorem 2.1.13), (Klenke, 2008, Theorem 5.4).

But the Chernoff bound has nothing to do with *products* of random variables, it is about *sums* of random variables. So one trick we could try is to convert sums into products using the exponential function. Fix some parameter $\theta > 0$ whose value we will choose later. We will look at the random variables

$$\begin{aligned} & \exp(\theta X_i) \qquad \forall i \in [n] \\ \text{and} \qquad & \exp(\theta X) \;=\; \exp\Big(\theta \sum_{i=1}^{n} X_i\Big) \;=\; \prod_{i=1}^{n} \exp(\theta X_i) \end{aligned}$$

Since $X_1, \ldots, X_n$ are independent, it follows[2] that $e^{\theta X_1}, \ldots, e^{\theta X_n}$ are also independent. Therefore, using (21.2.1) repeatedly,

$$\mathrm{E}\left[\, e^{\theta X} \,\right] \;=\; \prod_{i=1}^{n} \mathrm{E}\left[\, e^{\theta X_i} \,\right]. \tag{21.2.2}$$

So far this all seems promising. We want to prove that X is small, which is equivalent to proving that $e^{\theta X}$ is small. Using (21.2.2), we can do this by showing that each $\mathrm{E}\left[\, e^{\theta X_i} \,\right]$ is small. Perhaps we can somehow show $\mathrm{E}\left[\, e^{\theta X_i} \,\right]$ is small by comparing it to $\mathrm{E}[\, X_i \,]$?

If we were forgetting the rules of probability, we might be tempted to say that $\mathrm{E}\left[\, e^{\theta X_i} \,\right]$ equals $e^{\theta\, \mathrm{E}[\, X_i \,]}$, but that is false. We might remember one useful probability trick called Jensen's inequality (Fact B.4.2) that says $f(\mathrm{E}[A]) \le \mathrm{E}[f(A)]$ for any random variable A and any convex function f. Applying this with $f(x) = e^{\theta x}$, we see that

$$e^{\theta\, \mathrm{E}[\, X_i \,]} \;\le\; \mathrm{E}\left[\, e^{\theta X_i} \,\right]. \tag{21.2.3}$$

So we get a *lower bound* on $\mathrm{E}\left[\, e^{\theta X_i} \,\right]$ in terms of $\mathrm{E}[\, X_i \,]$, but we actually wanted an *upper bound*.

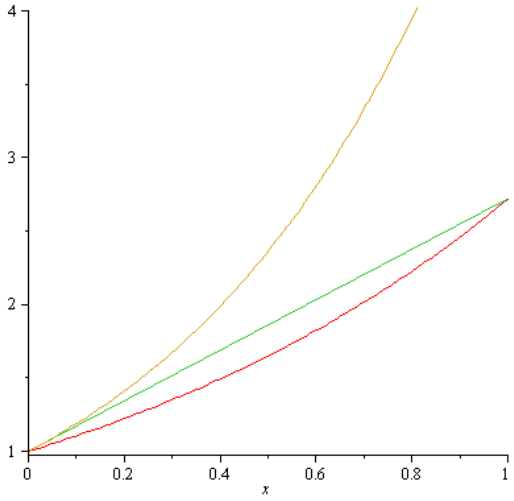
Claim 21.2.2 gives the desired upper bound; it shows that the inequality in (21.2.3) can almost be reversed. The proof is easy once we have the following convexity fact.

Claim 21.2.1. For all $\theta \in \mathbb{R}$ and all $x \in [0, 1]$,

$$\exp(\theta x) \;\le\; 1 + (e^{\theta} - 1)x \;\le\; \exp\big((e^{\theta} - 1)x\big).$$

The inequalities are illustrated by this plot.

[2] See (Lehman et al., 2018, Lemma 19.2.2), (Cormen et al., 2001, Equation (C.24)), (Grimmett and Stirzaker, 2001, Theorem 3.2.3), (Klenke, 2008, Remark 2.15(iii)).



Proof of Claim 21.2.1. Consider the first inequality $e^{\theta x} \le 1 + (e^\theta - 1)x$ for all $x \in [0,1]$. This follows from Fact B.3.6 by setting $c = e^\theta$. The second inequality $1 + (e^\theta - 1)x \le \exp((e^\theta - 1)x)$ follows from the familiar Fact A.2.5. □

Claim 21.2.2. Let $\theta \in \mathbb{R}$ be arbitrary. Then $\mathrm{E}\left[\, e^{\theta X_i} \,\right] \le \exp\left((e^\theta - 1)\,\mathrm{E}\left[\, X_i \,\right]\right)$.

Proof. The main idea is as follows. Although we cannot "pull the expectation inside the exponential", we can use Claim 21.2.1 to approximate the exponential by a linear function, then "pull the expectation inside" via linearity of expectation, then finally switch back to an exponential function.

The formal argument is

$$\mathrm{E}\left[\, e^{\theta X_i} \,\right] \;\le\; \mathrm{E}\left[\, 1 + (e^\theta - 1)X_i \,\right] \;=\; 1 + (e^\theta - 1)\,\mathrm{E}\left[\, X_i \,\right] \;\le\; \exp((e^\theta - 1)\,\mathrm{E}\left[\, X_i \,\right]),$$

where both inequalities follow from Claim 21.2.1. □

Now we are ready to prove the inequality (a) of the Chernoff bound.

$$\begin{aligned}
\Pr\left[\, X \ge (1+\delta)\hat{\mu} \,\right] &= \Pr\left[\, \exp(\theta X) \ge \exp\left(\theta(1+\delta)\hat{\mu}\right) \,\right] && \text{(by monotonicity)} \\
&\le \frac{\mathrm{E}\left[\, \exp(\theta X) \,\right]}{\exp(\theta(1+\delta)\hat{\mu})} && \text{(by Markov's inequality)} \\
&= \frac{\prod_{i=1}^n \mathrm{E}\left[\, e^{\theta X_i} \,\right]}{\exp(\theta(1+\delta)\hat{\mu})} && \text{(by (21.2.2))} \\
&\le \frac{\prod_{i=1}^n \exp((e^\theta - 1)\,\mathrm{E}\left[\, X_i \,\right])}{\exp(\theta(1+\delta)\hat{\mu})} && \text{(by Claim 21.2.2).}
\end{aligned}$$

Gathering everything inside one exponential we get

$$\Pr\left[\, X \ge (1+\delta)\hat{\mu} \,\right] \;\le\; \exp\left((e^\theta - 1)\sum_i \mathrm{E}\left[\, X_i \,\right] - \theta(1+\delta)\hat{\mu}\right).$$

Finally, substituting $\theta = \ln(1+\delta)$ and using $\sum_i \mathrm{E}\left[\, X_i \,\right] = \mathrm{E}\left[\, X \,\right] \le \hat{\mu}$ proves inequality (a).

References: Another exposition of inequality (a) can be found in (Lehman et al., 2018, Section 20.5.6).

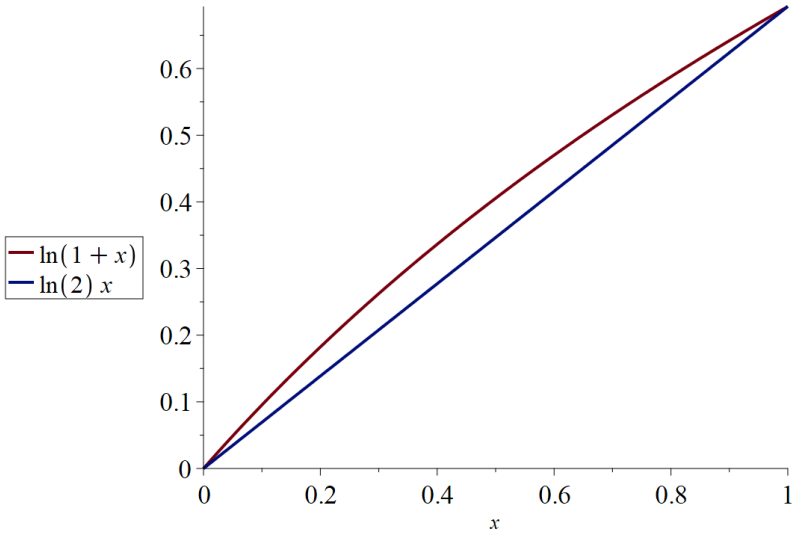
21.2.2 Proof of inequality (b), when $\delta \in [0,1]$

Claim 21.2.3. Then $(1+x)\ln(1+x) - x \geq \frac{\ln(2)}{2} \cdot x^2$ for all $x \in [0,1]$.

Proof. Note that the LHS and RHS both vanish at $x = 0$. So the claim holds if the derivative of the LHS is at least the derivative of the RHS on the interval $[0,1]$. By simple calculus,

$$\frac{d}{dx}\big[(1+x)\ln(1+x) - x\big] = \ln(1+x) \qquad \text{and} \qquad \frac{d}{dx}\frac{\ln(2)}{2}x^2 = \ln(2)x.$$

These derivatives are illustrated in the following figure.



They agree when $x = 0$ (both equal zero) and also agree when $x = 1$ (both equal $\ln 2$). Thus we have $\ln(1+x) \geq \ln(2)x$ for all $x \in [0,1]$, since the LHS is concave and the RHS is linear. □

Inequality (b) now follows straightforwardly because

$$\begin{aligned}
\frac{e^{\delta}}{(1+\delta)^{1+\delta}} &= \exp\Big(-\big((1+\delta)\ln(1+\delta) - \delta\big)\Big) \\
&\leq \exp\Big(-\frac{\ln 2}{2}\delta^2\Big) \qquad \text{(by Claim 21.2.3)} \\
&\leq e^{-\delta^2/3},
\end{aligned}$$

since $\ln(2) > 0.69 > 2/3$.

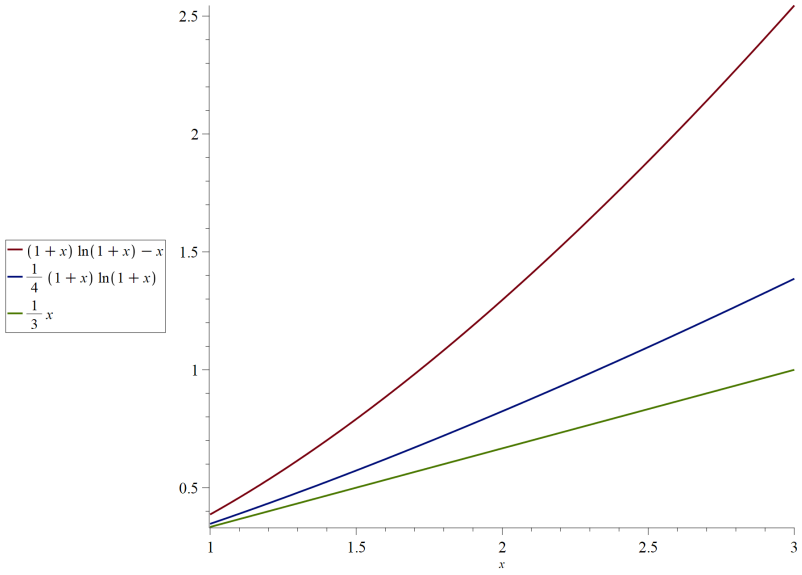
21.2.3 Proof of inequality (b), $\delta \geq 1$

Claim 21.2.4. Define

$$\begin{aligned}
f(x) &= (1+x)\ln(1+x) - x \\
g(x) &= (1+x)\ln(1+x)/4 \\
h(x) &= x/3
\end{aligned}$$

Then $f(x) > g(x) > h(x)$ for $x \geq 1$.

This claim is illustrated by the following plot.



Proof. First we observe that the claimed inequality holds at the point $x = 1$. We have

$$\begin{aligned} f(1) &= 2\ln(2) - 1 > 0.38 \\ g(1) &= \ln(2)/4 \approx 0.346 \\ h(1) &= 1/3 < 0.34. \end{aligned}$$

We will show that their derivatives satisfy

$$\frac{d}{dx}f(x) \overset{(1)}{\geq} \frac{d}{dx}g(x) \overset{(2)}{\geq} \frac{d}{dx}h(x) \qquad \forall x \geq 1,$$

from which the claim follows by integration. These derivatives have the following expressions.

$$\frac{d}{dx}f(x) = \ln(1+x) \qquad \frac{d}{dx}g(x) = (1+\ln(1+x))/4 \qquad \text{and} \qquad \frac{d}{dx}h(x) = 1/3.$$

Inequality (2) is straightforward. For $x \geq 1$, we have $(1+\ln(1+x))/4 \geq (1+\ln(2))/4 \approx 0.423$, which is greater than $1/3$.

For inequality (1), first observe that $y \geq (1+y)/4$ for $y \geq 1/3$. Substituting $y \leftarrow \ln(1+x)$ gives $\ln(1+x) \geq (1+\ln(1+x))/4$ for $x \geq e^{1/3} - 1 \approx 0.395$. This implies (1). □

We can now show inequality (b) of Theorem 21.1.1 in the case $\delta \geq 1$. Using the notation of Claim 21.2.4,

$$\begin{aligned} \frac{e^\delta}{(1+\delta)^{1+\delta}} &= \exp\Big(-\big((1+\delta)\ln(1+\delta) - \delta\big)\Big) \\ &= \exp\big(-f(\delta)\big) < \exp\big(-g(\delta)\big) < \exp\big(-h(\delta)\big). \end{aligned}$$

Substitituting the definition of g and h, we obtain

$$\Big(\frac{e^\delta}{(1+\delta)^{1+\delta}}\Big)^{\hat{\mu}} < \exp\big(-(1+\delta)\ln(1+\delta)\hat{\mu}/4\big) < \exp\big(-\delta\hat{\mu}/3\big).$$

21.2.4 Proof of inequality (c)

The argument is very similar to the proof of inequality (a). We will choose θ to be negative.

$$\begin{aligned}
\Pr[X \le (1-\delta)\check{\mu}] &= \Pr\left[\exp(\theta X) \ge \exp\left(\theta(1-\delta)\check{\mu}\right)\right] && \text{(by monotonicity and } \theta < 0) \\
&\le \frac{\mathrm{E}[\exp(\theta X)]}{\exp(\theta(1-\delta)\check{\mu})} && \text{(by Markov's inequality)} \\
&\le \frac{\prod_{i=1}^{n} \exp((e^{\theta}-1)\,\mathrm{E}[X_i])}{\exp(\theta(1-\delta)\check{\mu})} && \text{(by Claim 21.2.2)} \\
&= \exp\left((e^{\theta}-1)\sum_{i=1}^{n}\mathrm{E}[X_i] - \theta(1-\delta)\check{\mu}\right) \\
&\le \exp\left((e^{\theta}-1)\check{\mu} - \theta(1-\delta)\check{\mu}\right).
\end{aligned}$$

This last inequality holds because $\theta < 0$ so $e^{\theta} - 1 < 0$, and because $\check{\mu} \le \sum_{i=1}^{n} \mathrm{E}[X_i]$. Plugging in $\theta = \ln(1-\delta)$, which is negative, we obtain

$$\Pr[X \le (1-\delta)\check{\mu}] = \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\check{\mu}},$$

which is inequality (c).

21.2.5 Proof of inequality (d)

The statement and the proof are similar to Claim 21.2.3.

Claim 21.2.5. Then $(1-x)\ln(1-x) + x \ge \frac{1}{2}\cdot x^2$ for all $x \in [0,1)$.

Proof. Note that the LHS and RHS both vanish at $x = 0$. So the claim holds if the derivative of the LHS is at least the derivative of the RHS on the interval $[0,1)$. By simple calculus,

$$\frac{d}{dx}\big[(1-x)\ln(1-x) + x\big] = -\ln(1-x) \qquad \text{and} \qquad \frac{d}{dx}x^2/2 = x.$$

The linear approximation of $-\ln(1-x)$ at $x = 0$ is

$$x \cdot \tfrac{d}{dx}\big(-\ln(1-x)\big)\Big|_{x=0} = x \cdot \big(\tfrac{1}{1-x}\big)\Big|_{x=0} = x.$$

Furthermore, $-\ln(1-x)$ is convex on $[0,1)$ because its second derivative is $1/(1-x)^2 \ge 0$. Thus $-\ln(1-x) \ge x$ on $[0,1)$. □

Inequality (d) now follows straightforwardly because, using Claim 21.2.5,

$$\frac{e^{-\delta}}{(1-\delta)^{1-\delta}} = \exp\Big(-\big((1-\delta)\ln(1-\delta)+\delta\big)\Big) \le \exp\Big(-\delta^2/2\Big).$$

21.3 Proof of the Hoeffding Bound

The Hoeffding Bound was introduced in Section 9.3. We will prove the following slightly weaker result.

Theorem 21.3.1. Let $X_1, \ldots, X_n$ be independent random variables such that X_i always lies in the interval $[0,1]$. Define $X = \sum_{i=1}^{n} X_i$. Then

$$\Pr[\,|X - \mathrm{E}[X]| \ge t\,] \le 2\exp(-t^2/2n) \qquad \forall t \ge 0.$$

Simplifications. First of all, we will "center" the random variables, which cleans up the inequality by eliminating the expectation. Define $\hat{X}_i = X_i - \mathrm{E}[X_i]$ and $\hat{X} = \sum_{i=1}^n \hat{X}_i$. Note that[3] $\hat{X}_i \in [-1, 1]$. Our main argument is to prove that

$$\Pr\left[\hat{X} \geq t\right] \leq \exp(-t^2/2n). \tag{21.3.1}$$

The same argument also applies to $-\hat{X}$, so we get that

$$\Pr\left[-\hat{X} \geq t\right] = \Pr\left[\hat{X} \leq -t\right] \leq \exp(-t^2/2n).$$

Combining them with a union bound, we get

$$\Pr[\,|X - \mathrm{E}[X]| \geq t\,] = \Pr\left[|\hat{X}| \geq t\right] \leq \Pr\left[\hat{X} \geq t\right] + \Pr\left[-\hat{X} \geq t\right] \leq 2\exp(-t^2/2n).$$

This proves the theorem (with the weaker exponent).

Proof of (21.3.1). As in the proof of the Chernoff Bound (see (21.2.1)), we will use the fact that $\mathrm{E}[A \cdot B] = \mathrm{E}[A] \cdot \mathrm{E}[B]$ for any independent random variables A and B.

Key Idea #1: As in the proof of the Chernoff Bound, we will convert sums into products using the exponential function. Fix some parameter $\lambda > 0$ whose value we will choose later. Define

$$\begin{aligned} Y_i &= \exp(\lambda \hat{X}_i) \\ Y &= \exp(\lambda \hat{X}) = \exp\left(\lambda \sum_{i=1}^n \hat{X}_i\right) = \prod_{i=1}^n \exp(\lambda \hat{X}_i) = \prod_{i=1}^n Y_i. \end{aligned}$$

It is easy to check that, since $\{X_1, \ldots, X_n\}$ is mutually independent, so is $\left\{\hat{X}_1, \ldots, \hat{X}_n\right\}$ and $\{Y_1, \ldots, Y_n\}$. Therefore, as in (21.2.2),

$$\mathrm{E}[Y] = \prod_{i=1}^n \mathrm{E}[Y_i]. \tag{21.3.2}$$

So far this all seems quite good. We want to prove that $\hat{X}$ is small, which is equivalent to proving Y is small. Using (21.3.2), we can do this by showing that the $\mathrm{E}[Y_i]$ terms are small. Doing so involves an extremely useful tool.

Key Idea #2: The second main idea is a clever trick to bound terms of the form $\mathrm{E}[\exp(\lambda A)]$, where A is a mean-zero random variable. We discuss this idea in more detail in the next subsection. We will use[4] Lemma 21.3.2 to show

$$\mathrm{E}[Y_i] = \mathrm{E}\left[\exp(\lambda \hat{X}_i)\right] \leq \exp(\lambda^2/2). \tag{21.3.3}$$

Thus, combining this with (21.3.2),

$$\mathrm{E}[Y] \leq \prod_{i=1}^n \exp(\lambda^2/2) = \exp(\lambda^2 n/2). \tag{21.3.4}$$

[3] This step is where the argument is not careful enough to obtain the optimal exponent: $\hat{X}_i$ is actually supported on an interval of length 1, although our argument only assumes that it is supported on an interval of length 2.

[4] If we were more careful here and instead used Lemma 21.3.4, we could improve the constant in the exponent in (21.3.3) from 1/2 to 1/8. This would improve the constant in the exponent in (21.3.1) from 1/2 to 2.

Now we are ready to prove Hoeffding's inequality:

$$\begin{aligned}
\Pr\left[\hat{X} \geq t\right] &= \Pr\left[\exp(\lambda\hat{X}) \geq \exp\left(\lambda t\right)\right] \quad \text{(by monotonicity of } e^x\text{)} \\
&\leq \frac{\mathrm{E}\left[\exp(\lambda\hat{X})\right]}{\exp(\lambda t)} \quad \text{(by Markov's inequality)} \\
&= \mathrm{E}\left[Y\right] \cdot \exp(-\lambda t) \\
&\leq \exp(\lambda^2 n/2 - \lambda t) \quad \text{(by (21.3.4))} \\
&= \exp(-t^2/2n),
\end{aligned}$$

by optimizing to get $\lambda = t/n$. □

21.3.1 Hoeffding's Lemma

The second main idea of Hoeffding's inequality is the following claim.

Lemma 21.3.2 (Hoeffding's Lemma, symmetric version). Let A be a random variable such that $|A| \leq 1$ with probability 1 and $\mathrm{E}\left[A\right] = 0$. Then for any $\lambda > 0$, we have $\mathrm{E}\left[\exp(\lambda A)\right] \leq \exp(\lambda^2/2)$.

Intuitively, the expectation should be maximized by the random variable A that is uniform on $\{-1, +1\}$. In this case,

$$\mathrm{E}\left[\exp(\lambda A)\right] = \frac{1}{2}e^{\lambda} - \frac{1}{2}e^{-\lambda} \leq e^{\lambda^2/2}.$$

This inequality is a nice bound on the hyperbolic cosine function (Claim 21.3.3). The full proof of Lemma 21.3.2 basically reduces to the case of $A \in \{-1, 1\}$ using convexity of e^x.

Proof. Define $p = (1 + A)/2$ and $q = (1 - A)/2$. Observe that $p, q \geq 0$, $p + q = 1$, and $p - q = A$. By convexity,

$$\exp(\lambda A) = \exp\left(\lambda(p - q)\right) = \exp\left(\lambda p + (-\lambda)q\right) \leq p \cdot \exp(\lambda) + q \cdot \exp(-\lambda) = \frac{e^{\lambda} + e^{-\lambda}}{2} + \frac{A}{2}(e^{\lambda} - e^{-\lambda}).$$

Thus,

$$\mathrm{E}\left[\exp(\lambda A)\right] \leq \mathrm{E}\left[\frac{e^{\lambda} + e^{-\lambda}}{2} + \frac{A}{2}(e^{\lambda} - e^{-\lambda})\right] = \frac{e^{\lambda} + e^{-\lambda}}{2},$$

since $\mathrm{E}\left[A\right] = 0$. This last quantity is bounded by the following technical claim. □

Claim 21.3.3 (Approximation of Cosh). For any real x, we have $(e^x + e^{-x})/2 \leq \exp(x^2/2)$.

References: (Alon and Spencer, 2000, Lemma A.1.5).

Proof. First observe that the product of all the even numbers at most $2n$ does not exceed the product of all numbers at most $2n$. In symbols,

$$2^n(n!) = \prod_{i=1}^{n}(2i) \leq \prod_{i=1}^{2n} i = (2n)!$$

Now to bound $(e^x + e^{-x})/2$, we write it as a Taylor series and observe that the odd terms cancel.

$$\frac{e^x + e^{-x}}{2} = \sum_{n\geq 0}\frac{x^n}{n!} + \sum_{n\geq 0}\frac{(-x)^n}{n!} = \sum_{n\geq 0}\frac{x^{2n}}{(2n)!} \leq \sum_{n\geq 0}\frac{x^{2n}}{2^n(n!)} = \sum_{n\geq 0}\frac{(x^2/2)^n}{n!} = \exp(x^2/2)$$

□

A common scenario is that A is mean-zero, but lies in an "asymmetric" interval $[a,b]$, where $a<0<b$. A tighter version of Lemma 21.3.2 can be derived for this scenario, although its proof is more complicated.

Lemma 21.3.4 (Hoeffding's Lemma, asymmetric version). Let A be a random variable such that $A \in [a,b]$ with probability 1 and $\mathrm{E}[A]=0$. Then for any $\lambda>0$, we have $\mathrm{E}[\exp(\lambda A)] \le \exp\left(\lambda^2(b-a)^2/8\right)$.

References: (McDiarmid, 1998, Lemma 2.6), (Cesa-Bianchi and Lugosi, 2006, Lemma A.1), (Shalev-Shwartz and Ben-David, 2014, Lemma B.7), (Alon and Spencer, 2000, Theorem A.1.17), Wikipedia.

The proof uses ideas similar to the proof of Lemma 21.3.2, except we cannot use Claim 21.3.3 and must instead use an ad-hoc calculus argument.

21.3.2 Generalizations

Theorem 21.3.5 (Hoeffding's General Inequality). Let $X_1,\dots,X_n$ be independent random variables where $X_i \in [a_i,b_i]$. Let $X=\sum_{i=1}^n X_i$. Then

$$\begin{aligned}
\text{Left tail:} \qquad \Pr\left[\sum_{i=1}^n X_i \le \mathrm{E}[X]-s\right] &\le \exp\left(-2\frac{s^2}{\sum_{i=1}^n (b_i-a_i)^2}\right)\\
\text{Right tail:} \qquad \Pr\left[\sum_{i=1}^n X_i \ge \mathrm{E}[X]+s\right] &\le \exp\left(-2\frac{s^2}{\sum_{i=1}^n (b_i-a_i)^2}\right)\\
\text{Combined tails:} \qquad \Pr\left[\left|\sum_{i=1}^n X_i - \mathrm{E}[X]\right| \ge s\right] &\le 2\exp\left(-2\frac{s^2}{\sum_{i=1}^n (b_i-a_i)^2}\right).
\end{aligned}$$

In particular, for any desired $q\in(0,1)$, setting $s=\sqrt{\ln(2/q)\sum_{i=1}^n (b_i-a_i)^2/2}$ gives

$$\Pr\left[\left|\textstyle\sum_i X_i - \mathrm{E}[X]\right| \ge s\right] \le q.$$

References: (McDiarmid, 1998, Theorem 2.5), (Vershynin, 2018, Theorem 2.2.6), (Boucheron et al., 2012, Theorem 2.8), (Roch, 2020, Theorem 2.40), (Dubhashi and Panconesi, 2009, Problem 1.9), (Grimmett and Stirzaker, 2001, Theorem 12.2.3), Wikipedia.

21.4 Exercises

Exercise 21.1 **Massart's Lemma.** Let $P\subset\mathbb{R}^d$ be a set of points satisfying $\|p\|\le 1$ for all $p\in P$. Let $m=|P|$. Let $\xi_1,\dots,\xi_d\in\{-1,+1\}$ be uniform and independent random signs.

Part I. Prove that, for all $s\ge 0$,

$$\Pr\left[\max_{p\in P}\sum_{i=1}^d p_i\xi_i \ge s\right] \le m\cdot\exp(-s^2/2).$$

Part II. Prove that

$$\mathrm{E}\left[\max_{p\in P}\sum_{i=1}^d p_i\xi_i\right] \le 2\sqrt{\ln m}+O(1).$$

References: (Shalev-Shwartz and Ben-David, 2014, Lemma 26.8).

Chapter 22

More Applications of Concentration

22.1 Balls and Bins: The Heaviest Bin

Let us return to the topic of balls and bins, which was first introduced in Chapter 7. Consider throwing n balls into n bins, uniformly and independently. Let B_i be the number of balls in bin i. In Section 7.6 we used an ad hoc argument to analyze the load on the heaviest bin, namely $\max_i B_i$. Now we will give an alternative analysis using the Chernoff bound.

Theorem 22.1.1. Assume $n \geq 3$. Define $k = 16 \ln n / \ln \ln n$. With probability at least $1 - 1/n$, the heaviest bin has at most k balls. That is,

$$\Pr\left[\max_i B_i \leq k\right] \;\geq\; 1 - 1/n.$$

This theorem is optimal up to constant factors. It is known that $\max_i B_i \geq \ln n / \ln \ln n$ with probability at least $1 - 1/n$. See, e.g., (Mitzenmacher and Upfal, 2005, Lemma 5.12).

Proof. The first step is to give tail bounds on B_1. We decompose B_1 into indicator random variables as

$$B_1 \;=\; X_1 + X_2 + \cdots + X_n,$$

where X_j is 1 if the j^{th} ball lands in the first bin. For each $j \in [n]$ we have $\mathrm{E}[X_j] = 1/n$, so $\mathrm{E}[B_1] = \sum_j \mathrm{E}[X_j] = 1$. What is the probability that this first bin has more than k balls?

We will analyze this event using inequality (b) of the Chernoff bound. Specifically, letting $X = B_1$ and $\delta = k - 1$, we obtain

$$\begin{aligned}
\Pr[B_1 \geq k] &= \Pr[B_1 \geq k \,\mathrm{E}[B_1]] && \text{(since } \mathrm{E}[B_1] = 1\text{)} \\
&\leq \exp\big(-\mathrm{E}[B_1] \cdot k \ln(k)/4\big) && \text{(by the Chernoff bound)} \\
&= \exp\big(-k \ln(k)/4\big) && \text{(since } \mathrm{E}[B_1] = 1\text{).}
\end{aligned}$$

These inequalities require that $\delta \geq 1$, i.e., $k \geq 2$.

To proceed any further, we must understand $k \ln k$. A quick calculation gives

$$\begin{aligned} k \cdot \ln k &= 16\frac{\ln n}{\ln \ln n} \cdot \ln\left(\frac{16 \ln n}{\ln \ln n}\right) \\ &> 16\frac{\ln n}{\ln \ln n} \cdot \ln\left(\frac{\ln n}{\sqrt{\ln n}}\right) \qquad (\text{since } \sqrt{x} > \ln x \text{ for all } x > 0) \\ &= 16\frac{\ln n}{\ln \ln n} \cdot \ln\left(\sqrt{\ln n}\right) \\ &= 8\frac{\ln n}{\ln \ln n} \cdot \ln \ln n \;=\; 8 \ln n. \end{aligned}$$

Plugging that in,

$$\Pr[B_1 \geq k] \;\leq\; \exp\left(-k \ln(k)/4\right) \;\leq\; \exp\left(-2 \ln n\right) \;=\; n^{-2}. \tag{22.1.1}$$

So bin 1 is unlikely to have more than α balls.

Here we have analyzed bin 1, but the bins are all equivalent so the same analysis actually holds for all bins. The remainder of the argument is just the union bound.

$$\begin{aligned} \Pr[\,\textit{any} \text{ bin has} \geq k \text{ balls}\,] &= \Pr[B_1 \geq k \;\vee\; B_2 \geq k \;\vee\; \cdots \;\vee\; B_n \geq k] \\ &\leq \sum_{i=1}^{n} \Pr[B_i \geq k] && (\text{Fact A.3.8}) \\ &\leq \sum_{i=1}^{n} \frac{1}{n^2} && (\text{by (22.1.1)}) \\ &= \frac{1}{n}. \end{aligned}$$

Thus, with probability at least $1 - 1/n$, all bins have at most k balls. □

22.2 Congestion Minimization

One of the classically important areas in algorithm design and combinatorial optimization is *network flows*. A central problem in that area is the maximum flow problem. We now look at a generalization of this problem.

An instance of the problem consists of a directed graph $G = (V, A)$ and a sequence $(s_1, t_1), \ldots, (s_k, t_k)$ of pairs of vertices. Let $n = |V|$. (It is not crucial that the graph be directed; the problem is equally interesting in undirected graphs. However in network flow problems it is often more convenient to look at directed graphs. Feel free to think about whatever variant you find easier.)

A natural question to ask is: do there exist paths P_i from s_i to t_i for every i such that these paths share no arcs? This is called the **edge-disjoint paths** problem. Quite remarkably, it is NP-hard even in the case $k = 2$, assuming the graph is directed. For undirected graphs, it is polynomial time solvable if k is a fixed constant, but NP-hard if k is a sufficiently large function of n.

We will look at a variant of this problem called the **congestion minimization** problem. The idea is to allow each arc to be used in multiple paths, but not too many. The number of paths using a given arc is the "congestion" of that arc. We say that a solution has congestion C if it is a collection of paths P_i from s_i to t_i, where each arc is contained in at most C of the paths. The problem is to find the

minimum value of C such that there is a solution of congestion C. This problem is still NP-hard, since determining if $C = 1$ is the edge-disjoint paths problem.

We will look at the congestion minimization problem from the point of view of approximation algorithms. Let OPT be the minimum congestion of any solution. We would like to give an algorithm which can produce a solution with congestion at most $\alpha \cdot OPT$ for some $\alpha \geq 1$. This factor α is the called the approximation factor of the algorithm.

Theorem 22.2.1. There is an algorithm for the congestion minimization problem with approximation factor $O(\log n / \log \log n)$.

To design such an algorithm we will use linear programming. We write down an integer program (IP) which captures the problem exactly, relax that to a linear program (LP), then design a method for "rounding" solutions of the LP into solutions for the IP.

The Integer Program. Writing an IP formulation of an optimization problem is usually quite simple. That is indeed true for the congestion minimization problem. However, we will use an IP which you might find rather odd: our IP will have exponentially many variables. This will simplify our explanation of the rounding.

Let $\mathcal{P}_i$ be the set of *all* paths in G from s_i to t_i. (Note that $|\mathcal{P}_i|$ may be exponential in n.) For every path $P \in \mathcal{P}_i$, we create a variable x^i_P. This variable will take values only in $\{0, 1\}$, and setting it to 1 corresponds to including the path P in our solution.

The integer program is as follows

$$\begin{array}{lll}
\min & C & \\
\text{s.t.} & \displaystyle\sum_{P \in \mathcal{P}_i} x^i_P = 1 & \forall i = 1, \ldots, k \\
& \displaystyle\sum_i \sum_{P \in \mathcal{P}_i \text{ with } a \in P} x^i_P \leq C & \forall a \in A \\
& x^i_P \in \{0, 1\} & \forall i = 1, \ldots, k \text{ and } P \in \mathcal{P}_i
\end{array}$$

The last constraint says that we must decide for every path whether or not to include it in the solution. The first constraint says that the solution must choose exactly one path between each pair s_i and t_i. The second constraint ensures that the number of paths using each arc is at most C. The optimization objective is to find the minimum value of C such that a solution exists.

Every solution to the IP corresponds to a solution for the congestion minimization problem with congestion C, and vice-versa. Thus the optimum value of the IP is OPT, which we previously defined to be the minimum congestion of any solution to the original problem.

This IP is NP-hard to solve, so we relax it into a linear program, by replacing the integrality constraints with non-negativity constraints. It turns out to be convenient also to add the constraint $C \geq 1$. The resulting linear program is:

$$\begin{array}{lll}
\min & C & \\
\text{s.t.} & \displaystyle\sum_{P \in \mathcal{P}_i} x^i_P = 1 & \forall i = 1, \ldots, k \\
& \displaystyle\sum_i \sum_{P \in \mathcal{P}_i \text{ with } a \in P} x^i_P \leq C & \forall a \in A \\
& C \geq 1 & \\
& x^i_P \geq 0 & \forall i = 1, \ldots, k \text{ and } P \in \mathcal{P}_i
\end{array}$$

Remarkably, this LP can be solved in time polynomial in n (the number of nodes of G), even though its

number of variables could be exponential in n. The details are best left for a course on optimization[1]. Our algorithm will solve this LP and obtain a solution where the number of non-zero x_P^i variables is only polynomial in n. Let C^* be the optimum value of the LP.

Claim 22.2.2. $C^* \leq OPT$.

Proof. The LP was obtained from the IP by *removing* constraints. Therefore any feasible solution for the IP is also feasible for the LP. In particular, the optimal solution for the IP is feasible for the LP. So the LP has a solution with objective value equal to OPT. □

The Rounding. Our algorithm will solve the LP and most likely obtain a "fractional" solution — a solution with some non-integral variables, which is therefore not feasible for the IP. The next step of the algorithm is to "round" that fractional solution into a solution which is is feasible for the IP. In doing so, the congestion might increase, but we will ensure that it does not increase too much.

The technique we will use is called **randomized rounding**. For each each $i = 1, \ldots, k$, we randomly choose *exactly one path* P_i by setting $P_i = P$ with probability x_P^i. (The LP's constraints ensure that these are indeed probabilities: they are non-negative and sum up to 1.) The algorithm outputs the chosen paths $P_1, \ldots, P_k$.

Analysis. All that remains is to analyze the congestion of these paths. Let Y_i^a be the indicator random variable that is 1 if $a \in P_i$ and 0 otherwise. Let $Y^a = \sum_i Y_i^a$ be the congestion on arc a. The expected value of Y^a is easy to analyze:

$$\mathrm{E}[\,Y^a\,] \;=\; \sum_i \mathrm{E}[\,Y_i^a\,] \;=\; \sum_i \sum_{P \in \mathcal{P}_i \text{ with } a \in P} x_P^i \;\leq\; C^*,$$

where the inequality comes from the LP's second constraint. (Recall we assume that the fractional solution is optimal for the LP, and therefore $C = C^*$.)

The Chernoff bound says, if X is a sum of independent random variables each of which take values in $[0, 1]$, and μ is an *upper bound* on $\mathrm{E}[\,X\,]$, then

$$\Pr[\,X \geq (1+\delta)\mu\,] \;\leq\; \exp\Big(-\mu \cdot \big((1+\delta)\ln(1+\delta) - \delta\big)\Big) \qquad \forall \delta > 0.$$

We apply this to Y^a, taking $\mu = C^*$ and $\alpha = 1 + \delta = 6 \log n / \log\log n$. Following our balls-and-bins argument from last time,

$$\begin{aligned} \Pr[\,Y^a \geq \alpha C^*\,] \quad &\leq \quad \exp\Big(-C^*\big(\alpha \ln \alpha - (\alpha - 1)\big)\Big) \\ &\leq \quad \exp\big(-\alpha \ln \alpha + \alpha - 1\big) \\ &\leq \quad \exp\big(-(6/2)\ln n\big) \;=\; 1/n^3. \end{aligned}$$

We now use a union bound to analyze the probability of *any* arc having congestion greater than αC^*.

$$\Pr[\,\text{any } a \text{ has } Y^a \geq \alpha C^*\,] \;\leq\; \sum_{a \in A} \Pr[\,Y^a \geq \alpha C^*\,] \;\leq\; \sum_{a \in A} 1/n^3 \;\leq\; 1/n,$$

[1] There are two ways to do this. The first way is to solve the dual LP using the ellipsoid method. This can be done in poly(n) time even though it can have exponentially many constraints. The second way is to find a "compact formulation" of the LP which uses fewer variables, much like the usual LP that you may have seen for the ordinary maximum flow problem.

since the graph has at most n^2 arcs. So, with probability at least $1 - 1/n$, the algorithm produces a solution for which every arc has congestion at most αC^*, which is at most $\alpha \cdot OPT$ by Claim 22.2.2. So our algorithm has approximation factor $\alpha = O(\log n / \log\log n)$.

Further Remarks. The *rounding* algorithm that we presented is actually optimal: there are graphs for which $OPT/C^* = \Omega(\log n / \log\log n)$. Consequently, every rounding algorithm which converts a fractional solution of LP to an integral solution of IP must necessarily incur an increase of $\Omega(\log n / \log\log n)$ in the congestion.

That statement does not rule out the possibility that there is a better algorithm which behaves completely differently (i.e., one which does not use IP or LP at all). But sadly it turns out that there is no better algorithm (for the case of directed graphs). It is known that every efficient algorithm must have approximation factor $\alpha = \Omega(\log n / \log\log n)$, assuming a reasonable complexity theoretic conjecture (NP $\not\subseteq$ BPTIME($n^{O(\log\log n)}$)). So the algorithm that we presented is optimal, up to constant factors.

22.3 Error-correcting codes

Suppose a person (the sender) wants to transmit a message to another person (the receiver). This message might somehow be corrupted during transmission. The goal is for the receiver to determine the original message, even if these corruptions occur.

There are various ways to model corruptions, such as stochastic noise (or Shannon model), stochastic erasures, stochastic deletions, adversarial erasures, etc.

We will consider the ***adversarial noise model*** (or Hamming model), in which it is assumed that at most t symbols of the message are be corrupted. The values and locations of those corruptions can be arbitrary.

The sender and receiver will first agree upon a ***code***, which we define to be a set of binary strings of length n. These strings are called ***codewords***. In symbols, the ***Hamming cube*** is $\{0,1\}^n$, and a code is a set

$$\text{Code:} \qquad \mathcal{C} \subseteq \{0,1\}^n .$$

Since the codewords are binary strings, $\mathcal{C}$ is sometimes called a ***binary code***. The set $\mathcal{C}$ can be ordered, for example by viewing the codewords as binary numbers. Thus we may write $\mathcal{C} = \{C_1, \ldots, C_M\}$ for some value M. Note that $|\mathcal{C}| \le 2^n$, so

$$2^n \;\ge\; M \tag{22.3.1}$$

The communication scheme is as follows. Suppose that there are M different messages that the sender could send. We can think of a message simply as being an integer $m \in [M]$. (For example, the messages might be arbitrary binary strings of length $\lg M$.) Each message m can be identified with a codeword in a canonical way; for example, message m corresponds to codeword C_m.

Transmitting m directly would take $\lceil \lg M \rceil$ bits. Instead, the sender transmits C_m, which takes n bits. This requires more bits to be transmitted:

$$n \;\ge\; \lceil \lg M \rceil \tag{22.3.2}$$

because of (22.3.1). This value n is called the ***block length*** of the code. The efficiency of the code is measured by how tight the inequality (22.3.2) is. Quantatively, the ***rate*** of $\mathcal{C}$ is defined to be

$$R \;=\; \frac{\lg|\mathcal{C}|}{n} \;=\; \frac{\lg M}{n}.$$

The main question: if the codeword is corrupted during transmission, can the receiver still determine the original message m?

22.3.1 Decoding

Let $\oplus$ denote Binary Xor. The ***Hamming distance*** between $x, y \in \{0,1\}^n$ is

$$\Delta(x,y) \;=\; \sum_{i=1}^{n}(x_i \oplus y_i) \;=\; \sum_{i=1}^{n}|x_i - y_i| \;=\; \|x-y\|_1 . \tag{22.3.3}$$

Observe that Δ satisfies the triangle inequality

$$\Delta(x,z) \;\le\; \Delta(x,y) + \Delta(y,z) \tag{22.3.4}$$

because $\|\cdot\|_1$ is a norm. (This also follows by applying Fact A.2.4 component-wise.)

The ***minimum distance*** (or simply ***distance***) of $\mathcal{C}$ is

$$d \;=\; \min_{x \neq y \in \mathcal{C}} \Delta(x,y).$$

The ***relative distance*** is d/n.

Claim 22.3.1 (Unique decoding). Let $x \in \mathcal{C}$ be arbitrary. Produce y from x by flipping at most $\lfloor \frac{d-1}{2} \rfloor$ bits. Then, given y, one can determine x.

Proof. By construction of y, the Hamming distance $\Delta(x,y)$ satisfies

$$\Delta(x,y) \;\le\; \left\lfloor \frac{d-1}{2} \right\rfloor .$$

However, for any other codeword $z \in \mathcal{C}$, where $z \neq x$, we have

$$\begin{aligned}
\Delta(z,y) \;&\ge\; \Delta(x,z) - \Delta(x,y) \qquad \text{(by the triangle inequality (22.3.4))} \\
&\ge\; d - \left\lfloor \frac{d-1}{2} \right\rfloor \\
&\ge\; d/2 + 1/2 \;>\; \left\lfloor \frac{d-1}{2} \right\rfloor .
\end{aligned}$$

We have shown that y's Hamming distance to x is smaller than to any other z. □

22.3.2 A random code construction

Theorem 22.3.2. For any $\epsilon > 0$, there exists a binary code of relative distance $1/2 - \epsilon$ and rate $\Omega(\epsilon^2)$.

References: MIT course notes.

Remark 22.3.3.

- You might wonder why we aim for relative distance nearly $1/2$ instead of relative distance nearly 1. This is because binary codes with relative distance nearly 1 have rate going to zero with n. Specifically, if $\delta > 1/2 + \epsilon$ then $M \le 1/\epsilon$ so the rate is $R \le \lg(1/\epsilon)/n$, which vanishes as $n \to \infty$. See (Guruswami et al., 2019, Theorem 4.4.1).

- The Gilbert-Varshamov greedy code construction also achieves relative distance $1/2 - \epsilon$ and rate $\Omega(\epsilon^2)$. See (Guruswami et al., 2019, Theorem 4.2.1 and Proposition 3.3.5).
- For codes with relative distance $1/2 - \epsilon$, the rate of $\Omega(\epsilon^2)$ is nearly optimal. The so-called Linear Programming bound implies a $O(\epsilon^2 \log(1/\epsilon))$ upper bound on the rate. See (Guruswami et al., 2019, Section 8.2). For the special case of "balanced codes", the $O(\epsilon^2 \log(1/\epsilon))$ bound follows from an algebraic argument due to Alon.

Claim 22.3.4. Let y, z be independent and uniformly random points in $\{0,1\}^n$. Then

$$\Pr\left[\,\Delta(y,z)/n < 1/2 - \epsilon\,\right] \;\leq\; \exp(-\epsilon^2 n).$$

Proof. The Hamming distance $\Delta(y,z)$ can be decomposed into indicators as

$$\Delta(y,z) \;=\; \sum_{i=1}^{n} X_i$$

where $X_i = y_i \oplus z_i$. Note that X_i is an unbiased Bernoulli RV — it is 0 or 1 with probability 1/2. (This actually follows from Corollary A.3.25.) The expected distance is $\mu = \mathrm{E}\left[\,\Delta(y,z)\,\right] = n/2$. Then, by the Chernoff bound,

$$\begin{aligned}
\Pr\left[\,\Delta(y,z)/n < 1/2 - \epsilon\,\right] &= \Pr\left[\,\Delta(y,z) < (1-2\epsilon)\mu\,\right] \\
&< \exp\left(-(2\epsilon)^2\mu/2\right) \\
&= \exp(-\epsilon^2 n).
\end{aligned}$$

□

Proof of Theorem 22.3.2. Let $M = \exp(\epsilon^2 n/2)$ and pick $C_1, ..., C_M$ in $\{0,1\}^n$ independently and uniformly at random. The previous claim show that, for any pair of these points, they are unlikely to have small Hamming distance. This is extended to all pairs of points by the union bound.

$$\begin{aligned}
\Pr\left[\,\exists i,j \text{ s.t. } \Delta(C_i,C_j)/n < 1/2 - \epsilon\,\right] &\leq \binom{M}{2} \cdot \exp(-\epsilon^2 n) && \text{(union bound)} \\
&< \frac{M^2}{2} \cdot \exp(-\epsilon^2 n) && \text{(by Claim 22.3.4)} \\
&= \frac{1}{2} \cdot \Big(\underbrace{\exp(\epsilon^2 n/2)}_{=M}\Big)^2 \cdot \exp(-\epsilon^2 n) \\
&= 1/2.
\end{aligned}$$

We define the code $\mathcal{C} = \{C_1, \ldots, C_M\}$. It definitely has rate

$$\frac{\lg M}{n} \;=\; \frac{\epsilon^2 n/2}{\ln(2) n} \;=\; \Omega(\epsilon^2).$$

Furthermore, with probability at least 1/2, we have $\Delta(C_i, C_j)/n \geq 1/2 - \epsilon$ for all $i \neq j$, which implies that $\mathcal{C}$ has relative distance at least $1/2 - \epsilon$.

Since this construction has probability at least 1/2 of producing a code with the desired rate and distance, it follows that such a code must exist. This style of argument is an example of the probabilistic method. □

Research Questions. There are several open questions relating to this topic. See (Guruswami et al., 2019, Section 8.3).

- Is the $\Omega(\epsilon^2)$ rate optimal?
- Can one explicitly describe a code achieving rate $\Omega(\epsilon^2)$?
- Is there a code with rate $\Omega(\epsilon^2)$ that can be efficiently decoded?

Recent research. Very recently, startling progress has been made on the second and third questions. A breakthrough of Ta-Shma in STOC 2017 has shown that there are explicit codes with distance $1/2-\epsilon$ and rate $\Omega(\epsilon^{2+\alpha})$ where $\alpha \to 0$ as $\epsilon \to 0$. Moreover, in STOC 2021, Jeronimo et al. showed that these codes can be decoded in nearly-linear time.

Exercises

Exercise 22.1. For this exercise, define the ***Hamming ball***, for any $x \in \{0,1\}$ and $d \geq 0$, to be

$$B(x,d) \;=\; \{\, y \in \{0,1\}^n \,:\, \Delta(x,y) \leq d \,\}\,.$$

Here $\Delta(x,y)$ is the Hamming distance, defined in (22.3.3). Use the Chernoff bound to prove an upper bound on the volume $|B(x,d)|$.

Exercise 22.2. The Hoeffding bound shows that a sum of n independent indicator RVs deviates from its expectation by $\sqrt{n}$ with constant probability. This question (roughly) shows that the norm of a random vector deviates from its expectation by $O(1)$ with constant probability!

Let X be uniformly distributed on $\{0,1\}^n$.

Part I. Prove that $\mathrm{E}\left[\, \|X\|_2^2 \,\right] = n/2$.

Part II. Note that $\mathrm{E}\left[\, \|X\|_2 \,\right] \neq \sqrt{\mathrm{E}\left[\, \|X\|_2^2 \,\right]}$ in general. Prove that $\mathrm{E}\left[\, \|X\|_2 \,\right] \leq \sqrt{\mathrm{E}\left[\, \|X\|_2^2 \,\right]} = \sqrt{n/2}$.

Part III. Prove that, for all $t \geq 0$,

$$\Pr\left[\, \|X\|_2 \geq \sqrt{n/2} + t \,\right] \;\leq\; \exp(-4t^2).$$

Exercise 22.3 Sparse strategies for zero-sum games.

Background. In a zero-sum game[2] with payoff matrix $A \in \mathbb{R}^{m \times n}$, there are two players that we will name Alice and Bob. Alice has to choose one action in $[m]$ while, simultaneously, Bob has to pick one action in $[n]$. Both Alice and Bob know that if Alice picks action $i \in [m]$ and Bob picks $j \in [n]$, then Alice will have to pay $A_{i,j}$ dollars to Bob. In such a game, Alice wants a strategy that minimizes how

[2]For more details about zero-sum games, check out this video or Wikipedia.

much money she has to pay Bob, while Bob wants a strategy that maximizes how much money he receives from Alice.

From everyday experience (e.g., rock paper scissors), we know that it often make sense to use a randomized strategy. Let

$$\Delta_n \;=\; \left\{\, p \in [0,1]^n \;:\; \textstyle\sum_{j=1}^n p_j = 1 \,\right\}$$

be the set of all probability distributions over these n choices. Suppose that Alice chooses $r \in \Delta_m$ then takes action $i \in [m]$ with probability r_i. Similarly, Bob chooses $p \in \Delta_n$ then takes action $j \in [n]$ with probability p_j. With these randomized strategies, one can easily verify that the expected amount of money Alice pays Bob is $r^\mathsf{T} A p$.

The central result of zero-sum games, due to von Neumann, proves that there are randomized strategies that are simultaneously optimal for both Alice and Bob! Formally, there exist $p^* \in \Delta_n$ and $r^* \in \Delta_m$ such that

$$\min_{r\in\Delta}\max_{p\in\Delta} r^\mathsf{T} A p \;=\; \max_{p\in\Delta}\min_{r\in\Delta} r^\mathsf{T} A p \;=\; (r^*)^\mathsf{T} A p^*. \tag{22.3.5}$$

Although one can efficiently compute these optimal strategies using linear programming, they may give positive probability to almost all actions, so they are *dense*.

In this exercise, we will see that there are nearly-optimal strategies that are *sparse*. More concretely, let us assume that $A_{i,j} \in [0,1]$ for all i, j. Fix some parameter $\epsilon > 0$. We will show Bob has a strategy giving positive probability to $O(\log(m)/\epsilon^2)$ actions. By symmetry, the same is true for Alice.

Henceforth, fix $p \in \Delta_n$. Let $X_1, \ldots, X_k$ be i.i.d. random variables satisfying

$$\Pr[\, X_s = j \,] \;=\; p_j \qquad \forall j \in [n].$$

This is called a categorical distribution.

Part I. Show that for any $i \in [m]$ we have

$$\Pr\left[\sum_{j=1}^n A_{i,j} p_j - \frac{1}{k}\sum_{s=1}^k A_{i,X_s} > \epsilon \right] \le e^{-2\epsilon^2 k}.$$

Part II. Define $q \in [0,1]^n$ by $q_j = k_j/k$, where $k_j = |\{\, s \in [k] \;:\; X_s = j \,\}|$. Show that

$$\Pr\left[\sum_{j=1}^n A_{i,j} p_j - \sum_{j=1}^n A_{i,j} q_j > \epsilon \text{ for some } i \in [m] \right] \le m e^{-2\epsilon^2 k}.$$

Part III. Show that if $k = \left\lceil \frac{\ln(m+1)}{2\epsilon^2} \right\rceil$, then there is $q \in \Delta_n$ with at most k non-zero entries such that

$$\sum_{j=1}^n A_{i,j} p_j - \sum_{j=1}^n A_{i,j} q_j \le \epsilon \qquad \forall i \in [m].$$

Part IV. Let μ^* be the optimal value given by (22.3.5). Conclude that there is a randomized strategy $q \in \Delta_n$ for Bob that puts positive probability on at most $O(\frac{\log m}{\epsilon^2})$ actions and satisfies $\min_{r\in\Delta_m} r^\mathsf{T} A q \ge \mu^* - \epsilon$. That is, in expectation the optimal strategy for Bob gains at most ϵ more money than the strategy given by q.

Chapter 23

Dimensionality Reduction

Dimensionality reduction is the process of mapping a high dimensional dataset to a lower dimensional space, while preserving much of the important structure. In statistics and machine learning, this often refers to the process of finding a few directions in which a high dimensional random vector has maximum variance. Principal component analysis is a standard technique for that purpose.

In this chapter we consider a different sort of dimensionality reduction. Given a set of high-dimensional points, the goal is to find lower-dimensional points whose *pairwise distances* approximately match the original points. We present a technique, known as the ***random projection method*** or ***Johnson-Lindenstrauss method***, for solving this problem.

In the previous chapters, our main tool has been the Chernoff bound. Here we will not directly use the Chernoff bound, but the main proof uses very similar ideas.

23.1 Intuition

Let us begin with some three-dimensional examples. We will measure lengths using the Euclidean norm. Our notation for the length of a vector v is $\|v\| = \sqrt{\sum_i v_i^2}$.

In our first example, the dimension can be reduced while exactly preserving pairwise distances.

Example 23.1.1. Consider the points

$$x_1 = \begin{pmatrix} 0.3237 \\ -2.1870 \\ -0.3354 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 0.1327 \\ -3.5215 \\ -0.7627 \end{pmatrix} \qquad x_3 = \begin{pmatrix} -1.6022 \\ -2.2986 \\ -1.4660 \end{pmatrix}$$

They have pairwise distances

$$\|x_1 - x_2\| \approx 1.4142 \qquad \|x_1 - x_3\| \approx 2.2361 \qquad \|x_2 - x_3\| \approx 2.2361$$

We now define a linear map

$$L = \begin{pmatrix} -0.7056 & -0.4820 & -0.5193 \\ 0.5146 & -0.8525 & 0.0920 \end{pmatrix}$$

Then define two-dimensional points y_i by $y_i = Lx_i$. This yields the points

$$y_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad y_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad y_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

They have pairwise distances

$$\|y_1 - y_2\| \approx \|x_1 - x_2\| \approx 1.4142 \quad \|y_1 - y_3\| \approx \|x_1 - x_3\| \approx 2.2361 \quad \|y_2 - y_3\| \approx \|x_2 - x_3\| \approx 2.2361.$$

This is not so impressive because the points x live in a two-dimensional subspace, so we have simply performed an orthogonal change of coordinates to obtain their two-dimensional representation.

In our second example, it is not possible to reduce the dimension while exactly preserving pairwise distances.

Example 23.1.2. Define the points

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad x_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad x_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

They have pairwise distances

$$\|x_i - x_j\| \;=\; \sqrt{2} \qquad \forall i, j.$$

Thus, they form a regular 3-simplex (or tetrahedron) in $\mathbb{R}^3$. It can be shown that there is no representation for the 3-simplex in $\mathbb{R}^1$ or $\mathbb{R}^2$. So there is no way to reduce the dimension of these points while exactly preserving the pairwise distances.

Nevertheless, in the next section we will show that we can reduce the dimension of *any* high-dimensional point set if we are allowed to *approximate* the pairwise distance.

23.2 The Johnson-Lindenstrauss Theorem

Suppose we have n points $x_1, \ldots, x_n \in \mathbb{R}^d$. We would like to find n points $y_1, \ldots, y_n \in \mathbb{R}^t$, where $t \ll d$, such that the lengths and pairwise distances of the y vectors are approximately the same as for the x vectors. We will show that this can be accomplished while taking t to be surprisingly small.

Each vector y_i will be obtained from x_i by a linear map. Let R be a random $t \times d$ ***Gaussian matrix***. This means that each entry of R is an independent standard Gaussian random variable. Gaussians are defined in Appendix B.4.3.

Theorem 23.2.1 (Johnson-Lindenstrauss 1984). Let $x_1, \ldots, x_n \in \mathbb{R}^d$ be arbitrary. Pick any $\epsilon = (0, 1)$. There exists $t = O(\log(n)/\epsilon^2)$ such that, if R is a $t \times d$ Gaussian matrix, and $y_i = Rx_i/\sqrt{t}$ then, with probability at least $1 - 1/n$,

$$\begin{array}{rcccll} (1-\epsilon)\,\|x_j\| & \leq & \|y_j\| & \leq & (1+\epsilon)\,\|x_j\| & \forall j \\ (1-\epsilon)\,\|x_j - x_{j'}\| & \leq & \|y_j - y_{j'}\| & \leq & (1+\epsilon)\,\|x_j - x_{j'}\| & \forall j, j'. \end{array} \tag{23.2.1}$$

References: (Shalev-Shwartz and Ben-David, 2014, Section 23.2), (Dubhashi and Panconesi, 2009, Section 2.5), (Vershynin, 2018, Theorem 5.3.1), (Boucheron et al., 2012, Theorem 2.13), (Roch, 2020, Section 2.4.6), (Blum et al., 2018, Section 2.7), (Wainwright, 2019, Example 2.12).

Observations.

- The linear map R is *oblivious*: it does not depend on the points $x_1, \ldots, x_n$ at all.

- A standard Gaussian is also called a $N(0,1)$ random variable, meaning that has zero mean and variable 1. Instead of defining $y_i = Rx_i/\sqrt{t}$, we could equivalently let the entries of R have distribution $N(0, 1/t)$ and define $y_i = Rx_i$. (This follows from Fact B.4.6 with $d = 1$ and $\sigma_1 = 1/\sqrt{t}$.)

Whereas principal component analysis is only useful when the original data points $\{x_1, \ldots, x_n\}$ are nearly low dimensional, this theorem requires *absolutely no assumption* on the original data. Also, note that the final data points $\{y_1, \ldots, y_n\}$ have no dependence on d. The original data could live in an arbitrarily high dimension. (Although one can always assume $d \leq n$ since $x_1, \ldots, x_n$ lie in their linear span, which is a Euclidean space of dimension at most n.)

23.2.1 Overview of proof

To prove the theorem, let us define the random linear map that is used to construct the points y_j. The parameter t will be chosen appropriately below. Let R is a $t \times d$ matrix whose entries are independently drawn from $N(0,1)$, i.e., Gaussians with mean 0 and variance 1. The point y_j in Theorem 23.2.1 is simply by multiplication with R and then rescaling

$$y_j \leftarrow \frac{R \cdot x_j}{\sqrt{t}}.$$

In pseudocode, we can write the algorithm as follows.

Algorithm 23.1 The Johnson-Lindenstrauss algorithm. Each input vector x_i is in $\mathbb{R}^d$.

```
function JL(vector x_1, ..., x_n, int d, float ε)
    Let t ← O(ln(n)/ε^2)
    Let R be a random Gaussian matrix of size t × d
    for i = 1, ..., n
        Let y_i ← R · x_i/√t
    return y_1, ..., y_n
end function
```

Our proof of Theorem 23.2.1 will show that this algorithm produces vectors $y_1, \ldots, y_n$ satisfying (23.2.1) with probability at least $1 - 1/n$. (There is a caveat: the definition of t involves a constant that we have not specified.)

The key to proving Theorem 23.2.1 is to prove the following lemma.

Lemma 23.2.2 (Distributional JL). Let R be a $t \times d$ Gaussian matrix. Let $\delta \in (0, 1]$ be arbitrary. Let $t = 8\ln(2/\delta)/\epsilon^2$. Then for all vectors $v \in \mathbb{R}^d$ with $\|v\| = 1$,

$$\Pr\left[\frac{\|Rv\|}{\sqrt{t}} \notin (1-\epsilon, 1+\epsilon) \right] \leq \delta. \tag{23.2.2}$$

Proof of Theorem 23.2.1. Set $\delta = 1/n^3$. Consider the set of vectors

$$W = \{ x_i : i = 1, \ldots, n \} \cup \{ x_i - x_j : i \neq j \}.$$

There are at most n^2 vectors in W.

For any non-zero $w \in W$, we may apply the DJL lemma to $v = w/\|w\|$. The event that w's norm is *not* adequately preserved is

$$\mathcal{E}_w \;=\; \left\{ \frac{\|Rw\|}{\sqrt{t}} \notin [1-\epsilon, 1+\epsilon] \cdot \|w\| \right\} \;=\; \left\{ \frac{\|Rv\|}{\sqrt{t}} \notin [1-\epsilon, 1+\epsilon] \right\},$$

because $\|Rw\| = \|Rv \cdot \|w\|\| = \|w\| \cdot \|Rv\|$. The condition (23.2.1) holds if and only if no event $\mathcal{E}_w$ occurs. Thus

$$\begin{aligned}
\Pr[\,\text{condition (23.2.1) fails to hold}\,] \;&=\; \Pr\left[\bigcup_{w \in W} \mathcal{E}_w\right] \\
&\le\; \sum_{w \in W} \Pr[\,\mathcal{E}_w\,] \qquad \text{(the union bound)} \\
&\le\; |W| \cdot \delta \qquad \text{(by Lemma 23.2.2)} \\
&\le\; 1/n. \quad \square
\end{aligned}$$

23.2.2 Random Dot Products

The idea of taking a random dot product is crucial to our discussion. Consider an arbitrary vector $v \in \mathbb{R}^d$. Let $g \in \mathbb{R}^d$ be a random vector where $\mathrm{E}[\,g_i\,] = 0$ for each i. We are interested in the analyzing the dot product $g^\mathsf{T} v$. This quantity may not seem so interesting at first glance because, by linearity of expectation,

$$\mathrm{E}\left[\,g^\mathsf{T} v\,\right] \;=\; \sum_{i=1}^{d} \mathrm{E}[\,g_i\,]\, v_i \;=\; 0. \tag{23.2.3}$$

So let us also introduce the notion of ***variance***, which is defined in Definition B.4.3. We require the following fact.

Fact B.4.5. Let $g_1, \ldots, g_d$ be independent random variables with finite variance. Let $\sigma_1, \ldots, \sigma_d \in \mathbb{R}$ be arbitrary. Then $\mathrm{Var}\left[\sum_{i=1}^d \sigma_i g_i\right] = \sum_{i=1}^d \sigma_i^2 \,\mathrm{Var}[\,g_i\,]$.

We will additionally assume that $\mathrm{Var}[\,g_i\,] = 1$ for each i. Then we have

$$\begin{aligned}
\mathrm{E}\left[\,(g^\mathsf{T} v)^2\,\right] \;&=\; \mathrm{Var}\left[\,g^\mathsf{T} v\,\right] && \text{(by (23.2.3) and (B.4.2))} \\
&=\; \mathrm{Var}\left[\sum_{i=1}^{d} v_i g_i\right] \;=\; \sum_{i=1}^{d} v_i^2 \,\mathrm{Var}[\,g_i\,] && \text{(by Fact B.4.5)} \\
&=\; \|v\|^2 .
\end{aligned}$$

This shows that $(g^\mathsf{T} v)^2$ is an unbiased estimator for $\|v\|^2$. Thus, we can estimate $\|v\|^2$ by estimating $\mathrm{E}\left[\,(g^\mathsf{T} v)^2\,\right]$. If the estimate is sufficiently good, we can then take the square root and estimate $\|v\|$.

Which random vector? So far, the discussion is valid for *any* independent RVs g_i with mean 0 and variance 1. For example, we could take g_i to be uniform on $\{-1, 1\}$.

Our subsequent analysis will be made easier by choosing g_i to have the Gaussian distribution $N(0,1)$. This is due to a key property: *the sum of Gaussians is also Gaussian.*

Fact B.4.6. Let $g_1, \ldots, g_d$ be independent random variables where g_i has distribution $N(0,1)$. Then, for any scalars $\sigma_1, \ldots, \sigma_d$, the sum $\sum_{i=1}^d \sigma_i g_i$ has distribution $N(0, \sum_{i=1}^d \sigma_i^2) = N(0, \|\sigma\|_2^2)$.

Consequently, our random dot product $g^\mathsf{T} v$ has the distribution

$$\text{Distribution of } g^\mathsf{T} v: \quad N(0, \sum_{i=1}^d v_i^2) = N(0, \|v\|^2). \tag{23.2.4}$$

Improved estimates by averaging. A common idea in randomized algorithms is to average multiple independent estimates in order to get a higher quality estimate. (See, e.g., Section 12.2.2.) Perhaps we can get a high-quality estimate of $\|v\|^2$ by picking several random vectors g, then averaging the values of $(g^\mathsf{T} v)^2$?

This idea is exactly what the Johnson-Lindenstrauss lemma does. Our matrix R is a $t \times d$ Gaussian matrix. Let r_i refer to the i^{th} row of R. Then $r_1, \ldots, r_t$ are independent Gaussian vectors, so each component of Rv is a random dot product:

$$(Rv)_i \;=\; r_i^\mathsf{T} v \qquad \forall i \in [t].$$

The squared norm of $Rv/\sqrt{t}$ is then

$$\left\| \frac{Rv}{\sqrt{t}} \right\|^2 \;=\; \sum_{i=1}^t \frac{(Rv)_i^2}{t} \;=\; \sum_{i=1}^t \frac{(r_i^\mathsf{T} v)^2}{t}, \tag{23.2.5}$$

which is the average of t independent random dot products.

23.2.3 Proof of Lemma 23.2.2

Let us rewrite (23.2.2) as follows.

$$\begin{aligned}
\Pr\left[\frac{\|Rv\|}{\sqrt{t}} \notin (1-\epsilon, 1+\epsilon) \right] &= \Pr\left[\|Rv\|^2 \notin \left((1-\epsilon)^2, (1+\epsilon)^2\right) \cdot t \right] && \text{(squaring both sides)} \\
&= \Pr\left[\sum_{i=1}^t (r_i^\mathsf{T} v)^2 \notin \left((1-\epsilon)^2, (1+\epsilon)^2\right) \cdot t \right] && \text{(by (23.2.5))} \\
&\le \Pr\left[\sum_{i=1}^t (r_i^\mathsf{T} v)^2 \notin (1-\epsilon, 1+\epsilon) \cdot t \right]
\end{aligned}$$

Here we have used the simple bounds

$$(1-\epsilon)^2 \le 1-\epsilon \quad \text{and} \quad (1+\epsilon)^2 \ge 1+\epsilon.$$

We have shown in (23.2.4) that $r_i^\mathsf{T} v$ has the distribution $N(0, \|v\|^2)$, which is $N(0,1)$ since we assume that v is a unit vector. It now turns out that $\sum_{i=1}^t (r_i^\mathsf{T} v)^2$ has a well-known distribution too. It is the sum of the *squares* of t independent standard normal random variables, which is called the chi-squared distribution with parameter t.

Question 23.2.3. What is $\mathrm{E}\left[\|Rv\|^2 \right]$?

Answer.

We have

$$\mathrm{E}\left[\,\|Rv\|^2\,\right] \;=\; \sum_{i=1}^{t} \mathrm{E}\left[\,(r_i^{\mathsf{T}} v)^2\,\right] \;=\; \sum_{i=1}^{t} \mathrm{Var}\left[\,r_i^{\mathsf{T}} v\,\right] \;=\; t,$$

since $r_i^{\mathsf{T}} v$ is $N(0,1)$.

To summarize, our desired inequality is

$$\Pr\left[\;\underbrace{\sum_{i=1}^{t} (r_i^{\mathsf{T}} v)^2}_{X} \notin (1-\epsilon,\,1+\epsilon)\cdot \underbrace{t}_{\mathrm{E}[\,X\,]}\;\right] \;\le\; \delta \tag{23.2.6}$$

where X is a chi-squared RV with parameter t. Fortunately, tail bounds for these RVs are known.

Lemma 23.2.4 (Tail bound for chi-squared). Let X have the chi-squared distribution with parameter t. Then

$$\Pr\left[\,|X-t| \ge \epsilon t\,\right] \;\le\; 2\exp(-\epsilon^2 t/8) \qquad \forall \epsilon \in (0,1).$$

References: (Shalev-Shwartz and Ben-David, 2014, Lemma B.12), (Wainwright, 2019, Example 2.11 and Example 2.28).

This lemma easily allows us to prove (23.2.6), which proves (23.2.2).

$$\Pr\left[\,X \notin (1-\epsilon,\,1+\epsilon)\cdot t\,\right] \;=\; \Pr\left[\,|X-t| \ge \epsilon t\,\right] \;\le\; 2\exp(-\epsilon^2 t/8) \;=\; \delta, \tag{23.2.7}$$

since $t = 8\ln(2/\delta)/\epsilon^2$.

23.2.4 Example of a chi-squared tail bound

To illustrate one strategy for proving Lemma 23.2.4, let us prove the following bound on the right tail. This even yields the better constant in the exponent of $1/2$ rather than $1/8$.

Claim 23.2.5. Let X have the chi-squared distribution with parameter t. Then $\Pr\left[\,X \ge t(1+\epsilon)^2\,\right] \le \exp(-t\epsilon^2/2)$.

Proof. Our proof will follow the Chernoff bound strategy. For any $\alpha \in \mathbb{R}$ and $\theta \ge 0$, we have

$$\Pr\left[\,X \ge \alpha\,\right] \;=\; \Pr\left[\,e^{\theta X} \ge e^{\theta\alpha}\,\right] \;\le\; e^{-\theta\alpha}\,\mathrm{E}\left[\,e^{\theta X}\,\right]. \tag{23.2.8}$$

The quantity $\mathrm{E}\left[\,e^{\theta X}\,\right]$ is called the moment generating function, and for many standard distributions it has a known closed form. We now cheat by referring to Wikipedia, where we find that the moment generating function for the chi squared distribution is $\mathrm{E}\left[\,e^{\theta X}\,\right] = (1-2\theta)^{-t/2}$ (for $\theta < 1/2$), so

$$\Pr\left[\,X > \alpha\,\right] \;\le\; e^{-\theta\alpha}(1-2\theta)^{-t/2} \qquad (\text{if } \theta \in [0,1/2)).$$

The next step is to plug in an appropriate choice of θ. We set $\theta = (1 - t/\alpha)/2$, giving

$$\begin{aligned}
\Pr[\,X \geq \alpha\,] &\leq e^{(t-\alpha)/2}(t/\alpha)^{-t/2} \\
&= \exp\left(\frac{t}{2}\Big(1-(1+\epsilon)^2\Big) - \frac{t}{2}\ln\left(\frac{1}{(1+\epsilon)^2}\right)\right) \qquad \text{(setting } \alpha = t(1+\epsilon)^2\text{)} \\
&= \exp\Big(-t\big(\epsilon + \epsilon^2/2 - \ln(1+\epsilon)\big)\Big) \\
&\leq \exp\Big(-t\big(\epsilon + \epsilon^2/2 - \epsilon\big)\Big) \qquad \text{(using } \ln(1+x) \leq x\text{; see Exercise B.2)} \\
&\leq \exp(-t\epsilon^2/2). \qquad \square
\end{aligned}$$

Question 23.2.6. Improve the bound $\log(1+\epsilon) \leq \epsilon$ to $\log(1+\epsilon) \leq \epsilon - \epsilon^2/4$ for $\epsilon \in [0,1]$.

Answer.

It follows from Fact B.3.5 by replacing $z \to x/2$ and integration.

23.2.5 Broader context

In this section we have switched from the world of discrete probability to continuous probability. This is to make our lives easier. The same theorem would be true if we picked the coordinates of r_i to be uniform in $\{+1,-1\}$ rather than Gaussian. But the analysis of the $\{+1,-1\}$ case is trickier, and most proofs analyze that case by showing that its failure probability is not much worse than in the Gaussian case. So the Gaussian case is really the central problem.

Second of all, you might be wondering where the name ***random projection method*** comes from. Earlier versions of the Johnson-Lindenstrauss theorem used a slightly different linear map. Specifically, they used the map Rv where R^TR is a *projection* onto a uniformly random subspace of dimension t. (Recall that an orthogonal projection matrix is any symmetric, positive semidefinite matrix whose eigenvalues are either 0 or 1.) One advantage of that setup is its symmetry: one can argue that the failure probability in Lemma 23.2.2 would be the same if one instead chose a *fixed* subspace of dimension t and a *random* unit vector v. The latter problem can be analyzed by choosing the subspace to be the most convenient one of all: the span of the first t vectors in the standard basis.

So how is our map $R/\sqrt{t}$ different? It is almost a projection, but not quite. If we chose R to be a matrix of independent Gaussians, it turns out that the range of R^TR/t is indeed a uniformly random subspace, but its eigenvalues are not necessarily in $\{0,1\}$. If we had insisted that the random vectors r_i that we choose were *orthonormal*, then we would have obtained a projection matrix. We could explicitly orthonormalize them by the Gram-Schmidt method, but fortunately that turns out to be unnecessary: the Johnson-Lindenstrauss theorem is true, even if we ignore orthonormality of the r_i's.

Our linear map $R/\sqrt{t}$ turns out to be a bit more convenient in some algorithmic applications, because we avoid the awkward Gram-Schmidt step.

Optimality. Recently there has been much progress on understanding the optimality of these results. The DJL lemma is actually optimal, up to constant factors.

Theorem 23.2.7 (Jayram-Woodruff 2013, Kane-Meka-Nelson 2011). Any f satisfying the DJL lemma must satisfy $t = \Omega(\log(1/\delta)/\epsilon^2)$.

But, this does not necessarily imply that Theorem 23.2.1 is optimal; perhaps the theorem can be proven without using the DJL lemma. Alon proved the following lower bound. (See Theorem 9.3 of this paper.)

Theorem 23.2.8 (Alon). Let $x_1, \ldots, x_{n+1} \in \mathbb{R}^n$ be the vertices of a simplex, i.e., $\|x_i - x_j\| = 1$ for all $i \neq j$. If $y_1, \ldots, y_{n+1} \in \mathbb{R}^t$ satisfy (23.2.1), then $t = \Omega(\frac{\log(n)}{\epsilon^2 \log(1/\epsilon)})$.

This shows that Theorem 23.2.1 is almost optimal, up to the factor $\log(1/\epsilon)$ in the denominator. Actually, for this particular set of points (the vertices of a simplex), Theorem 23.2.1 is *not* optimal and Alon's bound is the right one. However, there is a different point set showing that Theorem 23.2.1 is in fact optimal.

Theorem 23.2.9 (Larsen-Nelson FOCS 2017). There exist points $x_1, \ldots, x_n \in \mathbb{R}^d$ such that the following is true. Consider any map $L : \mathbb{R}^d \to \mathbb{R}^t$, let $y_j = L(x_j)$, and suppose that (23.2.1) is satisfied. Then $t = \Omega(\log(n)/\epsilon^2)$.

Other norms and metrics. The Johnson-Lindenstrauss lemma very strongly depends on properties of the Euclidean norm. For other norms, this remarkable dimensionality reduction is not necessarily possible. For example, for the ℓ_1 norm $\|x\|_1 := \sum_i |x_i|$, it is known that any map into $\mathbb{R}^d$ that preserves pairwise ℓ_1-distances between n points up to a factor $c \geq 1$ must have $d = \Omega(n^{1/c^2})$. If $c = 1 + \epsilon$, then there are upper bounds of $d = O(n \log n/\epsilon^2)$ and $d = O(n/\epsilon^2)$.

References: Talagrand Proc. AMS 1990, Brinkman-Charikar FOCS 2003, Lee-Naor 2004, Newman-Rabinovich SODA 12.

For more on this subject, see the survey of Indyk and Matousek or the tutorial of Indyk.

23.3 Fast Johnson-Lindenstrauss

In the previous section we saw the Johnson-Lindenstrauss theorem on dimensionality reduction. Suppose we have n points in $\mathbb{R}^d$ and we map them to $\mathbb{R}^t$, where $t = O(\log(n)/\epsilon^2)$, simply by applying a $t \times d$ matrix of Gaussians (scaled appropriately). Then all lengths and pairwise distances are preserved up to a factor $1 + \epsilon$, with high probability. This is a very useful tool in algorithm design.

Let's consider the efficiency of such a mapping. Directly applying matrix-vector multiplication, the time to map a single point from $\mathbb{R}^d$ to $\mathbb{R}^t$ is $O(td)$. There has been much work on trying to make this faster.

One direction of research considered using a slightly *sparse* matrix instead of a dense matrix of Gaussians. The state of the art (Kane-Nelson JACM 2014) allows the matrix to have only an ϵ fraction of non-zero entries, so the time to map a single point becomes $O(\epsilon td)$. Their result is optimal to with a factor of $O(\log(1/\epsilon))$.

Today we discuss a different line of research. Instead of using sparse matrices, we will use *structured* matrices, for which multiplication can be done faster than the naive algorithm. (As a trivial example, consider the matrix of all ones. It is dense, but multiplying by it is very easy.) Such matrices are called "Fast Johnson-Lindenstrauss Transforms", and they have been used extensively in the algorithms, compressed sensing, machine learning and numerical linear algebra communities.

The first result of this type is stated below. Is is of the Distributional JL type, in that it preserves the length of any fixed vector with good probability.

Theorem 23.3.1 (Ailon-Chazelle STOC 2006). There is a $t \times d$ random matrix that satisfies the DJL lemma and for which matrix-vector multiplication takes time $O(d \log d + t^3)$.

To understand if this runtime is good, consider the scenario where we are applying dimensionality reduction to n data points. The Ailon-Chazelle result is only interesting for certain parameters n, d and t. First, suppose n is very small, say $d = n$, so $t \approx \log n = \log d$. Then the original JL theorem takes time roughly $O(td) = O(d \log d)$ per vector, which is the same as Ailon-Chazelle. Second, suppose n is

very large, say $n = 2^{\sqrt{d}}$, so $t \approx \log n = d^{1/2}$. Then the original JL theorem takes time $O(td) = O(d^{3/2})$ per vector and Ailon-Chazelle also takes time $O(d \log d + t^3) = O(d^{3/2})$. The Ailon-Chazelle result is interesting in the intermediate range, where $d \ll n \ll 2^{d^{1/2}}$

We will prove the following theorem, which is a slightly simplified form of the Ailon-Chazelle result.

Theorem 23.3.2. There is a random matrix R of size $t \times d$ with $t = O(\log(d/\delta)^2 \log(1/\delta)/\epsilon^2)$ such that, for each $x \in \mathbb{R}^d$,

$$\|Rx\| \in [1-\epsilon, 1+\epsilon] \cdot \|x\|$$

holds with probability at least $1 - \delta$. Matrix-vector multiplication with R takes time $O(d \log d + t)$.

23.3.1 A Simple Start: Super-Sparse Sampling

Let's start with simple idea: given a vector x, we will sample it using a very sparse sampling matrix. This matrix is denoted S and it has size $t \times d$. Each row of S has a single non-zero entry of value $\sqrt{d/t}$ in a uniformly random location.

For any vector $x \in \mathbb{R}^d$, we have

$$\mathrm{E}\left[\,(Sx)_i^2\,\right] \;=\; \sum_{j=1}^{d} \underbrace{\Pr\left[\,i^{\text{th}} \text{ row's nonzero entry is in location } j\,\right]}_{=1/d} \cdot (\sqrt{d/t})^2 \cdot x_j^2 \;=\; (1/t) \cdot \|x\|_2^2$$

$$\implies \quad \mathrm{E}\left[\,\|Sx\|^2\,\right] \;=\; \mathrm{E}\left[\,\sum_{i=1}^{t} (Sx)_i^2\,\right] \;=\; \|x\|_2^2 \tag{23.3.1}$$

So this works well in expectation, even if we have $t = 1$. Unfortunately the variance can be terrible.

Example 23.3.3 (Sparse case). Suppose x has just one non-zero coordinate, say $x = e_1$. The expected number of non-zero coordinates in Sx is t/d. So Sx is very likely to be 0 unless $t = \Omega(d)$. So there is very little dimensionality reduction.

Example 23.3.4 (Dense case). Suppose $x = [1, 1, \cdots, 1]/\sqrt{d}$, which has $\|x\|_2 = 1$. Then $Sx = \sqrt{1/t}x$, so $\|Sx\| = \sqrt{1/t} \cdot \sqrt{t} = 1$. Thus the norm of x is preserved exactly, regardless of the value of t.

Let us try to extract some general principles from these examples.

- *Sparse case.* If one coordinate is much larger than the others in absolute value (as in Example 23.3.3) then this approach seems unlikely to work.
- *Dense case.* If all coordinates have similar absolute value (as in Example 23.3.4, then the approach seems promising. In this case, we also say that x is "uncorrelated with the standard basis".

To make these principles mathematically precise, we turn to the language of norms. Let us recall the following standard inequality.

Fact B.2.1. For all $x \in \mathbb{R}^d$,

$$\|x\|_p \;\le\; \|x\|_r \;\le\; d^{1/r - 1/p} \cdot \|x\|_p \qquad \forall 1 \le r \le p \le \infty.$$

In particular, the most useful cases are

$$\begin{aligned} \|x\|_1 \;&\ge\; \|x\|_2 \;\ge\; \|x\|_\infty \\ \frac{1}{\sqrt{d}} \|x\|_1 \;&\le\; \|x\|_2 \;\le\; \sqrt{d} \cdot \|x\|_\infty . \end{aligned}$$

Let us now consider the norms in the preceding examples.

- *Sparse case.* In Example 23.3.3, we have $\|x\|_2 = \|x\|_\infty = 1$.
- *Dense case.* In Example 23.3.3, we have $\|x\|_2 = 1$ but $\|x\|_\infty = 1/\sqrt{d}$.

Inspired by the examples, we use the ratio $\|x\|_\infty / \|x\|_2$ as a measure of sparsity of the vector x.

$$\begin{aligned} x \text{ is sparse:} \quad & \frac{\|x\|_\infty}{\|x\|_2} \gtrapprox 1 \\ x \text{ is dense:} \quad & \frac{\|x\|_\infty}{\|x\|_2} \lessapprox \frac{1}{\sqrt{d}}. \end{aligned}$$

By Fact B.2.1, these are the most extreme values of this ratio.

The following claim formally proves that super-sparse sampling works when x is dense. Define

$$\lambda \;=\; \sqrt{\frac{2\ln(4d/\delta)}{d}}. \tag{23.3.2}$$

Claim 23.3.5. Let y be a fixed vector in $\mathbb{R}^d$ with $\|y\|_2 = 1$ and $\|y\|_\infty \le \lambda$. Let S be a $t \times d$ super-sparse sampling matrix with $t = 2\ln(4d/\delta)^2 \ln(4/\delta)/\epsilon^2$. Then

$$\Pr\Big[\, \|Sy\|_2^2 \notin (1-\epsilon, 1+\epsilon) \,\Big] \;\le\; \delta/2.$$

Proof. See Exercise 23.3. □

23.3.2 Idea: Rotating the basis

Stating the problematic scenario in these terms leads to a useful idea. We said that super-sparse sampling will require large t when $\|x\|_\infty / \|x\|_2 \approx 1$. Now we recall an important property of the Euclidean norm: it is invariant under rotations and reflections. Formally, $\|Mx\|_2 = \|x\|_2$ for any orthogonal matrix M. Another way to say this is that $\|\cdot\|_2$ is invariant under an orthogonal change of basis, and indeed $\|\cdot\|_2$ can be defined without reference to any basis (using an inner product). On the other hand, the infinity norm $\|\cdot\|_\infty$ is *heavily* dependent on the choice of basis. For example,

$$\begin{aligned} u = (1,0,0,\ldots,0) \in \mathbb{R}^d \qquad & \text{has} \qquad \|u\|_2 = \|u\|_\infty = 1 \\ v = (1,1,\ldots,1)/\sqrt{d} \in \mathbb{R}^d \qquad & \text{has} \qquad \|v\|_2 = 1 \text{ and } \|v\|_\infty = 1/\sqrt{d} \end{aligned}$$

even though v is a rotation of u. Intuitively, the quantity $\|x\|_\infty / \|x\|_2$ is telling us how well the vector x "aligns" with the standard basis.

In order for our super-sparse sampling to work we need x to be dense, which means that x is not aligned with the standard basis. However we have no control over x because our theorem needs to work for all x. The key idea is to control our basis instead. Why not choose a random basis that is unlikely to align with the given vector x? Indeed, that is basically what is accomplished by the dense matrix of Gaussians in the previous section.

The only issue with the dense matrix of Gaussians is that matrix-vector multiplications are too slow. So let's think if there is a quicker way to randomly rotate the basis. Whenever I think of vectors that "disagree" with the standard basis, the first object that comes to mind is a Hadamard matrix.

Definition 23.3.6. A ***Hadamard matrix*** is a $d \times d$ real matrix H that is orthogonal (meaning $H^\mathsf{T} H = I$) and has all entries in $\left\{\pm 1/\sqrt{d}\right\}$.

It is not a priori clear that Hadamard matrices exist, and indeed it is not fully known for what values of d they exist[1]. In the case $d = 2$, we have the Hadamard matrix

$$H_2 \;=\; \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} / \sqrt{2},$$

which amounts to a 45-degree rotation of the standard basis. In fact, we can build on this recursively to obtain a Hadamard matrix whenever d is a power of two.

$$H_d \;=\; \begin{pmatrix} H_{d/2} & H_{d/2} \\ H_{d/2} & -H_{d/2} \end{pmatrix} / \sqrt{2}. \tag{23.3.3}$$

This is called a Walsh-Hadamard matrix, and it has many nice properties.

First of all H_d is symmetric, so $H_d^\mathsf{T} = H_d$. To see that H_d is indeed a Hadamard matrix, we must make two observations. First, induction shows that every entry of H_d is $\pm(1/\sqrt{2})^{\log_2 d} = \pm 1/\sqrt{d}$. Second, we have

$$H_d^\mathsf{T} H_d \;=\; H_d H_d \;=\; \begin{pmatrix} H_{d/2} & H_{d/2} \\ H_{d/2} & -H_{d/2} \end{pmatrix} \cdot \begin{pmatrix} H_{d/2} & H_{d/2} \\ H_{d/2} & -H_{d/2} \end{pmatrix} /2 \;=\; \begin{pmatrix} 2H_{d/2}H_{d/2} & 0 \\ 0 & 2H_{d/2}H_{d/2} \end{pmatrix} /2 \;=\; I,$$

so H_d is orthogonal.

Another nice property of H_d is that matrix-vector multiplication can be performed in $O(d \log d)$ time. This follows from an easy divide-and-conquer algorithm.

23.3.3 Randomized Hadamard Matrix

Using H_d to change our basis will not guarantee that x is dense. It is obvious that, for *any* fixed orthogonal matrix M, there exist vectors x for which $\|Mx\|_\infty / \|Mx\|_2 = 1$: simply let x be any row of M. This is why the randomization is necessary.

Instead, we must pick a *random* change of basis M and argue that

$$\text{each } x \in \mathbb{R}^d \text{ satisfies} \qquad \left(\frac{\|Mx\|_\infty}{\|Mx\|_2} \approx \frac{1}{\sqrt{d}} \quad \text{with high probability} \right). \tag{23.3.4}$$

So far our Hadamard matrix $H = H_d$ has no randomness. How can we "randomize" it? Well, in the recursive definition (23.3.3) we quite arbitrarily put the minus sign in the lower-right quadrant. The construction works just as well if we put the minus sign in any quadrant, so this suggests that we should be able to randomize the construction using random signs.

We will introduce random signs in a slightly more convenient way. Let D be a diagonal matrix whose i^{th} diagonal entry is a random sign $\xi_i \in \{-1, 1\}$, and these are independent. Our random Hadamard matrix is the product $M = HD$. This is indeed a Hadamard matrix: its entries are still $\pm 1/\sqrt{d}$, and M is orthogonal because $M^\mathsf{T} M = DH^\mathsf{T} HD = D^2 = I$.

The following claim shows that, with this randomized Hadamard matrix M, every vector x satisfies (23.3.4).

[1] One open question is that a $d \times d$ Hadamard matrix exists whenever d is a multiple of 4.

Claim 23.3.7. Let $x \in \mathbb{R}^d$ be non-zero. Let $y = HDx$, where HD is the random Hadamard matrix. Then

$$\Pr_D \left[\frac{\|y\|_\infty}{\|y\|_2} \geq \underbrace{\sqrt{\frac{2\ln(4d/\delta)}{d}}}_{\lambda} \right] \leq \delta/2.$$

Proof. It suffices to consider the case $\|x\|_2 = 1$, because $\|HDx\|_\infty / \|HDx\|_2$ is invariant under rescaling x. Note that $\|HDx\|_2 = \|x\|_2 = 1$ as HD is orthogonal, so it suffices to show that all coordinates of HDx are likely small. We will follow our usual approach of showing that each coordinate is very likely to be small, then union bounding over all coordinates.

Consider y_1, the first coordinate of $y = HDx$. It is obtained by multiplying each coordinate of x by a random sign, then taking the dot-product with the first row of H. So

$$y_1 = \sum_j \xi_j H_{1,j} x_j, \tag{23.3.5}$$

Note that the terms of this sum are independent random variables. It is tempting to apply the Chernoff bound to analyze y_1, but the Chernoff bound that we have used so far is only valid for sums of non-negative random variables. Instead, we will use the generalized form of the Hoeffding bound, Theorem 21.3.5.

We apply that theorem with $X_j = \xi_j H_{1,j} x_j$. Since $\xi_j \in \{-1, +1\}$, we have $X_j = \pm H_{1,j} x_j$. So X_j lies in the interval $[a_j, b_j]$ where $-a_j = b_j = |H_{1,j} x_j|$. Note that $\mathrm{E}[X_j] = 0$ and

$$\sum_{j=1}^d (b_j - a_j)^2 = 4\sum_{j=1}^d H_{1,j}^2 x_j^2 = 4\sum_{j=1}^d (1/d) x_j^2 = 4\|x\|_2^2/d = 4/d.$$

Defining $q = \delta/2d$, the quantity s in Theorem 21.3.5 becomes $s = \sqrt{2\ln(4d/\delta)/d}$, which equals the value of λ defined in (23.3.2). Consequently, (23.3.5) and Theorem 21.3.5 yield

$$\Pr[|y_1| \geq \lambda] = \Pr\left[\left| \textstyle\sum_j X_j - \underbrace{\mathrm{E}\left[\sum_j X_j\right]}_{=0} \right| \geq s \right] \leq \delta/2d.$$

A union bound over all coordinates of y shows that

$$\Pr[\|y\|_\infty \geq \lambda] \leq \sum_{j=1}^d \Pr[|y_j| \geq \lambda] \leq d \cdot (\delta/2d) = \delta/2. \qquad \square$$

23.3.4 Putting it all together

Earlier we said that super-sparse sampling should work as long as x is dense holds. We have just shown x is likely to be dense after applying the randomized Hadamard matrix; more precisely, (23.3.4) holds with $M = HD$. The final step is to apply the super-sparse sampling matrix S to Mx. To summarize, the overall linear map is $R = SHD$ where S is a super-sparse sampling matrix of size $t \times d$, H is a $d \times d$ Hadamard matrix, and D is a diagonal matrix of random signs.

Proof of Theorem 23.3.2. Fix any vector x with $\|x\|_2 = 1$. Construct the random matrix $R = SHD$ as explained above and let $y = HDx$. Define the events

$$\begin{aligned} \mathcal{E}_1 &= \{\, \|y\|_\infty \geq \lambda \,\} \\ \mathcal{E}_2 &= \{\, \|Sy\|_2 \notin (1-\epsilon, 1+\epsilon) \,\} \end{aligned}$$

We have the following bounds.

$$\begin{aligned} \Pr[\,\mathcal{E}_1\,] &\leq \delta/2 \qquad \text{(by Claim 23.3.7)} \\ \Pr\left[\,\mathcal{E}_2 \mid \overline{\mathcal{E}_1}\,\right] &\leq \delta/2 \qquad \text{(by Claim 23.3.5).} \end{aligned}$$

Thus, by Exercise A.4, we obtain

$$\Pr[\,\|Rx\| \notin (1-\epsilon, 1+\epsilon)\,] = \Pr[\,\mathcal{E}_2\,] \leq \Pr[\,\mathcal{E}_1\,] + \Pr\left[\,\mathcal{E}_2 \mid \overline{\mathcal{E}_1}\,\right] \leq \delta.$$

Finally, let us check that matrix-vector multiplication with $R = SHD$ is efficient.

- Multiplication by D is trivial and takes $O(d)$ time.
- Multiplication by H requires $O(d \log d)$ time as explained above.
- Multiplication by S is trivial and takes $O(t)$ time.

So the total time is $O(d \log d + t)$. □

23.4 Subspace Embeddings

The Johnson-Lindenstrauss Theorem shows how to reduce the dimension while preserving distances between a finite set of points. Now we will consider how to reduce the dimension for the *infinitely many* points in a subspace.

Let U be a subspace of $\mathbb{R}^d$ with dimension k. We would like to find a matrix R of size $t \times d$ such that $t \approx k$ and

$$\frac{\|Rx\|}{\|x\|} \in [1-\epsilon, 1+\epsilon] \qquad \forall x \in U,\, x \neq 0. \tag{23.4.1}$$

Question 23.4.1. Is this trivial?

Answer.

Yes. Let $t = k$ and let the rows of R be an orthonormal basis of U. This satisfies (23.4.1) with $\epsilon = 0$.

What makes the problem non-trivial is that we want R to be *oblivious to* U, just like Johnson-Lindestrauss is oblivious to the points whose distance are preserved.

Theorem 23.4.2. For any integer k and any $\epsilon \in (0,1)$, let $t = O(k \log(1/\epsilon)/\epsilon^2)$. Let R be a $t \times d$ matrix where the entries are independent with distribution $N(0, 1/t)$. Then, for any subspace U of dimension k, we have

$$\Pr\left[\, \frac{\|Rx\|}{\|x\|} \in [1-\epsilon, 1+\epsilon] \;\; \forall x \in U,\, x \neq 0 \,\right] \geq 1 - 2e^{-k}.$$

To prove the JL Theorem, we proved the DJL Lemma for individual vectors x, then extended that to a finite collection of vectors by taking a union bound. Now Theorem 23.4.2 requires that we approximate the norm for all vectors in U simultaneously. Since U is infinitely large, a naive union bound will not suffice.

A simplifying reduction. Instead of considering an arbitrary subspace U, it suffices to consider the case that $U = \text{span}\{e_1, \ldots, e_k\}$. The reason stems from the following fact.

Fact 23.4.3. Let R be a $t \times d$ random Gaussian matrix and let A be any $d \times d$ orthogonal matrix. Then RA has the same distribution as R.

References: See (Vershynin, 2018, Proposition 3.3.2).

Imagine letting the first k columns of A be an orthonormal basis of U, then extending that to an orthonormal basis of $\mathbb{R}^d$. Then (23.4.1) is equivalent to

$$\frac{\|RAx\|}{\|x\|} \in [1-\epsilon, 1+\epsilon] \qquad \forall x \in \text{span}\{e_1, \ldots, e_k\}. \tag{23.4.2}$$

Since RA and R have the same distribution (by Fact 23.4.3), our goal is equivalent to showing

$$\Pr\left[\frac{\|Rx\|}{\|x\|} \in [1-\epsilon, 1+\epsilon] \;\; \forall x \in \mathbb{R}^k,\, x \neq 0\right] \geq 1 - e^{-k} \tag{23.4.3}$$

where R has now shrunk from size $t \times d$ to $t \times k$, since we only cared about span $\{e_1, \ldots, e_k\}$.

23.4.1 First approach: Appling a standard theorem

The event in (23.4.3) uses important quantities from linear algebra. For any $t \times k$ matrix R, define

$$\text{Maximum singular value:} \quad s_1(R) = \max_{x \in \mathbb{R}^k, x \neq 0} \frac{\|Rx\|}{\|x\|}$$
$$\text{Minimum singular value:} \quad s_k(R) = \min_{x \in \mathbb{R}^k, x \neq 0} \frac{\|Rx\|}{\|x\|}.$$

References: Wikipedia.

Using these definitions, (23.4.3) is equivalent to showing that

$$\Pr[\, 1 - \epsilon \leq s_k(R) \;\wedge\; s_1(R) \leq 1 + \epsilon \,] \geq 1 - 2e^{-k}. \tag{23.4.4}$$

The singular values of Gaussian matrices are very well-studied, and the following bound is known.

Theorem 23.4.4. If R has size $t \times k$ where the entries are independent $N(0, 1/t)$ then, for all $z > 0$,

$$\Pr\left[1 - \frac{\sqrt{k}+z}{\sqrt{t}} \leq s_k(R) \;\wedge\; s_1(R) \leq 1 + \frac{\sqrt{k}+z}{\sqrt{t}}\right] \geq 1 - 2\exp(-z^2/2).$$

References: See Theorem 2.13 in Davidson-Szarek, (Wainwright, 2019, Example 6.2 and Exercise 5.14), (Vershynin, 2018, Theorem 4.6.1).

Taking $t = k/\epsilon^2$ and $z = \sqrt{k}$ this proves (23.4.4) (up to constant factors) which proves Theorem 23.4.2.

23.4.2 Second approach: A direct argument

Next we observe that the function $x \mapsto \|Rx\| / \|x\|$ is a continuous function on non-zero x. Thus if $\|Rx\| / \|x\| \in [1-\epsilon, 1+\epsilon]$ holds for some x, then it approximately holds for all nearby x. Also, since $\|Rx\| / \|x\|$ only depends on the direction of x, not its norm, it suffices to consider vectors with $\|x\| = 1$.

Define the Euclidean sphere

$$S = \left\{ x \in \mathbb{R}^k \ : \ \|x\| = 1 \right\}.$$

Since we only need to consider points $y \in S$, it seems conceivable that we could find *finitely* many points $P = \{p_1, \ldots, p_\ell\}$ such that *every* point in S is "nearby" to some p_i.

Definition 23.4.5. An ϵ-***net*** of a set S is a set $P \subseteq S$ satisfying $\min_{p \in P} \|p - x\| \leq \epsilon$ for all $x \in S$.

References: (Vershynin, 2018, Definition 4.2.1), (Wainwright, 2019, Definition 5.1), Wikipedia.

The following fact is known.

Fact 23.4.6 (ϵ-net of sphere). The sphere S in $\mathbb{R}^k$ has an ϵ-net of size $(3/\epsilon)^k$.

References: (Vershynin, 2018, Corollary 4.2.13), (Wainwright, 2019, Example 5.8).

Let P be an ϵ-net of size $(3/\epsilon)^k = \exp\big(k \ln(3/\epsilon)\big)$. Then we apply the Johnson-Lindenstrauss lemma with error parameter ϵ to the points in P. Note that $\|p\| = 1$ for all $p \in P$, since $P \subseteq S$. By (23.2.7) and a union bound, we have

$$\|Rp\| \ \in \ [1-\epsilon, 1+\epsilon] \qquad \forall p \in P \tag{23.4.5}$$

with probability at least

$$1 - |P| \cdot \exp(-\epsilon^2 t/8) \ \geq \ 1 - \exp\big(k \ln(3/\epsilon) - \epsilon^2 t/8\big).$$

This probability is at least $1 - e^{-k}$ if $t = 16k \ln(3/\epsilon)/\epsilon^2$.

Now we show that, if the norm is (approximately) preserved on the ϵ-net, then it is preserved on the entire sphere.

Claim 23.4.7. Suppose that (23.4.5) holds. Then $\|Rx\| \leq 1 + O(\epsilon)$ for all $x \in S$.

References: (Vershynin, 2018, Lemma 4.4.1).

Proof. Consider any $x^* \in \arg\max_{x \in S} \|Rx\|$. The existence of x^* comes from the Weierstrass extreme value theorem, since $x \mapsto \|Rx\|$ is continuous and S is a closed, bounded set.

Since P is an ϵ-net, there exists $p^* \in P$ such that $\|x^* - p^*\| \leq \epsilon$. By (23.4.5), we have $\|Rp^*\| \in [1-\epsilon, 1+\epsilon]$. We now bound $\|Rx^*\|$ by relating it to $\|Rp^*\|$.

$$\begin{aligned}
\|Rx^*\| \ &\leq \ \|Rp^*\| + \|R(x^* - p^*)\| && \text{(by the triangle inequality)} \\
&\leq \ (1+\epsilon) + \|R(x^* - p^*)\| && \text{(by (23.4.5))} \\
&= \ (1+\epsilon) + \left\| R \frac{(x^* - p^*)}{\|x^* - p^*\|} \right\| \cdot \|x^* - p^*\| && \text{(normalizing the vector)} \\
&\leq \ (1+\epsilon) + \|Rx^*\| \cdot \epsilon && \text{(since } x^* \text{ is a maximizer).}
\end{aligned}$$

Rearranging, we get $\|Rx^*\| \leq \frac{1+\epsilon}{1-\epsilon}$. Since $\frac{1}{1-\epsilon} \leq 1 + 2\epsilon$ (see Fact B.3.5), the result follows. □

A similar argument yields the following claim, which proves Theorem 23.4.2.

Claim 23.4.8. Suppose that (23.4.5) holds. Then $\|Rx\| \geq 1 - O(\epsilon)$ for all $x \in S$.

23.4.3 Subspace Embeddings with Fast JL

In the preceding discussion we have assumed that R is a Gaussian matrix, which simplified matters due to the rotational invariance (Fact 23.4.3), but unfortunately multiplication by R is slow. We now discuss how to replace R with a Fast JL matrix.

The desired conclusion of Theorem 23.4.2 is that

$$\Pr\left[\ \frac{\|Rx\|}{\|x\|} \in [1-\epsilon, 1+\epsilon]\ \ \forall x \in U,\, x \neq 0\ \right] \geq 1 - 2e^{-k}.$$

Now let A be a $d \times k$ matrix whose columns are an orthonormal basis of U. Recall that P is an ϵ-net for S, the sphere in $\mathbb{R}^k$. Since A is an ***isometry*** (a distance-preserving bijective map) between $\mathbb{R}^k$ and U,

- $A \cdot S = \{\ Ax\ :\ x \in S\ \}$ is the sphere in U.
- $A \cdot P = \{\ Ap\ :\ p \in P\ \}$ is an ϵ-net for $A \cdot S$.

Now apply Theorem 23.3.2 with $\delta = e^{-k}/|P| \geq (\epsilon/3e)^k$ and $t = O(\log(d/\delta)^2 \log(1/\delta)/\epsilon^2)$. It follows that

$$\Pr\left[\ \|Rx\| \in [1-\epsilon, 1+\epsilon]\ \forall x \in A \cdot P\ \right]\ \ \geq\ \ 1 - 2e^{-k}.$$

An argument similar to Claim 23.4.7 implies that

$$\begin{aligned} &\Pr\left[\ \|Rx\| \in [1-O(\epsilon), 1+O(\epsilon)]\ \forall x \in U\ \right] \\ =\ &\Pr\left[\ \|Rx\| \in [1-O(\epsilon), 1+O(\epsilon)]\ \forall x \in A \cdot S\ \right]\ \ \geq\ \ 1 - 2e^{-k}. \end{aligned}$$

This proves Theorem 23.4.2 with the weaker dimension of $t = O\big((k \log(d)/\epsilon)^3\big)$, but with a matrix R supporting multiplication in time $O(d \log d + t)$.

Discussion. See Section 2.1 of this survey, Section 8.7 of this survey or Section 6 of this survey.

23.5 Exercises

Exercise 23.1. Let R be a JL matrix of size $t \times d$, where the entries are independent $N(0, 1/t)$. Prove that $\mathrm{E}\left[R^{\mathsf{T}} R\right] = I$. This is similar to the notion of being in isotropic position.

Exercise 23.2. Let $x_1, \ldots, x_n$ be given vectors in $\mathbb{R}^d$, and let $\epsilon \in (0,1)$ be arbitrary. For some $t = O(\log(n)/\epsilon^2)$, define R to be a $t \times d$ matrix whose entries are independent $N(0, 1/t)$.

Part I. Suppose that $\|x_i\| \leq 1$ for all $i \in [n]$. Prove that, with probability at least $1 - 1/n$,

$$|x_i^{\mathsf{T}} x_j - x_i R^{\mathsf{T}} R x_j| \;\leq\; \epsilon \qquad \forall i, j.$$

Part II. Now assume no constraint on the norm of $x_1, \ldots, x_n$. Prove that, with probability at least $1 - 1/n$,

$$|x_i^{\mathsf{T}} x_j - x_i R^{\mathsf{T}} R x_j| \;\leq\; \epsilon \, \|x_i\| \, \|x_j\| \qquad \forall i, j.$$

Exercise 23.3. Prove Claim 23.3.5.

Hints:

- $\|Sy\|_2^2 = \sum_i (Sy)_i^2$, and these summands are independent.
- The expectation was already analyzed above.
- The Generalized Hoeffding bound (Theorem 21.3.5) is recommended.

Exercise 23.4. Let R be a random Fast JL matrix. Is R rotationally invariant?

Exercise 23.5 $\frac{n}{4}$-net for the Hamming Cube. Recall that Section 23.4 defined an ϵ-net with respect to the Euclidean norm for the sphere. In this exercise we will construct a $\frac{n}{4}$-net with respect to the Hamming distance for the Hamming cube $\{0,1\}^n$. (Recall that Hamming distance was defined in (22.3.3).)

For this exercise, define the ***Hamming ball***, for any $x \in \{0,1\}$ and $d \geq 0$, to be

$$B(x, d) \;=\; \{\, y \in \{0,1\}^n \;:\; \Delta(x, y) \leq d \,\}.$$

Part I. Fix some $z \in \{0,1\}^n$ and $d > 0$. Let $x_1, \cdots, x_M$ be points sampled independently and uniformly from $\{0,1\}^n$. Suppose that $\Pr[\, z \in B(x_i, d) \,] \geq p$ for some $p \in [0,1]$. Show that

$$\Pr\left[z \notin \bigcup_{i \in [M]} B(x_i, d) \right] \;\leq\; e^{-pM}.$$

Part II. Fix $z \in \{0,1\}^n$ and let x be a point uniformly sampled from $\{0,1\}^n$. Show that

$$\Pr\left[\, z \in B(x, \tfrac{n}{4}) \,\right] \;\geq\; 2^{-n/2}.$$

Hint: You will need some bounds on binomial coefficients.

Part III. Show that there is a $\frac{n}{4}$-net for $\{0,1\}^n$ of size $O(n \cdot 2^{n/2})$.

Chapter 24

Applications of Johnson-Lindenstrauss

In this chapter we will see several applications of the Johnson-Lindenstrauss lemma. These examples illustrate two key reasons why Johnson-Lindenstrauss is so useful.

- First, it's *oblivious* to the points being transformed. We can pick the matrix ahead of time, and nevertheless it is likely to work well on whatever points it is applied.
- If there is an algorithm that involves the Euclidean norm, and whose performance is exponential in the dimension, dimensionality reduction will improve its performance to polynomial in the dimension!

24.1 Streaming algorithms for ℓ_2

Recall the streaming model from Chapter 13. We will change the notation slightly. The algorithm receives a sequence of items $(a_1, a_2, \ldots, a_n)$, where each $a_i \in [d]$. The frequency vector $x \in \mathbb{Z}^d$ is

$$x_j \;=\; |\{\; i \;:\; a_i = j \;\}| \;=\; \text{number of occurrences of } j \text{ in the stream.}$$

The objective is to estimate some properties of x while using $O(\log(dn))$ space. Properties of interest include $\|x\|_p$, or the number of non-zero entries, etc.

Today we will give a simple algorithm to estimate $\|x\|_2$. As an example of a scenario where this would be useful, consider a database table $((a_1, b_1), \ldots, (a_n, b_n))$. A self-join with the predicate $a = a$ would output all triples (a, b, b') where (a, b) and (a, b') belong to the table. What is the size of this self-join? It is simply $\|x\|^2$, where x is the frequency vector for the a values in the table. So a streaming algorithm for estimating $\|x\|$ could be quite useful in database query optimization.

The Algorithm. The idea is very simple: instead of storing x explicitly, we will store a *dimensionality reduced* form of x. Let R be a random Gaussian matrix of size $t \times d$, divided by $\sqrt{t}$. (In other words, each entry is drawn from the distribution $N(0, 1/t)$. This is a rescaling of the linear map R defined in Section 23.2.) Instead of explictly maintaining the vector x, the algorithm maintains the vector $y = R \cdot x$.

Algorithm 24.1 Estimating the ℓ_2 norm of the frequency vector.

```
1: function EstimateL2(int n)
2:     Set t = O(1/ε^2)
3:     Let R be a t × d matrix with independent entries drawn from N(0, 1/t)
4:     Let y be the zero vector in R^t
5:     for i = 1, ..., m do
6:         Receive item a_i from the stream
7:         Add column a_i of R to y
8:     end for
9:     return ||y||
10: end function
```

At time step i, the algorithm receives the index $j = a_i$. This implicitly causes x_j to increase by 1. Since $y = R \cdot x$, the corresponding change in y is to add the j^{th} column of R to y.

To analyze this algorithm, we use the Johnson-Lindenstrauss lemma. In Eq. (23.2.7), we proved that

$$\Pr\left[\,(1-\epsilon)\|x\| \;\le\; \|y\| \;\le\; (1+\epsilon)\|x\|\,\right] \;\ge\; 1 - \exp(-\epsilon^2 t/8).$$

Thus, setting $t = \Theta(1/\epsilon^2)$, we conclude that $\|y\|$ gives a $(1+\epsilon)$ approximation of $\|x\|$ with constant probability. Alternatively, if we want y to give an accurate estimate at *each* of the n time steps, we can take $t = \Theta(\log(n)/\epsilon^2)$.

How much space does this algorithm take? The algorithm stores two objects: the vector y and the matrix R. The vector y uses $t = O(1/\epsilon^2)$ words of space, which is consistent with our goal of using very little space.

Unfortunately, the matrix R uses td words of space. Storing R explicitly is worse than storing the frequency vector x. (There is also the issue of how many bits of accuracy are needed when generating the Gaussian random variables, but we will ignore that.) However, it seems that there may be some benefits to storing R instead of x, because R contains purely random numbers. The values of R do not depend on values appearing in the stream.

- At first glance, it may seem that R needn't be stored at all: every time an entry of R is accessed, a new independent random variable could be generated. But this does not work: each time an item j appears in the stream, we must add the j^{th} column of R to y, and it must be *the same column* each time.

- Does it help to use the Fast JL transform $R = SHD$ introduced in Section 23.3? Storing this matrix R requires only $O(d+t)$ words of space, which is an improvement over the ordinary JL matrix, but still too much.

- In a practical implementation, R will be generated by a pseudorandom generator initialized by some seed. This has the advantage that we can regenerate columns of R at will by resetting the seed. This is likely to work well in practice, but may not have provable guarantees.

- There is another approach that has been studied by theorists, and has provable guarantees but is probably too complicated to use in practice. Theorists have developed provably good pseudo-random generators, but only for algorithms *that use a small amount of space*. Since streaming

algorithms do indeed use little space, this approach can indeed be used to regenerate the matrix R as necessary.

References: See this paper of Nisan.

- Another approach with theoretical guarantees is to generate R using special low-independence hash functions. A basic form of these ideas was discussed in Chapter 14. We may return to this in future chapters. See (Blum et al., 2018, Section 6.2.3) or Chapter 6 of these notes.

24.2 Euclidean nearest neighbor

The nearest neighbor problem is a classic problem involving high-dimensional data. Given points $P = \{p_1, ..., p_n\} \in \mathbb{R}^d$, the goal is to build a (static) data structure so that, given a query point $q \in \mathbb{R}^d$, we can quickly find i minimizing $\|q - p_i\|$. We focus on the Euclidean norm $\|\cdot\| = \|\cdot\|_2$, but this problem is interesting for many norms.

Trivial solutions. This problem can trivially be solved in polynomial time. We could do no processing of P, then for each query find the closest point by exhaustive search. This requires time $O(nd)$ for each query. An alternative approach is to use a k-d tree, which is a well-known data structure for representing geometric points. Unfortunately this could take $O(dn^{1-1/d})$ time for each query, which is only a substantial improvement over exhaustive search when the dimension d is a constant. This phenomenon, the failure of low dimensional methods when applied in high dimensions, is known as the "curse of dimensionality".

Overcoming the curse. We will show that the curse can be overcome through the use of randomization and approximation. The ***ϵ-approximate nearest neighbor*** problem (ϵ-NN) is defined as follows. Given a query point $q \in \mathbb{R}^d$, define

$$\text{OPT} \;=\; \min_{p \in P} \|p - q\|$$

we must find a point $\hat{p} \in P$ such that

$$\|\hat{p} - q\| \;\le\; \big(1 + O(\epsilon)\big) \cdot \text{OPT}. \tag{24.2.1}$$

Our goal is to preprocess P and produce a data structure of size $\text{poly}(n)$. Given a query point q, we wish to spend time say $O(d \log(n)/\epsilon^2)$ finding a point $p \in P$ satisfying (24.2.1). Imagine a setting where $n \approx 10^{10}$ is the number of web pages, $d \approx 10^3$ is the number of features describing those pages, and $\epsilon = 10^{-1}$ is the desired approximation. Our approach can answer a query in time $d \log(n)/\epsilon^2 \approx 10^6$, whereas the trivial approach requires time $nd \approx 10^{13}$.

The high-level idea of our solution is very simple. First, design an exhaustive search algorithm that requires space roughly $2^{O(d)}$ to solve ϵ-NN for d-dimensional data. Then, apply dimensionality reduction to reduce the data set to dimension $t \approx \log n$. Running the exhaustive search algorithm on the t-dimensional data only requires space $2^{O(t)} = n^{O(1)}$.

24.2.1 Point Location in Equal Balls

The first step is to reduce our problem to a simpler one, in which a query only needs to determine whether the closest point is at distance less than or greater than roughly r. This simpler problem is called the ***ϵ-Point Location in Equal Balls*** problem (ϵ-PLEB). It is defined as follows.

The input data is a collection of n points $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$. The notation $B(p, r)$, defined in (B.2.2), denotes the Euclidean ball of radius r around p. We will call $B(p, r)$ a small ball, and $B(p, (1+\epsilon)r)$ a big ball.

Given a query point $q \in \mathbb{R}^d$, the requirements are as follows:

- *Case 1:* $(1+\epsilon)r < \text{OPT}$.
  In this case, q is not in any big ball, i.e., $\nexists p_i$ with $q \in B(p_i, (1+\epsilon)r)$. We must output No.

- *Case 2:* $r < \text{OPT} \le (1+\epsilon)r$.
  In this case, q is in a big ball but not any small ball. We can either output No, or output Yes together with a point p_j with $q \in B(p_j, (1+\epsilon)r)$.

- *Case 3:* $\text{OPT} \le r$.
  In this case, q is in a small ball (i.e., $\exists p_i \in P$ with $q \in B(p_i, r)$). We must output Yes and a point p_j with $q \in B(p_j, (1+\epsilon)r)$.

Although there is some uncertainty about what happens in Case 2, we can draw some conclusions from the output.

- If the output is Yes, then we are always provided with a point p_j such that $q \in B(p_j, (1+\epsilon)r)$.

- If the output is No, then either Case 1 or Case 2 could apply. In either case,

$$\text{there is no } p \in P \text{ s.t. } q \in B(p, r). \tag{24.2.2}$$

24.2.2 Reduction of ϵ-NN to ϵ-PLEB

We now explain how to solve the ϵ-NN problem using a solution to the ϵ-PLEB problem. First scale the point set P so that the minimum interpoint distance is at least 1, then let Δ be the maximum interpoint distance. So $1 \le \|p - p'\| \le \Delta$ for all $p, p' \in P$.

Initialization. To initialize the data structure, we create an instance of ϵ-PLEB(r) for every radius r in

$$\mathcal{R} \;=\; \left\{(1+\epsilon)^0, (1+\epsilon)^1, (1+\epsilon)^2, \ldots, \Delta\right\}. \tag{24.2.3}$$

Question 24.2.1. How many different radii are there?

Answer.

There are $\log_{1+\epsilon}(\Delta) = \frac{\ln(\Delta)}{\ln(1+\epsilon)} \le O(\log(\Delta)/\epsilon)$, by Exercise B.2.

Queries. Fix a query point $q \in \mathbb{R}^d$. Let us consider what the different outputs of ϵ-PLEB(r) might be for the radii in $\mathcal{R}$.

Example 1:

Radius	$(1+\epsilon)^0$	$(1+\epsilon)^1$	$(1+\epsilon)^2$	$(1+\epsilon)^3$	$(1+\epsilon)^4$	$\cdots$	Δ
Output	No	No	No	Yes	Yes	$\cdots$	Yes
Case	1	1	2	3	3	$\cdots$	3

Example 2:

Radius	$(1+\epsilon)^0$	$(1+\epsilon)^1$	$(1+\epsilon)^2$	$(1+\epsilon)^3$	$(1+\epsilon)^4$	$\cdots$	Δ
Output	No	No	No	Yes	Yes	$\cdots$	Yes
Case	1	1	1	2	3	$\cdots$	3

Figure 24.1: Some possible outcomes of the ϵ-PLEB subroutine. Note that there can be at most one radius that results in Case 2.

Claim 24.2.2. For the radii in $\mathcal{R}$, all No outputs occur for smaller radii than all Yes outputs.

Proof. If radius r falls into Case 1, then clearly all smaller radii also fall into Case 1 and therefore produce the output No.

If radius r falls into Case 3, then clearly all larger radii also fall into Case 3 and therefore produce the output Yes.

It remains to consider Case 2. Clearly *at most one* radius in $\mathcal{R}$ can satisfy the condition of Case 2. With this radius, either Yes or No could be output, but both possibilities are consistent with the claim. □

Given any query point q, let $\hat{r}$ be the minimum radius r for which ϵ-PLEB(r) says Yes. Let $\hat{p} \in P$ be the returned point for which $q \in B(\hat{p}, (1+\epsilon)\hat{r})$. By Claim 24.2.2, $\hat{r}$ can be found by binary search.

Claim 24.2.3. $\hat{p}$ satisfies (24.2.1), and therefore is a solution to the ϵ-NN problem.

Proof. The requirements of the ϵ-PLEB$(\hat{r})$ subroutine ensure that that $\|\hat{p} - q\| \le \hat{r}(1+\epsilon)$.

Recall that $\hat{r}$ is the *minimum* radius that said Yes. If $\hat{r} = 1$ then clearly p satisfies (24.2.1). Otherwise, ϵ-PLEB$(\hat{r}/(1+\epsilon))$ output No. It follows (from (24.2.2)) that there is no point $p' \in P$ with $q \in B(p', r/(1+\epsilon))$. In other words,

$$\frac{r}{1+\epsilon} \;<\; \min_{p' \in P} \left\|p' - q\right\| .$$

Thus $\hat{p}$ satisfies

$$\|\hat{p} - q\| \;\le\; (1+\epsilon)\hat{r} \;<\; (1+\epsilon)^2 \cdot \min_{p' \in P} \left\|p' - q\right\| .$$

Since $(1+\epsilon)^2 \le 1 + 3\epsilon$ (see Fact B.1.3), this proves (24.2.1). □

Question 24.2.4. How many radii must the binary search consider in order to determine r^*?

Answer.

$$\cdot\Big(\big(\epsilon/(\nabla)\text{gol}\big)\,\text{gol}\Big)O \;=\; |\mathcal{R}|^{2}\text{gol}$$

24.2.3 Solving PLEB

The main idea here is quite simple. We discretize the space, then use a hash table to identify locations belonging to a ball.

Preprocessing. In more detail, the preprocessing step for ϵ-PLEB(r) proceeds as follows. We first partition the space into cuboids (d-dimensional cubes) of side length $\epsilon r/\sqrt{d}$. Note that[1] the Euclidean diameter of a cuboid is its side length times $\sqrt{d}$, which is ϵr. Each cuboid is identified by a canonical point, say the minimal point contained in the cuboid. We then create a hash table, initially empty. For each point p_i and each cuboid C that intersects $B(p_i, r)$, we insert the (key, value) pair (C, p_i) into the hash table.

Queries. Now consider how to perform a query for a point q. The first step is to determine the cuboid C that contains q, by simple arithmetic. Next, we look up C in the hash table. If there are no matches, that means that no ball $B(p_i, r)$ intersects C, and therefore q is not contained in any ball of radius r (a small ball). So, by the requirements of ϵ-PLEB(r), we can say No.

Suppose that C is in the hash table. Then the hash table can return an arbitrary pair (C, p_j), which tells us that $B(p_j, r)$ intersects C. By the triangle inequality, the distance from p_j to q is at most r plus the diameter of the cuboid, which is ϵr. So $\|p_j - q\| \le (1+\epsilon)r$, which means that q is contained in the big ball around p_j. By the requirements of ϵ-PLEB(r), we can say Yes and we can return the point p_j.

Time and Space Analysis. To analyze this algorithm, we first need to determine the number of cuboids that intersect a ball of radius r. From (B.2.3), the volume of $B(p, r)$ is $2^{O(d)} r^d / d^{d/2}$. On the other hand, the volume of a cuboid is $(\epsilon r/\sqrt{d})^d$. So the number of cuboids that intersect this ball is roughly

$$\frac{2^{O(d)} r^d / d^{d/2}}{(\epsilon r/\sqrt{d})^d} \;=\; O(1/\epsilon)^d.$$

Therefore the time and space used by the preprocessing step is roughly $O(1/\epsilon)^d$.

To perform a query, we just need to compute the cuboid containing q then look up that cuboid in the hash table. This takes $O(d)$ time if we use the perfect hashing of Section 15.4. This cannot be improved — $\Omega(d)$ time is clearly necessary, since we must examine most coordinates of the vector q.

Unfortunately the preprocessing time and space is exponential in d, which is another example of the curse of dimensionality. The next section addresses this via dimensionality reduction.

24.2.4 Applying Dimensionality Reduction

The next step is to apply the Johnson-Lindenstrauss lemma to map our points to a low-dimensional space. Let $t = O(\log(n)/\epsilon^2)$. Let R be a random $t \times d$ matrix whose entries have distribution $N(0, 1/t)$. For any fixed query point q, Theorem 23.2.1 says that the linear map R approximately preserves pairwise distances between all points in $P \cup \{q\}$ with probability $1 - 1/n$.

The analysis of ϵ-PLEB changes as follows. The preprocessing step must apply the matrix to all points in P, which takes time $O(dnt)$. The time to set up the hash table improves to $O(1/\epsilon)^t = n^{O(\log(1/\epsilon)/\epsilon^2)}$. So assuming ϵ is a constant, the preprocessing step runs in polynomial time. Each query must also apply the Johnson-Lindenstrauss matrix to the query point, which takes time $O(td) = O(d \log(n)/\epsilon^2)$.

Finally, we analyze the reduction which allowed us to solve ϵ-NN. Recall that the preprocessing step simply instantiates ϵ-PLEB(r) for all values of r in $\mathcal{R}$. As discussed in Question 24.2.1, the number of different instances is $|\mathcal{R}| = O\big(\log(\Delta)/\epsilon\big)$, so the total preprocessing time is

$$n^{O(\log(1/\epsilon)/\epsilon^2)} \;\cdot\; O\big(\log(\Delta)/\epsilon\big).$$

[1] This can be seen from Fact B.2.1, because the cuboid has ℓ_∞ diameter $\epsilon r/\sqrt{d}$.

This is polynomial time assuming $\Delta \le 2^{n^{\text{poly}(1/\epsilon)}}$ and ϵ is a constant. Each query must perform binary search over different radii to find $\hat{r}$. Each query to ϵ-PLEB now takes time $O(t)$. So the total query time is

$$O\big(dt\big) + O(\log|\mathcal{R}|) \cdot O(t) \;=\; O\Big(t \cdot \big(d + \log(\log(\Delta)/\epsilon)\big)\Big).$$

This is roughly $O(d \log(n)/\epsilon^2)$ so long as Δ is reasonable, say $\Delta \le 2^{2^d}$.

24.2.5 Discussion

Some of the earliest papers on approximate nearest neighbour are Kleinberg, Indyk and Motwani and Kushilevitz-Ostrovsky-Rabani. The algorithm we present is due to Indyk and Motwani. It seems quite inefficient, but of course there have been numerous improvements. The Indyk-Motwani paper itself also proposed the alternative approach of "locality sensitive hashing", which was touched upon in Section 12.3. Some modern techniques for nearest neighbor are described in the survey of Andoni and Indyk.

24.3 Fast Least-Squares Regression

Given matrix A of size $d \times m$ (with $d \ge m$) and a vector $b \in \mathbb{R}^d$, the goal is to find

$$x^* \in \operatorname*{argmin}_{x \in \mathbb{R}^m} \|Ax - b\| .$$

For some background motivation, see CPSC 340 Lecture 12 and Lecture 13.

Mathematically, this is quite a simple problem to solve: $\|Ax - b\|_2^2$ is a convex, differentiable quadratic function, so the minimizers are the points with zero gradient. After some simple vector calculus, this amounts to solving to the linear system

$$A^\mathsf{T} A x = A^\mathsf{T} b. \tag{24.3.1}$$

References: UBC MATH 307 notes, (Trefethen and Bau, 1997, Theorem 11.1).

The focus of this section is algorithms for computationally solving (24.3.1).

Conventional Solutions. There are several standard approaches to solving (24.3.1).

- Naively, one could compute $A^\mathsf{T} A$ and $A^\mathsf{T} b$ explicitly and use Gaussian elimination to solve the problem exactly.

  Question 24.3.1. How much time does this take?

  Answer.
  $O(dm^2)$ time to compute $A^\mathsf{T} A$, $O(dm)$ time to compute $A^\mathsf{T} b$, and $O(m^3)$ time to perform Gaussian elimination. Since $d \ge m$, this is $O(dm^2)$ time.
  This can be improved to $O(dm^{1.38})$ by a galactic algorithm.

- Since $A^\mathsf{T} A$ is symmetric, we could use the Cholesky decomposition instead of Gaussian elimination. This approach can also solve the problem exactly in $O(dm^2)$ time.

  References: (Trefethen and Bau, 1997, Algorithm 11.1).

Theorem 24.3.2. There is a randomized algorithm to find $\hat{x}$ satisfying

$$\|A\hat{x} - b\| \;\leq\; (1 + O(\epsilon)) \cdot \|Ax^* - b\| \tag{24.3.2}$$

in time

$$O(dm \log d) + \text{poly}(m \log d/\epsilon).$$

The failure probability is at most $2e^{-m}$.

The key point is that the leading term has improved from $O(dm^2)$ to only $O(dm \log d)$. This gives an improvement over conventional results if $\log d \ll m \ll d$ and ϵ is modest.

The main idea is as follows. We will use a Fast JL matrix R to reduce the number of rows of A and b to $t = O((m \log d/\epsilon)^3)$; see Section 23.4.3. After the dimension reduction, we then solve the problem by conventional methods.

Algorithm 24.2 Fast algorithm for least-squares regression. A has size $d \times m$ and b has length d.

1: **function** FastLeastSquares(matrix A, vector b, float ϵ)
2: Let $t \leftarrow O\big((m \log d/\epsilon)^3\big)$
3: Let R be a Fast JL matrix of size $t \times d$
4: Compute the matrix RA and the vector Rb
5: Find $\hat{x} \in \text{argmin}_{x \in \mathbb{R}^m} \|RAx - Rb\|$, by conventional methods.
6: **return** $\hat{x}$
7: **end function**

Runtime analysis. The key computations are on line 4. Computing RA by ordinary matrix multiplication would take time $O(tdm)$, which is too slow. However, since R is a Fast JL matrix, we can do this much more quickly. Recall from Theorem 23.3.2 that matrix-vector multiplication with R takes time $O(d \log d + t)$. So we can construct RA by separately multiplying R by each column of A. Since there are m columns, this takes $O(dm \log d + tm)$ time. The vector Rb can be computed in the same way.

Next, on line 5, we can find $\hat{x}$ by conventional methods in $O(tm^2)$ time. Plugging in the definition of t, the total runtime is

$$O(dm \log d + tm) + O(tm^2) \;=\; O(dm \log d) + \text{poly}(m \log d/\epsilon).$$

Analysis. Define

$$U \;=\; \text{span}\big(\{\, Ax \,:\, x \in \mathbb{R}^m \,\} \cup \{b\}\big). \tag{24.3.3}$$

The vectors in U all have length d, but U is a subspace of dimension at most $m + 1$. By (**??**), the matrix R approximately preserves the norm of all vectors in U, with probability at least $1 - 2e^{-m}$. By definition of $\hat{x}$, we have

$$\|RA\hat{x} - Rb\| \;\leq\; \|RAx^* - Rb\| \;=\; \|R(Ax^* - b)\| \;\leq\; (1 + \epsilon) \cdot \|Ax^* - b\|\,,$$

since $Ax^* - b \in U$. On the other hand,

$$\|RA\hat{x} - Rb\| \;=\; \|R(A\hat{x} - b)\| \;\geq\; (1 - \epsilon)\,\|A\hat{x} - b\|\,,$$

since $A\hat{x} - b \in U$. Putting these inequalities together, we obtain

$$\|A\hat{x} - b\| \;\leq\; \frac{1 + \epsilon}{1 - \epsilon} \cdot \|Ax^* - b\|\,.$$

Since $\frac{1}{1-\epsilon} \leq 1 + 2\epsilon$ (see Fact B.3.5), this completes the proof of Theorem 24.3.2.

Discussion. This algorithm is due to Tamas Sarlos. See also Section 10.3 of this survey.

There have been many improvements. Let $\mathrm{nnz}(A)$ denote the number of non-zero entries in the matrix A. It is known that the guarantee (24.3.2) can be achieved in time

$$O(\mathrm{nnz}(A)) + \mathrm{poly}(m/\epsilon).$$

See, for example, this paper or Theorem 21 of this survey.

24.4 Approximate Matrix Multiplication

Let A and B be matrices of size $n \times n$. Computing the product AB exactly takes $O(n^3)$ time by conventional methods, or $O(n^{2.38})$ by a galactic algorithm. The following algorithm is much faster, and is based on a very simple idea:

> *reduce the dimension of the matrices before multiplying them.*

Algorithm 24.3 Algorithm to estimate AB. The matrices A and B have size $n \times n$.

```
1: function FastMatMul(matrix A, B, float ε)
2:     Set t = O(log(n)/ε^2)
3:     Let R be a t × n matrix with independent entries drawn from N(0, 1/t)
4:     return AR^T RB
5: end function
```

Question 24.4.1. What is the runtime of this algorithm?

Answer.
It depends on the order in which the matrices are multiplied. Multiplying $R^\mathsf{T}R$ first will give runtime $O(n^3)$. However, first multiplying AR^T, then RB takes time $O(n^2t)$, and finally combining them takes time $O(n^2t) = O(n^2 \log(n)/\epsilon^2)$.

One appealing aspect of this algorithm is the following claim.

Claim 24.4.2. FastMatMul(A, B, ϵ) is an unbiased estimator of AB.

The proof of this claim is Exercise 24.1.

Notation. The entry of A in the i^{th} row and j^{th} column is denoted $A_{i,j}$. To refer to rows and columns, we will use the following handy notation. Thinking of $*$ as a "wildcard" character, we will let $A_{i,*}$ denote the i^{th} row of A and let $A_{*,k}$ denote the k^{th} column of A. Using this notation, the entries of the product AB can be expressed as a dot product as follows.

$$(AB)_{i,k} \;=\; \sum_{j=1}^{n} A_{i,j}B_{j,k} \;=\; A_{i,*}B_{*,k}$$

References: (Murphy, 2022, Section 7.2.3).

Definition 24.4.3. For a matrix A, its ***Frobenius norm*** $\|A\|_F$ is defined by

$$\|A\|_F^2 \;=\; \sum_{i,j\in[n]} (A_{i,j})^2 \;=\; \sum_{i\in[n]} \|A_{i,*}\|_2^2 \;=\; \sum_{j\in[n]} \|A_{*,j}\|_2^2. \tag{24.4.1}$$

This is just the Euclidean norm of A when viewed as a vector of length n^2.

References: (Trefethen and Bau, 1997, page 22), (Vershynin, 2018, Section 4.1.3), (Murphy, 2022, Section 7.1.3.2), (Blum et al., 2018, Section 12.8.5), Wikipedia.

Our main theorem is that FastMatMul(A, B, ϵ) is a good approximation for AB, with error measured in the Frobenius norm.

Theorem 24.4.4. With probability at least $1 - 1/n$,

$$\|AB - AR^{\mathsf{T}}RB\|_F \;\le\; \epsilon \|A\|_F \|B\|_F .$$

Proof. We apply dimensionality reduction to the rows of A and the columns of B. Define

$$\mathcal{X} \;=\; \{\, (A_{i,*})^{\mathsf{T}} \,:\, i \in [n] \,\} \cup \{\, B_{*,k} \,:\, k \in [n] \,\} .$$

Exercise 23.2 shows that, with probability $1 - 1/n$,

$$\begin{aligned} & |x^{\mathsf{T}}x' - x^{\mathsf{T}}R^{\mathsf{T}}Rx'| \;\le\; \epsilon \|x\| \|x'\| && \forall x,\, x' \in \mathcal{X} \\ \Longrightarrow \quad & |A_{i,*}B_{*,k} - A_{i,*}R^{\mathsf{T}}RB_{*,k}| \;\le\; \epsilon \|A_{i,*}\| \|B_{*,k}\| && \forall i,\, k \in [n]. \end{aligned}$$

Squaring and summing over i, k, we have

$$\begin{aligned} \sum_{i,k\in[n]} \big(\underbrace{A_{i,*}B_{*,k} - A_{i,*}R^{\mathsf{T}}RB_{*,k}}_{=(AB - AR^{\mathsf{T}}RB)_{i,k}} \big)^2 \;&\le\; \epsilon^2 \sum_{i,k\in[n]} \|A_{i,*}\|^2 \|B_{*,k}\|^2 \\ &=\; \epsilon^2 \Big(\sum_{i\in[n]} \|A_{i,*}\|^2 \Big) \Big(\sum_{k\in[n]} \|B_{*,k}\|^2 \Big). \end{aligned}$$

The theorem follows by taking the square root and using the identity (24.4.1). □

24.5 Exercises

Exercise 24.1. Prove Claim 24.4.2.

Chapter 25

Polynomial Methods

In Section 15.1 we discussed the problem of testing equality of two bitstrings in a distributed setting. We solved that problem by comparing the hash values when using the ***polynomial hash*** functions of Section 14.3. That was our first taste of polynomial methods.

25.1 Polynomial Identity Testing

The ***polynomial identity testing*** problem (henceforth, PIT) can roughly be defined as follows. Given a multivariate polynomial $p(x_1, \ldots, x_n)$ with coefficients in some field $\mathbb{F}$, decide if p equals the zero polynomial. This problem might sound dull and algebraic, but it is perhaps better to think of it as very abstract and general. Many interesting non-algebraic problems, such as testing equality of strings, are related to PIT.

Let us now explain PIT in more detail. Every polynomial $p(x_1, \ldots, x_n)$ can be expanded into a sum of monomials with coefficients in $\mathbb{F}$. For example, the polynomial $p(x, y, z) = (x + 2y)(3y - z)$ can be expanded into a sum of monomials as

$$p(x, y, z) \;=\; 3xy + 6y^2 - xz - 2yz.$$

If p is expanded into a sum of monomials, and all coefficients of those monomials are zero, then p is called the ***zero polynomial***, or ***identically zero***.

Example 25.1.1. Is

$$p(x, y, z) \;=\; (x + y)^2 - (x - y)^2 - 4xy$$

identically zero?

Answer.

Yes, by trivial expansion.

Example 25.1.2. Is

$$p(x, y, z) \;=\; (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \;-\; \det \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$$

identically zero?

Answer.

Yes, by the formula for a Vandermonde determinant.

Univariate polynomials are commonly classified by their degree. For a multivariate polynomial, this requires some explanation. A monomial is any expression of the form $c \cdot \prod_{i=1}^{n} x_i^{\alpha_i}$ where $c \in \mathbb{F}$ and $\alpha_1, \ldots, \alpha_n$ are non-negative integers. The ***total degree*** of this monomial is $\sum_{i=1}^{n} \alpha_i$. The total degree of a polynomial is the maximum total degree of its monomials.

We can now state the PIT question mathematically as follows. If p is a polynomial of total degree d, and we expand

$$p(x_1, \ldots, x_n) \;=\; \sum_{\substack{\alpha_1,\ldots,\alpha_n \geq 0 \\ \alpha_1+\cdots+\alpha_n \leq d}} c_{\alpha_1,\ldots,\alpha_n} \prod_{i=1}^{n} x_i^{\alpha_i},$$

then are all coefficients $c_{\alpha_1,\ldots,\alpha_n}$ equal to zero?

Computational considerations. As stated above, PIT is a mathematical question: is p equal to the zero polynomial, or isn't it? We are interested in the computational aspects of PIT. This requires discussing how the data is represented and what operations are permitted. In particular, we must clarify what it means to be "given p".

We will assume that p is represented as a "black box" which, given explicit values $x_1, \ldots, x_n$, returns the value $p(x_1, \ldots, x_n)$ in one unit of time. This model is useful because p might have an implicit representation, such as in Example 25.1.2. Nevertheless, given numeric values for x_1, x_2, x_3 we can evaluate $p(x_1, x_2, x_3)$ by computing the determinant numerically.

Complexity Status. The complexity status of PIT is quite interesting. We will show that there is a randomized algorithm to decide PIT. However, there is no known efficient deterministic algorithm to decide PIT. Furthermore, if such an algorithm existed then there would be significant consequences in complexity theory. It is perhaps fair to say that PIT is the canonical problem in RP that is not known to be in P.

25.1.1 The Schwartz-Zippel Lemma

The main tool that we will use in this chapter is the following.

Lemma 25.1.3 (Schwartz-Zippel Lemma). Let $\mathbb{F}$ be any field. Let $p(x_1, \ldots, x_n)$ be a polynomial of total degree d with coefficients in the field $\mathbb{F}$. Assume that p is not identically zero. Let $S \subseteq \mathbb{F}$ be any finite set. Then, if we pick $y_1, \ldots, y_n$ independently and uniformly from S,

$$\Pr[\; p(y_1, \ldots, y_n) = 0 \;] \;\leq\; \frac{d}{|S|}.$$

Interestingly, the right-hand side $\frac{d}{|S|}$ does not depend on n!

References: (Motwani and Raghavan, 1995, Theorem 7.2), Wikipedia.

To help understand the lemma, let p be a polynomial of total degree d over $\mathbb{R}$. Fix any set S of size $2d$, and pick each y_i uniformly and independently from S. According to the lemma, $(y_1, \ldots, y_n)$ is a root of p with probability at most $1/2$.

Question 25.1.4. Can we conclude that polynomials over $\mathbb{R}$ have finitely many roots?

Answer.
No! Consider the polynomial $p(x_1, x_2) = x_1$ which has total degree $d = 1$. It has *infinitely* many roots because setting $x_1 = 0$ and x_2 to anything gives a root. However, if we fix $S = \{0, 1\}$ and randomly choose $y_1, y_2 \in S$ then indeed $\Pr[\, p(y_1, y_2) = 0\,] = 1/2$, so Lemma 25.1.3 is tight.

Proof of Lemma 25.1.3. We proceed by induction on n.

The base case is the case $n = 1$. Over any field $\mathbb{F}$, a univariate polynomial of degree d has at most d roots; see Fact A.2.13. So the probability that y_1 is a root is at most $d/|S|$.

Now we assume the theorem is true for polynomials with $n-1$ variables, and we prove it for those with n variables. The main idea is to obtain polynomials with fewer variables by factoring out the variable x_1 from p. Let k be the largest power of x_1 appearing in any monomial of p. We may write

$$p(x_1, \ldots, x_n) \;=\; \sum_{i=0}^{k} x_1^i \cdot q_i(x_2, \ldots, x_n).$$

By our choice of k, the polynomial q_k is not identically zero, and its total degree is at most $d - k$.

Now randomly choose the values of $y_2, \ldots, y_n \in S$ and define the event

$$\mathcal{E}_1 \;=\; \text{“}q_k(y_2, \ldots, y_n) = 0\text{”}.$$

By induction

$$\Pr[\,\mathcal{E}_1\,] \;=\; \Pr[\; q_k(y_2, \ldots, y_n) = 0\;] \;\le\; \frac{d-k}{|S|}. \tag{25.1.1}$$

We have already chosen the values $y_2, \ldots, y_n \in S$. What remains is the *univariate* polynomial

$$f(x_1) \;=\; \sum_{i=0}^{k} x_1^i \cdot q_i(y_2, \ldots, y_n) \;=\; p(x_1, y_2, \ldots, y_n).$$

Conditioning on the event $\overline{\mathcal{E}_1}$, the coefficient of x_1^k in f is non-zero, and so f is not identically zero. Now choose y_1 uniformly at random from S, and define the event

$$\mathcal{E}_2 \;=\; \text{“}f(y_1) = 0\text{”} \;=\; \text{“}p(y_1, \ldots, y_n) = 0\text{”}.$$

By the univariate argument of the base case,

$$\Pr\left[\,\mathcal{E}_2 \;\middle|\; \overline{\mathcal{E}_1}\,\right] \;=\; \Pr\left[\, f(y_1) = 0 \;\middle|\; \overline{\mathcal{E}_1}\,\right] \;\le\; \frac{k}{|S|}. \tag{25.1.2}$$

We can combine the two types of errors in a familiar way.

$$\begin{aligned}
\Pr[\,p(y_1, \ldots, y_n) = 0\,] \;&=\; \Pr[\,\mathcal{E}_2\,] \\
&\le\; \Pr[\,\mathcal{E}_1\,] + \Pr\left[\,\mathcal{E}_2 \;\middle|\; \overline{\mathcal{E}_1}\,\right] && \text{(by Exercise A.4)} \\
&\le\; \frac{d-k}{|S|} + \frac{k}{|S|} && \text{(by (25.1.1) and (25.1.2))} \\
&=\; \frac{d}{|S|} && \square
\end{aligned}$$

25.1.2 Solving PIT

Algorithm 25.1 Randomized algorithm for PIT. It is assumed that p has coefficients in $\mathbb{F}$ and $S \subseteq \mathbb{F}$.

```
1: function SolvePIT(polynomial p(x_1, ..., x_n), set S)
2:     Pick y_1, ..., y_n uniformly and independently from S
3:     if p(y_1, ..., y_n) = 0 then return True
4:     else return False
5: end function
```

Recall the definitions of an ***RP-algorithm*** and a ***coRP-algorithm*** from Section 1.2.

Theorem 25.1.5. Let d be the total degree of p. If $|S| \geq 2d$ then SolvePIT is a coRP algorithm for the PIT problem.

Proof. If the correct output is True then p is identically zero, so $p(y_1, \ldots, y_n)$ will always equal 0, and the algorithm will always output True. So the algorithm has no false negatives.

If the correct output is False then the algorithm will incorrectly output True if $p(y_1, \ldots, y_n) = 0$. By Lemma 25.1.3, this happens with probability at most $d/|S| \leq 1/2$. □

25.2 Bipartite Matching

Let $G = (U \cup V, E)$ be a bipartite graph, meaning that U and V are disjoint sets of vertices, and every edge in E has exactly one endpoint in U and exactly one endpoint in V. A ***matching*** in G is a set of edges that share no endpoints. A ***perfect matching*** in G is a set of edges $M \subseteq E$ such that every vertex is contained in exactly one edge of M.

Polynomial time algorithms are known to decide if G has a perfect matching, and even to construct such a matching. We will give a randomized algorithm to decide if G has a perfect matching, by reducing that problem to PIT.

Let A be the matrix whose rows are indexed by the vertices in U and columns are indexed by the vertices in V. The entries of A are:

$$A_{u,v} \;=\; \begin{cases} x_{u,v} & (\text{if } uv \in E) \\ 0 & (\text{if } uv \notin E) \end{cases}, \tag{25.2.1}$$

where $\{\, x_{u,v} \,:\, uv \in E \,\}$ are distinct variables.

Another consequence of Claim 25.2.1 is that $\det A$ can be viewed as a polynomial over *any* field $\mathbb{F}$. This is because all the monomials have coefficient either $+1$ or -1, which are numbers in every field $\mathbb{F}$.

Claim 25.2.1.

$$\det A \;=\; \sum_{\text{perfect matching } m:U\to V} \pm \prod_{u\in U} x_{u,m(u)}. \tag{25.2.2}$$

Corollary 25.2.2. $\det A$ is identically zero if and only if G has no perfect matching.

Proof of Claim 25.2.1. By the Leibniz formula for determinants,

$$\det A \;=\; \sum_{\text{bijection } \pi:U\to V} \pm \prod_{u\in U} A_{u,\pi(u)}, \tag{25.2.3}$$

where the signs are irrelevant for our purposes.

The first observation is that the monomials in the right-hand side of (25.2.3) all involve distinct sets of variables, and therefore there can be no cancellations amongst these monomials.

Next, observe that a *bijection* from U to V is simply a pairing $\{ (u, \pi(u)) : u \in U \}$ of elements in U and elements in V such that each vertex appears in exactly one pair. In contrast, a *perfect matching* is such a bijection in which every pair $(u, \pi(u))$ is also an edge in E.

Lastly, observe that a monomial $\prod_{u \in U} A_{u,\pi(u)}$ is non-zero if and only if every pair $(u, \pi(u))$ is an edge in E. This is because $A_{u,\pi(u)} = 0$ if there is no edge from u to $\pi(u)$. It follows that the non-zero summands in (25.2.3) are precisely the summands in (25.2.2). □

25.2.1 A perfect matching algorithm

Algorithm 25.2 Randomized algorithm for deciding if G has a perfect matching. Here $G = (U, V, E)$ is a bipartite graph and $n = |U| = |V|$.

```
1: function DecidePM(graph G)
2:     Let 𝔽 be any field and S any set where S ⊆ 𝔽 and |S| ≥ 2n
3:     Let A be the matrix described in (25.2.1)
4:     Replace every non-zero entry of A with an independent random number uniform in S
5:     if det A ≠ 0 then return True else return False
6: end function
```

Question 25.2.3. What is the running time of this algorithm?

Answer.

Assume that operations in $\mathbb{F}$ take $O(1)$ time. Using Gaussian elimination, one can compute $\det A$ in time $O(n^3)$. This can be improved to $O(n^{2.38})$ by a galactic algorithm.

Theorem 25.2.4. DecidePM is an RP-algorithm for deciding if G has a perfect matching.

Proof. After line 3, $\det A$ is a polynomial of total degree at most n. As observed above, it can be viewed as a polynomial over any field $\mathbb{F}$.

If G has no perfect matching then by Corollary 25.2.2, $\det A$ is identically zero and so the algorithm will always return False. Therefore there are no false positives.

If G does have a perfect matching then, by Corollary 25.2.2, $\det A$ is not identically zero. By the Schwartz-Zippel lemma (Lemma 25.1.3), the false negative probability is at most

$$\frac{\text{total degree}}{|S|} \leq \frac{n}{2n} = \frac{1}{2}.$$

□

Question 25.2.5. How can we decrease the failure probability to δ?

Answer.

One approach is to use probability amplification by repeated trials, as in Chapter ??. Another approach is to increase S (and $\mathbb{F}$) to have size at least n/δ.

25.3 Exercises

Exercise 25.1. Let $a = (a_1, \ldots, a_s)$ be a string of characters in $[\![q]\!]$, Let p be a prime with $p \geq q$. Let

$$h(a) \;=\; \Big(\sum_{i=1}^{s} a_i X_i + Y\Big) \bmod p.$$

be a linear hash function as defined in (14.1.1). Use Lemma 25.1.3 to prove that collisions are unlikely:

$$\Pr\left[\, h(a) = h(b) \,\right] \;\leq\; \frac{1}{p}.$$

This is similar to Corollary 14.1.7.

Exercise 25.2. Give a coRP algorithm for the PIT problem under the assumption that $|\mathbb{F}| > d$.

Exercise 25.3. Give an example of a field $\mathbb{F}$ and a polynomial $p(x_1, \ldots, x_n)$ such that p is *not* the zero polynomial, but nevertheless $p(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in \mathbb{F}$.

Conclude that, without the assumption $|\mathbb{F}| > d$, it is *impossible* to solve PIT when p is represented as a black box.

Exercise 25.4. Let $p(x_1, \ldots, x_n)$ be a polynomial over $\mathbb{F}_2$, represented explicitly with a formula. Prove that deciding whether there exist $x_1, \ldots, x_n \in \mathbb{F}_2$ such that $p(x_1, \ldots, x_n) \neq 0$ is NP-hard.

Exercise 25.5. Suppose that $p(x_1, \ldots, x_n)$ is represented as an explicit formula (using addition, subtraction, multiplication and parentheses). Let d be the total degree of p.

If $d \geq |\mathbb{F}|$, then Lemma 25.1.3 does not give any useful information about the number of roots. Despite that, give a coRP algorithm for PIT.

Exercise 25.6. Algorithm 25.2 is a randomized, polynomial time algorithm for deciding if a bipartite graph has a *perfect* matching.

Part I. Give a randomized, polynomial time algorithm for deciding if a bipartite graph has a matching of size at least k. You should use or modify Algorithm 25.2.

Note: k can depend on n.

Part II. BONUS: Give a randomized algorithm to find the size of a maximum matching in $O(n^{2.38})$ time.

Exercise 25.7. Let A, B, C be $n \times n$ matrices with entries in any field $\mathbb{F}$. We would like to decide if $A \cdot B \stackrel{?}{=} C$. Of course this can be solved exactly in $O(n^3)$ time by matrix multiplication. The following interesting algorithm has runtime only $O(n^2)$.

Algorithm 25.3 An algorithm for deciding if $A \cdot B \stackrel{?}{=} C$. Assume that $\mathbb{F}$ is a finite field.

```
1: function MatMult(matrix A, B, C)
2:     Pick vectors r and s in F^n uniformly at random
3:     if (r^T A) · (Bs) = r^T C s then return True
4:     else return False
5: end function
```

Part I. Define a polynomial p with $2n$ variables such that $A \cdot B \neq C$ if and only if p is not identically zero.

Part II. Assuming that $\mathbb{F}$ is finite with $|\mathbb{F}| > 2$, prove that MatMult can decide if $A \cdot B \stackrel{?}{=} C$ with failure probability at most $3/4$.

Part III. How can the algorithm be modified to handle the case $\mathbb{F} = \mathbb{R}$?

Part IV. BONUS: Even if $|\mathbb{F}| = 2$, show that the failure probability is still at most $3/4$.

Chapter 26

Martingales

Probability begins with the definition of independence.

"Probability: Theory and Examples", R. Durrett

26.1 Introduction

When one is faced with multiple random variables, perhaps the first question one should ask is whether they are independent. If so, this may simplify their analysis because they can be analyzed separately. Moreover, it is simpler to *generate* independent random variables; for example, they can be generated simultaneously, possibly even on different computers.

Computer programs running on a single machine have a temporal aspect that is not well captured by the notion of independence. For example, a program might generate at each time step a "new" random number, but one whose distribution somehow depends on the previous random numbers.

One common model for sequences of dependent random variables is a ***Markov chain***. These are typically used to model random sequences of "states" (categorical RVs), in which the distribution of the next state is determined once the current state is known.

References: (Anderson et al., 2017, Definition 10.42), (Motwani and Raghavan, 1995, Section 6.2), (Mitzenmacher and Upfal, 2005, Section 7.1), (Grimmett and Stirzaker, 2001, Section 6.1), Wikipedia.

In contrast, ***martingales*** are sequences of *real-valued* RVs in which the next RV can depend arbitrarily on all previous RVs; however, they must satisfy

the expectation of the next RV, conditioned on all previous RVs, must equal the current RV.

This is of course a vague description, but it conveys the key ideas and allows us to discuss some examples.

Question 26.1.1. Let $S_0, S_1, \ldots, S_T \in \{\textsc{Sunny}, \textsc{Cloudy}\}$ be random variables satisfying

$$\begin{aligned} \Pr[\, S_0 = \textsc{Sunny}\,] &= 1/2 \\ \Pr[\, S_0 = \textsc{Cloudy}\,] &= 1/2 \\ \Pr[\, S_t = S_{t-1}\,] &= 2/3 \qquad \forall t \in [T] \\ \Pr[\, S_t \neq S_{t-1}\,] &= 1/3 \qquad \forall t \in [T]. \end{aligned}$$

Are these a Markov chain or a martingale?

Answer.

These are categorical random variables, so they cannot be a martingale. They are a Markov chain because the distribution of S_t is completely determined by the value of S_{t-1}.

Example 26.1.2. Let $S_0, S_1, \ldots, S_T \in \mathbb{Z}$ satisfy

$$\begin{aligned} \Pr[\, S_0 = 1 \,] &= 1/2 \\ \Pr[\, S_0 = 2 \,] &= 1/2 \\ \Pr[\, S_t - S_{t-1} = S_0 \,] &= 1/2 \quad \forall t \in [T] \\ \Pr[\, S_t - S_{t-1} = -S_0 \,] &= 1/2 \quad \forall t \in [T] \end{aligned}$$

Are these a Markov chain or a martingale?

These are a martingale because

$$\mathrm{E}[\, S_t \mid S_0, \ldots, S_{t-1} \,] = \frac{1}{2} \cdot (S_{t-1} + S_0) + \frac{1}{2} \cdot (S_{t-1} - S_0) = S_{t-1}.$$

However, they are not a Markov chain. For example, suppose that $S_1 = 0$. Then S_2 might take values ± 1 with probability $1/2$ (if $S_0 = 1$), or might take values ± 2 with probability $1/2$ (if $S_0 = 2$). Since the distribution cannot be determined from S_1 alone, they do not form a Markov chain.

Question 26.1.3. Let $S_0, S_1, \ldots, S_T \in \mathbb{Z}$ be a ***standard random walk***, meaning that $S_0 = 0$ and

$$\Pr[\, S_t - S_{t-1} = \pm 1 \,] = 1/2 \quad \forall t \in [T]$$

Are these a Markov chain or a martingale?

Answer.

They are both! They are a Markov chain because the distribution of S_t depends only on S_{t-1}. They are a martingale because

$$\mathrm{E}[\, S_t \mid S_0, \ldots, S_{t-1} \,] = (S_{t-1} - 1) \cdot \frac{1}{2} + (S_{t-1} + 1) \cdot \frac{1}{2} = S_{t-1}.$$

26.2 Definitions

Consider a randomized algorithm that runs at discrete time steps $t \in [T]$. We will keep track of the ***full history***[1] of its random variables as follows.

Full history: F_t = (all random variables that have been generated by the end of time t) $\forall t \in [T]$.

As more RVs are generated over time, we have the obvious[2] containment

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_T.$$

Typically we set $F_0 = \emptyset$ for notational convenience.

With respect to this full history, a ***martingale*** is a sequence of random variables

$$S_0,\ S_1,\ S_2,\ \ldots,\ S_T$$

satisfying three conditions.

[1] A reader comfortable with measure theoretic probability will notice that this concept (and its notation) is building toward the idea of a filtration (Grimmett and Stirzaker, 2001, page 473) (Klenke, 2008, Definition 9.9).

[2] We will use the notation of sets for operations on F_t, although strictly speaking they should be viewed as vector-valued random variables.

- To rule out pathological situations, we will require that $\mathrm{E}[\,|S_t|\,]$ is finite for all t.

- The value of S_t is completely determined by time t. In other words, the value of S_t is completely determined by the vector F_t.

- Given everything that is known at time $t-1$, we expect that S_t will equal S_{t-1}. In symbols,

$$\mathrm{E}[\,S_t \mid F_{t-1}\,] \;=\; S_{t-1}. \tag{26.2.1}$$

References: (Motwani and Raghavan, 1995, Definition 4.11), (Grimmett and Stirzaker, 2001, Definition 12.1.8), (Roch, 2020, Definition 3.30), (Durrett, 2019, Section 4.2), (Klenke, 2008, Definition 9.24), Wikipedia.

The theory of martingales becomes much richer if we consider *infinite* sequences of random variables. However in computer science applications that is not always necessary, and we can illustrate many of the key ideas by focusing on the case of finite[3] sequences.

26.2.1 Background on expectations

In order to understand martingales, it is important to have a solid understanding of conditional expectations. Defining conditional expectations for continuous RVs requires considerable care[4], so we will assume for simplicity that all RVs are discrete. Let us first review the definitions.

Definition 26.2.1 (Expectation conditioned on an event). Without loss of generality we consider events of the form "$B = b$", where B is a RV and b is a fixed constant. The conditional expectation $\mathrm{E}[\,A \mid B = b\,]$ is a non-random scalar. It is defined to equal

$$\mathrm{E}[\,A \mid B = b\,] \;=\; \sum_a a \cdot \Pr[\,A = a \mid B = b\,]. \tag{26.2.2}$$

References: (Lehman et al., 2018, Definition 19.4.5), (Anderson et al., 2017, Definitions 10.2 and 10.7).

Definition 26.2.2 (Expectation conditioned on a RV). $\mathrm{E}[\,A \mid B\,]$ is a ***random variable*** that takes the value $\mathrm{E}[\,A \mid B = b\,]$ whenever $B = b$.

References: (Anderson et al., 2017, Definition 10.23), (Mitzenmacher and Upfal, 2005, Definition 2.7), (Motwani and Raghavan, 1995, Definition 4.4), (Grimmett and Stirzaker, 2001, Definition 3.7.3).

Expectation of conditional expectation. Since $\mathrm{E}[\,A \mid B\,]$ is a random variable, we can take its expectation. This amounts to eliminating the conditioning on B.

Claim 26.2.3 (Law of total expectation). $\mathrm{E}[\,\mathrm{E}[\,A \mid B\,]\,] = \mathrm{E}[\,A\,]$.

[3] In particular, this restricts our attention to *uniformly integrable martingales/Doob martingales*. See, e.g., (Grimmett and Stirzaker, 2001, Example 12.3.9), (Durrett, 2019, Section 4.6), (Klenke, 2008, Theorem 11.7).

[4] See, e.g., (Grimmett and Stirzaker, 2001, Section 10.2 and 10.4), (Grimmett and Stirzaker, 2001, Section 4.6), (Durrett, 2019, Section 4.1), (Klenke, 2008, Chapter 8).

Proof.

$$\begin{aligned}
\mathrm{E}[\,\mathrm{E}[A \mid B]\,] &= \sum_b \mathrm{E}[A \mid B=b]\cdot\Pr[B=b] && \text{(by Definition A.3.9)}\\
&= \sum_b \Big(\sum_a a\cdot\Pr[A=a \mid B=b]\Big)\cdot\Pr[B=b] && \text{(by (26.2.2))}\\
&= \sum_a a\cdot\Big(\sum_b \Pr[A=a \wedge B=b]\Big) && \text{(by Definition A.3.4)}\\
&= \sum_a a\cdot\Pr[A=a] && \text{(by Fact A.3.6)}\\
&= \mathrm{E}[A]. \quad \square
\end{aligned}$$

References: (Anderson et al., 2017, Equation (10.28) and (10.29)), (Mitzenmacher and Upfal, 2005, Theorem 2.7), (Grimmett and Stirzaker, 2001, Theorem 3.7.4), (Motwani and Raghavan, 1995, Lemma 4.9), (McDiarmid, 1998, equation (31)), Wikipedia.

In fact, the same argument works even if all expectations are conditioned on C.

Claim 26.2.4 (Tower property). $\mathrm{E}[\,\mathrm{E}[A \mid B,C]\ \mid C] = \mathrm{E}[A \mid C]$.

References: (Motwani and Raghavan, 1995, Lemma 4.14), (Durrett, 2019, Theorem 4.1.13(ii)), (Klenke, 2008, Theorem 8.14(iv)), (Grimmett and Stirzaker, 2001, Exercise 3.7.1(f)), (McDiarmid, 1998, equation (30)), Wikipedia.

With ordinary expectations, any constants can be pulled outside due to linearity of expectation. For example, $\mathrm{E}[aX] = a\,\mathrm{E}[X]$ if a is a constant. Interestingly, when using conditional expectations, it is possible to pull certain *random variables* outside of the expectation, since conditional expectations are themselves random variables.

Claim 26.2.5 (Pulling out known factors). Let A, B and C be random variables such that A is completely determined by C. (That is, $A = f(C)$ for some function f.) Then

$$\mathrm{E}[A\cdot B \mid C] \;=\; A\cdot\mathrm{E}[B \mid C].$$

References: (Anderson et al., 2017, Equation (10.36)), (Grimmett and Stirzaker, 2001, Exercise 3.7.1(e)), (Klenke, 2008, Theorem 8.14(iii)).

26.2.2 An equivalence for martingales

Martingales can be characterized as sequences of RVs that give the best guess about a future RV.

> *"for finite [sequences] all corresponding martingales may be obtained in this way."*
>
> *Colin McDiarmid, "Concentration".*

Theorem 26.2.6 (Informal).

For all t, S_t is the best guess for S_T, given everything known at time t

$\iff$ *$S_0,\ldots,S_T$ is a martingale*

To prove this we will separately consider the two directions. First we show that a martingale is completely determined by the final RV S_T, together with the sequence $F_0,\ldots,F_T$.

Claim 26.2.7. For any martingale $S_0, S_1, \ldots, S_T$,

$$S_t \;=\; \mathrm{E}[S_T \mid F_t] \qquad \forall t \le T.$$

References: (Durrett, 2019, Theorem 4.2.5), (Grimmett and Stirzaker, 2001, Exercise 12.1.2), (Klenke, 2008, Remark 9.27).

Proof. Observe that F_k, the RVs generated up to time k, can be regarded as a *pair* of RVs

$$F_k \;=\; F_{k-1},\; F_k \backslash F_{k-1}, \tag{26.2.3}$$

of which the latter describes the new RVs generated *at* time k.

The martingale property (26.2.1) says that

$$\mathrm{E}\left[\, S_T \;\middle|\; F_{T-1} \,\right] \;=\; S_{T-1}. \tag{26.2.4}$$

This proves the claim for $t = T-1$.

Now take the expectation conditioned on F_{T-2}, to obtain

$$\begin{aligned}
\mathrm{E}\left[\, S_T \;\middle|\; F_{T-2} \,\right] &= \mathrm{E}\left[\, \mathrm{E}\left[\, S_T \;\middle|\; F_{T-2},\, F_{T-1} \backslash F_{T-2} \,\right] \;\middle|\; F_{T-2} \,\right] && \text{(by Claim 26.2.4)} \\
&= \mathrm{E}\left[\, \mathrm{E}\left[\, S_T \;\middle|\; F_{T-1} \,\right] \;\middle|\; F_{T-2} \,\right] && \text{(by (26.2.3))} \\
&= \mathrm{E}\left[\, S_{T-1} \;\middle|\; F_{T-2} \,\right] && \text{(by (26.2.4))} \\
&= S_{T-2} && \text{(by (26.2.1)).}
\end{aligned}$$

This proves the claim for $t = T-2$. Iterating this argument proves the claim for all t. □

It is worth calling attention to the following special case.

Corollary 26.2.8. If $S_0, S_1, \ldots, S_T$ is a martingale then

$$\mathrm{E}\left[\, S_T \,\right] \;=\; \mathrm{E}\left[\, S_0 \,\right].$$

Moreover, if S_0 is not random (which would hold if $F_0 = \emptyset$, for example), then $\mathrm{E}\left[\, S_T \,\right] = S_0$.

References: (Mitzenmacher and Upfal, 2005, Lemma 12.1).

Proof. Simply apply Claim 26.2.7 with $t = 0$ then take the unconditional expectation. □

The following claim gives the other direction of Theorem 26.2.6.

Claim 26.2.9. Let S be any random variable with $\mathrm{E}\left[\, |S| \,\right]$ finite. Define

$$S_t \;=\; \mathrm{E}\left[\, S \;\middle|\; F_t \,\right] \qquad \forall t \in \{0, \ldots, T\}.$$

Then $S_0, S_1, \ldots, S_T$ is a martingale.

The proof is left as Exercise 26.1.

References: (Motwani and Raghavan, 1995, Theorem 4.13), (Roch, 2020, Example 3.34), (Klenke, 2008, Exercise 9.2.1).

26.2.3 Relaxing to inequalities

Sometimes we encounter random variables that satisfy (26.2.1) with *in*equality instead of equality. These can still be useful, so they deserve their own names.

$$\begin{aligned}
\text{A } \textbf{\textit{supermartingale}} \text{ satisfies:} \quad & \mathrm{E}\left[\, S_t \;\middle|\; F_{t-1} \,\right] \;\leq\; S_{t-1} \qquad \forall t \in [T] \\
\text{A } \textbf{\textit{submartingale}} \text{ satisfies:} \quad & \mathrm{E}\left[\, S_t \;\middle|\; F_{t-1} \,\right] \;\geq\; S_{t-1} \qquad \forall t \in [T].
\end{aligned}$$

These names might be the opposite of what you might expect: a supermartingale goes *down* in expectation, whereas a submartingale goes *up* in expectation. Oh well — the names are firmly established, and we're stuck with them.

References: (Grimmett and Stirzaker, 2001, Definition 12.1.9), (Motwani and Raghavan, 1995, Definition 4.7), (Durrett, 2019, Section 4.2), (Roch, 2020, Definition 3.30), (Klenke, 2008, Definition 9.24), Wikipedia.

The following claim gives a trivial generalization of Corollary 26.2.8 to these settings as well.

Claim 26.2.10. Suppose that $S_0, S_1, \ldots, S_T$ is:

$$\begin{aligned} \text{a } \textbf{\textit{supermartingale}}. \text{ Then} \quad & \mathrm{E}[S_T] \leq \mathrm{E}[S_0] \\ \text{a } \textbf{\textit{submartingale}}. \text{ Then} \quad & \mathrm{E}[S_T] \geq \mathrm{E}[S_0]. \end{aligned}$$

26.3 Azuma's Inequality

For our purposes, the main appeal of martingales is that they have powerful concentration properties, despite lacking independence. This is one reason that martingales find many uses in computer science.

The main martingale concentration result we will use is Azuma's inequality, which is shown in the following theorem. It is a generalization[5] of Hoeffding's inequality, which we discussed in Section 9.3 and Section 21.3.

Theorem 26.3.1. Let $S_0, S_1, \ldots, S_n$ be a martingale, as in Section 26.2. Suppose that $|S_t - S_{t-1}| \leq 1$ for all $t \in [n]$. For any $\alpha \geq 0$,

$$\begin{aligned} & \Pr[S_n \geq S_0 + \alpha] \leq \exp(-\alpha^2/2n) \\ \text{and} \quad & \Pr[S_n \leq S_0 - \alpha] \leq \exp(-\alpha^2/2n). \\ \text{Combining these} \quad & \Pr[|S_n - S_0| \geq \alpha] \leq 2\exp(-\alpha^2/2n). \end{aligned}$$

References: (McDiarmid, 1998, Theorem 3.10), (Cesa-Bianchi and Lugosi, 2006, Lemma A.7), (Motwani and Raghavan, 1995, Theorem 4.16), (Mitzenmacher and Upfal, 2005, Theorem 12.6), (Alon and Spencer, 2000, Theorem 7.2.1), (Roch, 2020, Theorem 3.52), (Grimmett and Stirzaker, 2001, Theorem 12.2.3), (Wainwright, 2019, Corollary 2.20), (Klenke, 2008, Exercise 9.2.4), Wikipedia.

26.4 Applications of Azuma's Inequality

26.4.1 Balls and bins, Bloom filters

Our first application is intended to illustrate two ideas.

- An approach to show strong concentration bounds for a RV that is *not* a sum of independent RVs.
- An argument that our best guess for a RV does not change much over time.

Consider the following algorithm that throws b balls independently into m bins. The number X of non-empty bins is returned at the end.

[5] In fact, this generalization was known to Hoeffding. In his paper, he wrote "the inequalities... remain true [with] the weaker assumption that the sequence... is a martingale."

```
1: function ThrowBalls(int b, int m)
2:     Create array B[1..m] of Booleans, initially all False
3:     for t = 1, ..., b
4:         Generate Y_t independently and uniformly in [m]        ▷ The bin for ball t
5:         Set B[Y_t] ← True
6:     Compute X, the number of True entries of B
7:     return X
8: end function
```

We remark that X can be expressed as

$$X \;=\; \text{NonEmpty}(Y) \;=\; |\{\, Y_t \,:\, t \in [b] \,\}|.$$

So the code needn't maintain the array B; that is just a conceptual convenience.

Our Bloom filter discussion in Section 12.4 relies on a balls and bins analysis. Specifically, when $b = 1.45m$ we showed in Claim 12.4.13 that $\mathrm{E}[\,X\,] < 0.499m$. The runtime analysis of the Bloom filter initialization relies on showing that $\Pr[\,X \geq m/2\,]$ is small. Since X is not a sum of independent RVs, we cannot use the Chernoff or Hoeffding inequalities. Nevertheless, we can give a tail bound for X using Azuma's inequality.

Theorem 26.4.1.

$$\Pr\left[\, X - \mathrm{E}[\,X\,] \geq c\sqrt{b} \,\right] \;\leq\; \exp(-c^2/2).$$

Following Section 26.2, we let F_t be all RVs created up to the end of iteration t of the outer loop. Then the full history is as follows.

$$\begin{aligned}
F_0 &= \emptyset \\
F_1 &= (Y_1) \\
F_2 &= (Y_1, Y_2) \\
F_3 &= (Y_1, Y_2, Y_3) \\
&\vdots \;=\; \vdots \\
F_t &= (\, Y_i \,:\, i \leq t \,) \qquad \forall t \in [b].
\end{aligned}$$

Our best guess for X changes over time as the full history unfolds. Define

$$S_t \;=\; \mathrm{E}[\,X \mid F_t\,] \qquad \forall t \in \{0, \ldots, b\}\,.$$

Then $S_0, S_1, \ldots, S_b$ is a martingale, by Claim 26.2.9.

Note that F_b contains all generated random variables. So the expectation conditioned on F_b does nothing at all — all RVs are known once F_b is known. Thus

$$S_n \;=\; \mathrm{E}[\,X \mid F_b\,] \;=\; X.$$

On the other hand F_0 contains no random variables. So the expectation conditioned on F_0 is just the *unconditional* expectation. Thus S_0 is *not* random, and simply equals

$$S_0 \;=\; \mathrm{E}[\,X \mid F_0\,] \;=\; \mathrm{E}[\,X\,].$$

A key observation is that the best guess for X changes by at most 1 in each time step.

Claim 26.4.2. Fix any $t \in [b]$. Then $|S_t - S_{t-1}| \leq 1$.

Proof sketch. Recall that

$$\begin{aligned} S_{t-1} &= \mathrm{E}\,[\,X \mid Y_1, \ldots, Y_{t-1}\,] \\ S_t &= \mathrm{E}\,[\,X \mid Y_1, \ldots, Y_{t-1}, Y_t\,]. \end{aligned}$$

Choosing a location for ball t can affect the number of non-empty bins by at most 1, so $|S_t - S_{t-1}| \leq 1$. □

Proof. Let $Y_1, \ldots, Y_b$ be the random choices generated by ThrowBalls(b, m). Now let $Z_1, \ldots, Z_b$ be a copy of Y where we generate a new random value for Z_t. Clearly

$$|\mathrm{NonEmpty}(Y) - \mathrm{NonEmpty}(Z)| \;\leq\; 1 \tag{26.4.1}$$

since only one ball has changed its location.

Now we will take the expectation conditioned on $F_t = \{Y_1, \ldots, Y_t\}$, but *not* conditioning on the new value Z_t. The key observation is that

$$\mathrm{E}\,[\,\mathrm{NonEmpty}(Z) \mid F_t\,] \;=\; \mathrm{E}\,[\,\mathrm{NonEmpty}(Y) \mid F_{t-1}\,]. \tag{26.4.2}$$

This is because Z has the same distribution as Y, and we have *not* conditioned on Z_t. We get

$$\begin{aligned} 1 &\geq \mathrm{E}\,[\,|\mathrm{NonEmpty}(Y) - \mathrm{NonEmpty}(Z)| \mid F_t\,] && \text{(by (26.4.1))} \\ &\geq |\,\mathrm{E}\,[\,\mathrm{NonEmpty}(Y) \mid F_t\,] - \mathrm{E}\,[\,\mathrm{NonEmpty}(Z) \mid F_t\,]\,| && \text{(by Jensen's inequality, Fact B.4.2)} \\ &= |\mathrm{E}\,[\,\mathrm{NonEmpty}(Y) \mid F_t\,] - \mathrm{E}\,[\,\mathrm{NonEmpty}(Y) \mid F_{t-1}\,]| && \text{(by (26.4.2))} \\ &= |S_t - S_{t-1}|. \quad \square \end{aligned}$$

Claim 26.4.2 shows that the hypotheses of Azuma's inequality (Theorem 26.3.1) are satisfied. We obtain

$$\Pr\left[\, X - \mathrm{E}\,[\,X\,] \geq c\sqrt{b} \,\right] \;=\; \Pr\left[\, S_b - S_0 \geq c\sqrt{b} \,\right] \;\leq\; \exp(-c^2/2).$$

This proves Theorem 26.4.1.

26.4.2 Vertex coloring

Our next application is intended to illustrate two ideas.

- It can be useful for RVs to enter the full history in a way that is not the most obvious.
- It is possible to show that a RV is concentrated around some value, without knowing what that value is!

A ***vertex coloring*** of a graph $G = (V, E)$ is a function $f : V \to [c]$ such that $f(u) \neq f(v)$ for all $uv \in E$. The ***chromatic number*** of G is the minimum integer c for which a coloring exists. This is often denoted $\chi(G)$.

Consider the following algorithm for computing the chromatic number of a random graph. The algorithm is not interesting, it just helps to understand the temporal aspect of the random variables.

Algorithm 26.1 An algorithm that generates a random graph in which each edge is added independently with probability p, then computes its chromatic number.

```
1: function RandGraph(int n, float p)
2:     Create graph G = (V, E) with V ← [n] and E ← ∅
3:     for t = 1, ..., n
4:         for s = 1, ..., t − 1
5:             Add edge st to E with probability p
6:     Compute χ(G) by exhaustive search
7:     return χ(G)
8: end function
```

Our main result shows that $\chi(G)$ is tightly concentrated around its expectation. Mysteriously, the theorem does not reveal what that expectation is.

Theorem 26.4.3.

$$\Pr\left[\,|\chi(G) - \mathrm{E}\left[\,\chi(G)\,\right]| \geq c\sqrt{n}\,\right] \;\leq\; 2\exp(-c^2/2).$$

References: (Alon and Spencer, 2000, Theorem 7.2.4), (Mitzenmacher and Upfal, 2005, Section 12.5.4), (Motwani and Raghavan, 1995, Exercise 4.11).

Following Section 26.2, we let F_t be all RVs created up to the end of iteration t of the outer loop. Let $X_{uv} \in \{0,1\}$ be the RV that indicates if edge $uv \in E$. Then the full history is as follows.

$$\begin{aligned}
F_0 &= \emptyset \\
F_1 &= \emptyset \qquad \text{(no edges are added when } t = 1) \\
F_2 &= (X_{12}) \\
F_3 &= (X_{12}, X_{13}, X_{23}) \\
&\vdots \;=\; \vdots \\
F_t &= (\, X_{uv} \;:\; 1 \leq u < v \leq t \,) \qquad \forall t \in [n].
\end{aligned}$$

Our best guess for $\chi(G)$ changes over time as the full history unfolds. Define

$$S_t \;=\; \mathrm{E}\left[\,\chi(G) \;\middle|\; F_t\,\right] \qquad \forall t \in \{0, \ldots, n\}\,.$$

Then $S_0, S_1, \ldots, S_n$ is a martingale, by Claim 26.2.9. Note that S_n (which equals $\chi(G)$) is a RV giving the chromatic number of a random graph with parameters n and p. On the other hand S_0 (which equals S_1) is *not* random and simply equals $\mathrm{E}\left[\,\chi(G)\,\right]$.

The key is to understand how much the best guess for $\chi(G)$ changes at each time step.

Claim 26.4.4. Fix any $t \in [n]$. Then $|S_t - S_{t-1}| \leq 1$.

Proof sketch. Recall that

$$\begin{aligned}
S_{t-1} &= \mathrm{E}\left[\,\chi(G) \;\middle|\; F_{t-1}\,\right] \\
S_t &= \mathrm{E}\left[\,\chi(G) \;\middle|\; F_t\,\right].
\end{aligned}$$

Choosing the edges for vertex t affects the chromatic number by at most one, because we can always give t a new color. □

Proof. Let G be the random graph generated by $\textsc{RandGraph}(n,p)$. Now let H be a copy of G where we generate fresh independent samples for all edges in $E_t = \{\, st \,:\, s < t \,\}$. Observe that

$$|\chi(G) - \chi(H)| \;\le\; 1. \tag{26.4.3}$$

This is because any coloring of G can be modified to a coloring of H by giving vertex t a new color, or vice versa. Now we take the expectation of (26.4.3) conditioned on the edges of G in F_t, but *not* conditioning on the resampled edges of H. We get

$$\begin{aligned} 1 \;&\ge\; \mathrm{E}\,[\,|\chi(G) - \chi(H)| \;\mid F_t\,] \\ &\ge\; |\,\mathrm{E}\,[\chi(G) \mid F_t\,] - \mathrm{E}\,[\chi(H) \mid F_t\,]\,|, \end{aligned}$$

by Jensen's inequality (Fact B.4.2). Now the key observation is that $\mathrm{E}\,[\chi(H) \mid F_t\,] = \mathrm{E}\,[\chi(G) \mid F_{t-1}\,]$. This is because H has the same distribution as G, and we have *not* conditioned on H's edges in E_t. So

$$\begin{aligned} 1 \;&\ge\; |\mathrm{E}\,[\chi(G) \mid F_t\,] - \mathrm{E}\,[\chi(G) \mid F_{t-1}\,]| \\ &=\; |S_t - S_{t-1}|. \end{aligned}$$

□

Claim 26.4.4 shows that the hypotheses of Azuma's inequality (Theorem 26.3.1) are satisfied. We obtain that

$$\Pr\left[\,|\chi(G) - \mathrm{E}\,[\chi(G)]| \ge c\sqrt{n}\,\right] \;=\; \Pr\left[\,|S_n - S_0| \ge c\sqrt{n}\,\right] \;\le\; 2\exp(-c^2/2).$$

This proves Theorem 26.4.3.

26.5 Proof of Azuma's inequality

Let $\lambda > 0$ be an arbitrary parameter. A useful viewpoint is to define the following random variables.

$$X_t \;=\; \exp\left(\lambda(S_t - S_0) - \lambda^2 t/2\right) \qquad \forall t \in \{0,\ldots,n\}$$

This might seem a strange definition, but there are some fundamental reasons why it is very natural. In more advanced contexts, it is called the stochastic exponential.

The key is the following claim, after which the remainder of the proof is straightforward.

Claim 26.5.1. The RVs $X_0, X_1, \ldots, X_n$ are a *super*martingale.

Now we follow Chernoff's approach of Section 21.2.1.

$$\begin{aligned} \Pr\,[\,S_n - S_0 \ge \alpha\,] \;&=\; \Pr\left[\,\exp\left(\lambda(S_n - S_0) - \lambda^2 n/2\right) \ge \exp(\lambda\alpha - \lambda^2 n/2)\,\right] && \text{(by monotonicity)} \\ &\le\; \frac{\mathrm{E}\,[X_n]}{\exp(\lambda\alpha - \lambda^2 n/2)} && \text{(by Markov's inequality)} \\ &\le\; \frac{\mathrm{E}\,[X_0]}{\exp(\lambda\alpha - \lambda^2 n/2)} && \text{(by Claims 26.5.1 and 26.2.10)} \\ &=\; \exp(-\lambda\alpha + \lambda^2 n/2) && \text{(since } X_0 = 1) \\ &=\; \exp(-\alpha^2/n + \alpha^2/2n) \;=\; \exp(-\alpha^2/2n) && \text{(by plugging in } \lambda = \alpha/n). \end{aligned}$$

This completes the proof of Theorem 26.3.1. It remains to prove the claim.

Proof of Claim 26.5.1. Consider any time t. Then, by definition of X_t,

$$\mathrm{E}\left[\, X_t \mid F_{t-1} \,\right] \;=\; \mathrm{E}\left[\, \exp\left(\lambda(S_t - S_0) - \lambda^2 t/2\right) \mid F_{t-1} \,\right]$$

Observe that the random variables S_{t-1} and S_0 are completely determined by F_{t-1}, and so $e^{\lambda(S_{t-1}-S_0)-\lambda^2 t/2}$ is also. By Claim 26.2.5 we can pull that outside the expectation.

$$= \; e^{\lambda(S_{t-1}-S_0)-\lambda^2 t/2} \cdot \mathrm{E}\left[\, \exp\left(\lambda(S_t - S_{t-1})\right) \mid F_{t-1} \,\right]$$

Since $|S_t - S_{t-1}| \le 1$ and $\mathrm{E}\left[\, S_t - S_{t-1} \mid F_{t-1} \,\right] = 0$, the hypotheses of Hoeffding's lemma (Lemma 21.3.2) are satisfied, so we obtain the upper bound

$$\le \; e^{\lambda(S_{t-1}-S_0)-\lambda^2 t/2} \cdot e^{\lambda^2/2} \;=\; X_{t-1}.$$

This shows that $X_0, \ldots, X_n$ form a supermartingale. □

26.6 Exercises

Exercise 26.1. Prove Claim 26.2.9.

Exercise 26.2. Let $S_0, S_1, \ldots, S_n$ be a martingale, as in Section 26.2. Suppose that there are non-random values $c_1, \ldots, c_n$ such that $|S_t - S_{t-1}| \le c_t$ for all $t \in [n]$. Define

$$X_t \;=\; \exp\left(\lambda(S_t - S_0) - \frac{\lambda^2}{2}\sum_{i=1}^{t} c_i^2\right) \qquad \forall t \in \{0, \ldots, n\}\,.$$

Prove that $X_0, \ldots, X_n$ form a supermartingale.

Chapter 27

Gradient Descent

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. We will assume that $\min_{x \in \mathcal{X}} f(x)$ is achieved by some point x^*. Our objective is to find a point $x \in \mathcal{X}$ for which $f(x) - f(x^*)$ is small.

TODO: Define subgradients.

Since f is convex, we have $\partial f(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$. Throughout this chapter we will use the Euclidean norm, and we will assume that f is 1-Lipschitz. This condition can be equivalently stated in terms of subgradients (via Fact B.3.8), which implies

$$\|g\|_2 \leq 1 \qquad \forall x \in \mathcal{X},\ g \in \partial f(x). \tag{27.0.1}$$

27.1 Unconstrained gradient descent

The algorithm is shown in Algorithm 27.1.

Algorithm 27.1 Gradient descent for minimizing a convex, 1-Lipschitz function over $\mathbb{R}^n$.

1: **procedure** GradientDescent(vector $x_1 \in \mathbb{R}^n$, int T)
2: Let $\eta = 1/\sqrt{T}$
3: **for** $i \leftarrow 1, \ldots, T$ **do**
4: $x_{i+1} \leftarrow x_i - \eta g_i$, where $g_i = \nabla f(x_i)$ if f is differentiable, or
5: g_i is any subgradient in $\partial f(x_i)$ if f is non-differentiable
6: **return** $\sum_{i=1}^{T} x_i / T$

Remark 27.1.1. Observe that the only way that Algorithm 27.1 accesses the function f is in line 4. We assume that f is presented to the algorithm as a ***subgradient oracle***, which receives a point $x \in \mathbb{R}^n$, and returns any subgradient $g \in \partial f(x)$.

Theorem 27.1.2. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and 1-Lipschitz. Fix an optimal solution $x^* \in \operatorname{argmin}_x f(x)$ and a starting point $x_1 \in \mathbb{R}^n$. Define $\eta = \frac{1}{\sqrt{T}}$. Suppose that $\|x_1 - x^*\|_2 \leq 1$. Then

$$f\left(\frac{1}{T}\sum_{i=1}^{T} x_i\right) - f(x^*) \ \leq \ \frac{1}{\sqrt{T}}.$$

References: (Bubeck, 2015, Theorem 3.2), (Shalev-Shwartz and Ben-David, 2014, Corollary 14.2, §14.2.3).

Proof. We bound the error on the i^{th} iteration as follows:

$$\begin{aligned} f(x_i) - f(x^*) &\le \langle\, g_i,\, x_i - x^* \,\rangle \qquad \text{(by the subgradient inequality (B.3.1))} \\ &= \frac{1}{\eta}\langle\, x_i - x_{i+1},\, x_i - x^* \,\rangle \qquad \text{(by the gradient step in line 4)} \\ &= \frac{1}{2\eta}\Big(\|x_i - x_{i+1}\|_2^2 + \|x_i - x^*\|_2^2 - \|x_{i+1} - x^*\|_2^2 \Big) \qquad \text{(by the cosine law (B.2.1)).} \end{aligned}$$

To analyze the average error, sum the previous displayed equation over i. The last two terms telescope, yielding

$$\begin{aligned} \sum_{i=1}^{T} \big(f(x_i) - f(x^*)\big) &\le \frac{1}{2\eta}\left(\Big(\sum_{i=1}^{T} \|x_i - x_{i+1}\|_2^2 \Big) + \|x_1 - x^*\|_2^2 - \|x_{T+1} - x^*\|_2^2 \right) \\ &\le \frac{1}{2\eta}\Big(\sum_{i=1}^{T} \|\eta g_i\|_2^2 + \|x_1 - x^*\|_2^2 \Big) \quad \text{(by the gradient step in line 4)} \\ &\le \frac{\eta T}{2} + \frac{1}{2\eta} \qquad \text{(by (27.0.1) and the assumption on } x_1) \end{aligned}$$

Dividing by T and using Jensen's inequality (Fact B.4.2) and the definition of η gives

$$f\left(\sum_{i=1}^{T} \frac{x_i}{T} \right) - f(x^*) \;\le\; \sum_{i=1}^{T} \frac{1}{T}\big(f(x_i) - f(x^*)\big) \;\le\; \frac{\eta}{2} + \frac{1}{2T\eta} \;=\; \frac{1}{\sqrt{T}},$$

as required. □

Remark 27.1.3. The error guarantee of Theorem 27.1.2 is optimal for *any* algorithm that only accesses f using a subgradient oracle (Bubeck, 2015, Theorem 3.13).

General reduction from arbitrary scaling. The analysis present above assumes that the given function f is 1-Lipschitz. How shall we handle a function that is L-Lipschitz? It also has a certain "scale assumption" $\|x_1 - x^*\|_2 \le 1$. How could we handle a general scale, say $\|x_1 - x^*\|_2 \le R$? In Section 27.4 we will discuss a general reduction that can handle such scenarios.

Theorem 27.1.4. Suppose that we have a theorem giving a convergence rate guarantee $c(T)$ for gradient descent assuming f is 1-Lipschitz and assuming the "scale" $\|x_1 - x^*\|_2 \le 1$. Suppose h is an L-Lipschitz function whose "scale" is bounded by R. Then there is a black-box reduction from h to f, showing that gradient descent on h achieves convergence rate $RL \cdot c(T)$.

27.2 Projected gradient descent

In this section we consider the problem $\min_{x \in \mathcal{X}} f(x)$ where $\mathcal{X}$ is a closed, convex set. Again, f is assumed to be convex and 1-Lipschitz.

The ordinary gradient descent algorithm does not ensure that the iterates remain in $\mathcal{X}$. In this section we modify the algorithm to project back onto $\mathcal{X}$. The algorithm now takes a gradient step from the iterate x_i to compute a new point y_{i+1}, then projects onto $\mathcal{X}$ to obtain the new iterate x_{i+1}.

The algorithm, shown in Algorithm 27.2, is a slight modification of Algorithm 27.1. The theorem is a slight modification of Theorem 27.1.2. The only changes are highlighted below.

Algorithm 27.2 Projected gradient descent for minimizing convex, 1-Lipschitz functions over a convex set.

1: **procedure** ProjectedGradientDescent(set $\mathcal{X} \subseteq \mathbb{R}^n$, vector $x_1 \in \mathcal{X}$, int T)
2: Let $\eta = 1/\sqrt{T}$
3: **for** $i \leftarrow 1, \ldots, T$ **do**
4: $y_{i+1} \leftarrow x_i - \eta g_i$, where $g_i \in \partial f(x_i)$.
5: $x_{i+1} \leftarrow \Pi_{\mathcal{X}}(y_{i+1})$
6: **return** $\sum_{i=1}^{T} x_i/T$

Theorem 27.2.1. Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and 1-Lipschitz (with respect to $\|\cdot\|_2$). Fix an optimal solution $x^* \in \operatorname{argmin}_{x \in \mathcal{X}} f(x)$ and a starting point $x_1 \in \mathcal{X}$. Define $\eta = \frac{1}{\sqrt{T}}$. Suppose that $\|x_1 - x^*\|_2 \le 1$. Then

$$f\left(\frac{1}{T}\sum_{i=1}^{T} x_i\right) - f(x^*) \;\le\; \frac{1}{\sqrt{T}}.$$

References: (Bubeck, 2015, Theorem 3.2), (Shalev-Shwartz and Ben-David, 2014, §14.4.1).

Proof. We bound the error on the i^{th} iteration as follows:

$$\begin{aligned}
f(x_i) - f(x^*) &\le \langle\, g_i,\, x_i - x^* \,\rangle && \text{(by the subgradient inequality (B.3.1))} \\
&= \frac{1}{\eta}\langle\, x_i - y_{i+1},\, x_i - x^* \,\rangle && \text{(by the gradient step in line 4)} \qquad (27.2.1) \\
&= \frac{1}{2\eta}\Big(\|x_i - y_{i+1}\|_2^2 + \|x_i - x^*\|_2^2 - \|y_{i+1} - x^*\|_2^2 \Big) && \text{(by the cosine law (B.2.1))} \\
&\le \frac{1}{2\eta}\Big(\|x_i - y_{i+1}\|_2^2 + \|x_i - x^*\|_2^2 - \|x_{i+1} - x^*\|_2^2 \Big).
\end{aligned}$$

The last line uses Fact B.3.10: since x_{i+1} is the projected point $\Pi_{\mathcal{X}}(y_{i+1})$ and $x^* \in \mathcal{X}$, we have $\|x_{i+1} - x^*\|_2^2 \le \|y_{i+1} - x^*\|_2^2$.

To analyze the average error, sum the previous displayed equation over i. The last two terms telescope, yielding

$$\begin{aligned}
\sum_{i=1}^{T} \big(f(x_i) - f(x^*)\big) &\le \frac{1}{2\eta}\left(\Big(\sum_{i=1}^{T} \|x_i - y_{i+1}\|_2^2\Big) + \|x_1 - x^*\|_2^2 - \|x_{T+1} - x^*\|_2^2 \right) \\
&\le \frac{1}{2\eta}\Big(\sum_{i=1}^{T} \|\eta g_i\|_2^2 + \|x_1 - x^*\|_2^2 \Big) \quad \text{(by the gradient step in line 4)} \\
&\le \frac{\eta T}{2} + \frac{1}{2\eta} \qquad \text{(by (27.0.1) and the assumption on } x_1)
\end{aligned}$$

Dividing by T and using Jensen's inequality (Fact B.4.2) and the definition of η gives

$$f\left(\sum_{i=1}^{T} \frac{x_i}{T}\right) - f(x^*) \;\le\; \sum_{i=1}^{T} \frac{1}{T}\big(f(x_i) - f(x^*)\big) \;\le\; \frac{\eta}{2} + \frac{1}{2T\eta} \;=\; \frac{1}{\sqrt{T}},$$

as required. □

27.3 Stochastic gradient descent

Stochastic gradient descent is a generalization of gradient descent in which we relax the notion of a subgradient oracle. Instead of requiring that the oracle return an *actual* subgradient, we will allow it to return a random vector that *in expectation* is a subgradient. (This is formalized below.) The stochastic gradient descent algorithm, shown in Algorithm 27.3, is a trivial modification of Algorithm 27.2 to use this stochastic oracle.

Algorithm 27.3 Stochastic gradient descent for minimizing a convex, Lipschitz function over a convex set $\mathcal{X}$.

1: **procedure** StochasticGradientDescent(set $\mathcal{X} \subseteq \mathbb{R}^n$, vector $x_1 \in \mathcal{X}$, int T)
2: Let $\eta = 1/\sqrt{T}$
3: **for** $i \leftarrow 1, \ldots, T$ **do**
4: Let $\hat{g}_i$ be a random vector obtained from the subgradient oracle at x_i
5: $y_{i+1} \leftarrow x_i - \eta\, \hat{g}_i$,
6: $x_{i+1} \leftarrow \Pi_{\mathcal{X}}(y_{i+1})$
7: **return** $\sum_{i=1}^{T} x_i/T$

The analysis of stochastic gradient descent is actually quite straightforward. Below we will prove Theorem 27.3.2, which is very similar to Theorem 27.2.1, it just requires some care with conditional expectations.

With that in mind, let us use the notation of Section 26.2, in which F_t consists of all RVs created up to the end of iteration t of the algorithm. Then the full history is as follows.

$$\begin{aligned} F_0 &= \emptyset \\ F_1 &= (\hat{g}_1) \\ F_2 &= (\hat{g}_1, \hat{g}_2) \\ F_3 &= (\hat{g}_1, \hat{g}_2, \hat{g}_3) \\ &\vdots \quad = \quad \vdots \\ F_t &= (\, \hat{g}_i \,:\, i \le t \,) \qquad \forall t \in [T]. \end{aligned} \tag{27.3.1}$$

Claim 27.3.1. For each $i \in [T]$,

- $\hat{g}_i$ is completely determined by F_i, and
- x_{i+1} is completely determined by F_i.

Proof sketch. The first statement is trivial from (27.3.1). By induction, x_i is determined by F_{i-1}. Thus, x_i and $\hat{g}_i$ are both determined by F_i, from which it follows that y_{i+1} and x_{i+1} are too. □

Randomized subgradient oracle: assumptions. To analyze the algorithm, we require two assumptions about the subgradient oracle. For notational convenience, we define

$$g_i \;=\; \mathrm{E}\,[\,\hat{g}_i \mid F_{i-1}\,].$$

Our first assumption is that

$$\underbrace{\mathrm{E}\left[\,\hat{g}_i \mid F_{i-1}\,\right]}_{=g_i} \;\in\; \partial f(x_i). \tag{27.3.2}$$

That is, the *expected* output of the oracle is a subgradient.

Note that both sides of (27.3.2) are completely determined by F_{i-1}. For the left, this follows from definition of conditional expectation (see Definition 26.2.2). For the right, this follows from Claim 27.3.1.

Our second assumption is that the vectors returned by the subgradient oracle are not too large. We formalize this assumption as:

$$\mathrm{E}\left[\,\|\hat{g}_i\|_2^2\,\right] \;\le\; 1 \qquad \forall i. \tag{27.3.3}$$

Theorem 27.3.2. Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex. Fix an optimal solution $x^* \in \operatorname{argmin}_{x\in\mathcal{X}} f(x)$ and a starting point $x_1 \in \mathcal{X}$. Define $\eta = \frac{1}{\sqrt{T}}$. Suppose that $\|x_1 - x^*\|_2 \le 1$. Under our assumptions on the subgradient oracle,

$$\mathrm{E}\left[\, f\left(\frac{1}{T}\sum_{i=1}^{T} x_i\right)\right] - f(x^*) \;\le\; \frac{1}{\sqrt{T}}.$$

References: (Shalev-Shwartz and Ben-David, 2014, Theorem 14.8), (Bubeck, 2015, Section 6.1).

Before proving the theorem, it is useful to define the *noise*.

$$\text{Noise in subgradient oracle:} \qquad \hat{z}_i \;=\; g_i - \hat{g}_i.$$

Lemma 27.3.3.

$$\mathrm{E}\left[\, f\Big(\frac{1}{T}\sum_{i=1}^{T} x_i\Big)\right] - f(x^*) \;\le\; \frac{1}{\sqrt{T}} + \frac{1}{T}\,\mathrm{E}\left[\,\sum_{i=1}^{T} \langle\, \hat{z}_i,\, x_i - x^* \,\rangle\right].$$

Proof. We bound the error on the i^{th} iteration as follows:

$$\begin{aligned}
f(x_i) - f(x^*) \;&\le\; \langle\, g_i,\, x_i - x^* \,\rangle \qquad \text{(by the subgradient inequality (B.3.1))} \\
&\le\; \langle\, \hat{g}_i,\, x_i - x^* \,\rangle + \langle\, \hat{z}_i,\, x_i - x^* \,\rangle \qquad \text{(by definition of } \hat{z}_i\text{)} \\
&=\; \frac{1}{\eta}\langle\, x_i - y_{i+1},\, x_i - x^* \,\rangle + \langle\, \hat{z}_i,\, x_i - x^* \,\rangle \qquad \text{(by the gradient step)}
\end{aligned}$$

The first term is identical to the quantity considered in the previous section, equation (27.2.1). Thus, summing over all T and telescoping, we obtain

$$\sum_{i=1}^{T} \big(f(x_i) - f(x^*)\big) \;\le\; \sum_{i=1}^{T} \frac{\|x_i - y_{i+1}\|_2^2}{2\eta} + \frac{\|x_1 - x^*\|_2^2}{2\eta} + \sum_{i=1}^{T} \langle\, \hat{z}_i,\, x_i - x^* \,\rangle$$

Now using that $x_i - y_{i+1} = \eta\hat{g}_i$ and $\|x_1 - x^*\| \le 1$, we obtain

$$\le\; \frac{\eta}{2}\sum_{i=1}^{T} \|\hat{g}_i\|_2^2 + \frac{1}{2\eta} + \sum_{i=1}^{T} \langle\, \hat{z}_i,\, x_i - x^* \,\rangle. \tag{27.3.4}$$

Now dividing by T and taking the expectation, we have

$$\mathrm{E}\left[\frac{1}{T}\sum_{i=1}^{T}\big(f(x_i)-f(x^*)\big)\right] \;\le\; \frac{\eta}{2T}\sum_{i=1}^{T}\mathrm{E}\left[\,\|\hat{g}_i\|_2^2\,\right]+\frac{1}{2\eta T}+\mathrm{E}\left[\frac{1}{T}\sum_{i=1}^{T}\langle\,\hat{z}_i,\,x_i-x^*\,\rangle\right].$$

Using our assumption (27.3.3) on the oracle, and Jensen's inequality (Fact B.4.2), we obtain

$$\mathrm{E}\left[f\Big(\sum_{i=1}^{T}\frac{x_i}{T}\Big)\right]-f(x^*) \;\le\; \frac{\eta}{2}+\frac{1}{2\eta T}+\mathrm{E}\left[\frac{1}{T}\sum_{i=1}^{T}\langle\,\hat{z}_i,\,x_i-x^*\,\rangle\right].$$

Plugging in $\eta = 1/\sqrt{T}$ completes the proof. □

Claim 27.3.4. Define $S_0 = 0$ and

$$S_i \;=\; \sum_{j\le i}\langle\,\hat{z}_j,\,x_j-x^*\,\rangle \qquad \forall i\in[T].$$

Then $S_0, S_1, \ldots, S_T$ is a martingale (with respect to the full history $F_0, F_1, \ldots, F_T$).

The proof is left as Exercise 27.1.

Proof of Theorem 27.3.2. Since $S_0, S_1, \ldots, S_T$ is a martingale, it follows from Corollary 26.2.8 that $\mathrm{E}[\,S_T\,] = \mathrm{E}[\,S_0\,] = 0$. So by Lemma 27.3.3, we have

$$\mathrm{E}\left[f\Big(\frac{1}{T}\sum_{i=1}^{T}x_i\Big)\right]-f(x^*) \;\le\; \frac{1}{\sqrt{T}}+\frac{1}{T}\,\mathrm{E}\Bigg[\underbrace{\sum_{i=1}^{T}\langle\,\hat{z}_i,\,x_i-x^*\,\rangle}_{=S_T}\Bigg] \;=\; \frac{1}{\sqrt{T}}.$$ □

27.3.1 Application: training ML models

To train a machine learning model, we often need to minimize a function $f:\mathbb{R}^n\to\mathbb{R}$ of the form

$$f(x) \;=\; \frac{1}{m}\sum_{i=1}^{m}f_i(x)$$

where each f_i is convex.

This setting commonly arises if $x\in\mathbb{R}^n$ gives some parameters of the model, there are m training examples, and f_i is a "loss function" that measures how well these parameters do at classifying the i^{th} example.

A prototypical example would be training a "soft" support vector machine, with the training set

$$\begin{aligned}\text{Examples:}\quad & z_1,\ldots,z_m\in\mathbb{R}^n\\ \text{Labels:}\quad & y_1,\ldots,y_m\in\{+1,-1\}\end{aligned}$$

and loss function $f_i(x)=\max\{0,1-y_i\langle\,x,\,z_i\,\rangle\}$. If $\langle\,x,\,z_i\,\rangle$ has the opposite sign of y_i then $f_i(x)$ gives a positive loss; otherwise, if they have the same sign and $|\langle\,x,\,z_i\,\rangle|\ge 1$, then $f_i(x)$ gives no loss. Thus minimizing f yields parameters x for which the sign of $\langle\,x,\,z_i\,\rangle$ tends to match the label y_i.

References: (Shalev-Shwartz and Ben-David, 2014, Equation (15.6)).

Suppose we have a (deterministic) subgradient oracle for each f_i. We can build upon these to give a randomized subgradient oracle for f, with which we can execute SGD.

The randomized subgradient oracle. The randomized subgradient oracle for f is very simple. Given a fixed point x, pick an index $I \in [m]$ uniformly at random, then return any subgradient of f_I at the point x.

Question 27.3.5. Suppose that each function f_i is 1-Lipschitz. Is assumption (27.3.3) necessarily satisfied?

Answer.
Yes. Each vector g_i has $\|g_i\|_2 \leq 1$ by Fact B.3.8, which implies that $\|\hat{g}\|_2 \leq 1$ with probability 1. So $\|\hat{g}\|_2^2 \leq 1$, which implies (27.3.3).

Claim 27.3.6. $\mathrm{E}[\,\hat{g}\,] \in \partial f(x)$.

Proof. Let $s_i \in \partial f_i(x)$ be the vector that would be returned by the i^{th} subgradient oracle, so that $\mathrm{E}[\,\hat{g}\,] = \frac{1}{m}\sum_{i=1}^m s_i$. Then

$$\begin{aligned}
\mathrm{E}[\,\hat{g}\,] \;&\in\; \frac{1}{m}\sum_{i=1}^m \partial f_i(x) && \text{(definition of Minkowski sum)} \\
&= \partial\Big(\frac{1}{m}\sum_{i=1}^m f_i(x)\Big) && \text{(by Fact B.3.9)} \\
&= \partial\big(f(x)\big). \quad \square
\end{aligned}$$

27.3.2 Application: geometric median

Let $p_1, \ldots, p_m$ be points in $\mathbb{R}^n$. A geometric median of these points is a point x^* that minimizes

$$f(x) \;=\; \frac{1}{m}\sum_{i=1}^m \|x - p_i\|\,.$$

We present an application of SGD to approximate the geometric median. Following the notation of the previous section, let $f_i(x) = \|x - p_i\|$.

Claim 27.3.7. f_i is 1-Lipschitz.

Proof sketch. This follows from the triangle inequality. $\square$

Claim 27.3.8. Define

$$g \;=\; \begin{cases} \frac{x-p_i}{\|x-p_i\|} & (\text{if } x \neq p_i) \\ 0 & (\text{if } x = p_i). \end{cases}$$

Then $g \in \partial f_i(x)$.

Proof sketch. If $x \neq p_i$ then f_i is differentiable at x, and one may verify that $g = \nabla f_i(x)$. Otherwise $x = p_i$, so x is the global minimizer of f_i, in which case $0 \in \partial f_i(x)$. $\square$

Imagine that we have a *warm start* for finding the geometric median. Specifically, we have a point $x_1 \in \mathbb{R}^n$ satisfying

$$\|x_1 - x^*\| \leq \lambda \qquad \text{where} \qquad \lambda = 40 f(x^*).$$

Interestingly, there is a randomized algorithm to find this warm start in $O(n)$ time. See Appendix C.1 of this paper. Given the warm start, SGD provides the desired approximation.

Theorem 27.3.9. Given the warm start x_1, run stochastic gradient descent with no constraints (i.e., $\mathcal{X} = \mathbb{R}^n$) for $T = 1600/\epsilon^2$ iterations. Then

$$\mathrm{E}[\, f(x_T)\,] \;\leq\; (1+\epsilon)\cdot f(x^*).$$

Proof. Theorem 27.3.2 yields

$$\mathrm{E}[\, f(x_T)\,] - f(x^*) \;\leq\; \frac{\lambda}{\sqrt{T}} \;=\; \frac{40 f(x^*)}{\sqrt{T}} \;=\; \frac{40 f(x^*)}{\sqrt{1600/\epsilon^2}} \;=\; \epsilon f(x^*). \qquad \square$$

Question 27.3.10. What is the runtime of this algorithm?

Answer.

Each iteration of SGD requires $O(n)$ time, so the total runtime is $O(T) = O(n/\epsilon^2)$.

27.4 Scaling reductions

Our analyses above make two assumptions

- *Scale of codomain:* the given function f is 1-Lipschitz, and
- *Scale of domain:* $\|x_1 - x^*\|_2 \leq 1$.

How can we handle a general scale, say an L-Lipschitz function with $\|x_1 - x^*\|_2 \leq R$? There is a general reduction that can handle such scenarios.

Meta-theorem. Suppose that we have a theorem giving a convergence rate guarantee $c(T)$ for gradient descent assuming that $f : \hat{\mathcal{X}} \to \mathbb{R}$ is 1-Lipschitz and assuming $\|x_1 - x^*\|_2 \leq 1$. Suppose that $h : \mathcal{X} \to \mathbb{R}$ is a convex function that is L-Lipschitz, and such that $\|x_1 - x^*\| \leq R$. Then there is a black-box reduction from h to f, showing that gradient descent on h achieves convergence rate $RL \cdot c(T)$.

Proof of meta-theorem. Let $OPT = \min_{x\in\mathcal{X}} h(x)$. Define $\hat{\mathcal{X}} = \mathcal{X}/R$ and $f : \hat{\mathcal{X}} \to \mathbb{R}$ by

$$f(x) \;=\; \frac{1}{RL}(h(Rx) - OPT).$$

Thus,

$$\begin{aligned} h(x) \;&=\; RL \cdot f(x/R) + OPT \\ \min_{x\in\hat{\mathcal{X}}} f(x) \;&=\; 0. \end{aligned} \tag{27.4.1}$$

Claim 27.4.1. $v \in \partial h(x)$ iff $v/L \in \partial f(x/R)$.

Consider running gradient descent on h with step sizes $\eta_t = \frac{R}{L\sqrt{t}}$ from the starting point x_1, producing iterates $x_2, x_3, \ldots$. Let g_i be the subgradient used in the i^{th} iteration. Define $\hat{g}_i = g_i/L$.

Simulataneously, imagine running gradient descent on f with step sizes $\hat{\eta}_t = \frac{1}{\sqrt{t}} = \frac{L}{R}\eta_t$ and vectors $\hat{g}_i = g_i/L$, from the starting point $\hat{x_1} = x_1/R$. Let $\hat{x_2}, \hat{x_3}, \ldots$ be the vectors produced.

Claim 27.4.2. $\hat{x_i} = x_i/R$ for all $i \geq 1$.

Proof. By induction, the case $i = 1$ true by definition. So suppose true up to i. By definition $g_i \in \partial h(x_i)$, so Claim 27.4.1 implies that $\hat{g}_i \in \partial f(x_i/R) = \partial f(\hat{x}_i)$. Then

$$\hat{x_{i+1}} \;=\; \hat{x_i} - \hat{\eta_i} \cdot \hat{g_i} \;=\; \frac{1}{R}x_i - \frac{L}{R}\eta_i \cdot \frac{1}{L}g_i \;=\; \frac{1}{R}(x_i - \eta_i g_i) \;=\; \frac{1}{R}x_{i+1}. \quad \square$$

To illustrate the meta-theorem, we apply it to Theorem 27.1.2, obtaining:

Theorem 27.4.3. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and L-Lipschitz (with respect to $\|\cdot\|_2$). Fix an optimal solution $x^* \in \operatorname{argmin}_x f(x)$ and a starting point $x_1 \in \mathbb{R}^n$. Define $\eta = \frac{L}{R\sqrt{T}}$. Suppose that $\|x_1 - x^*\|_2 \le R$. Then

$$\begin{aligned} h\left(\sum_{i=1}^{T} \frac{x_i}{T}\right) - h(x^*) \;&=\; RL \cdot f\left(\sum_{i=1}^{T} \frac{x_i}{RT}\right) \qquad \text{(by (27.4.1))} \\ &=\; RL \cdot f\left(\sum_{i=1}^{T} \frac{\hat{x_i}}{T}\right) \qquad \text{(by Claim 27.4.2)} \\ &\le\; \frac{RL}{\sqrt{T}} \qquad \text{(by Theorem 27.1.2).} \end{aligned}$$

27.5 Exercises

Exercise 27.1 The martingale in SGD.

Part I. Prove that $\mathrm{E}[\,\hat{z}_i \mid F_{i-1}\,] = 0$.

Part II. Prove Claim 27.3.4.

Exercise 27.2 High-probability bound for SGD. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex, 1/2-Lipschitz function. Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set with $\operatorname{diam}(\mathcal{X}) = 1$, and assume $x_1 \in \mathcal{X}$.

Suppose that the subgradient oracle satisfies (27.3.2). However, instead assuming (27.3.3), we assume that the noise satisfies $\|\hat{z}_i\| \le 1/2$.

Part I. Prove that $\|\hat{g}_i\|_2^2 \le 1$.

Part II. Prove that

$$\Pr\left[\; f\Big(\frac{1}{T}\sum_{i=1}^{T} x_i\Big) - f(x^*) \;\ge\; \frac{1 + \sqrt{2\ln(1/\delta)}}{\sqrt{T}} \;\right] \;\le\; \delta.$$

Chapter 28

The Lovász Local Lemma

The Lovász Local Lemma (LLL) is an intriguing method for analyzing events that are not independent, but have some restricted sort of dependencies. It is not as widely applicable as many of the other the techniques we have seen so far, but from time to time one encounters scenarios in which the LLL is the only technique that works.

28.1 Statement of the Symmetric LLL

Very often when designing randomized algorithms, we create a discrete probability space in which there are "bad events" $\mathcal{E}_1, \ldots, \mathcal{E}_n$ that we do not want to occur. For example, in the congestion minimization problem (Section 22.2), $\mathcal{E}_i$ could be the event that edge i has too much congestion. Often our analysis aims to show that we can avoid these bad events with high probability, i.e., $\Pr\left[\bigwedge_{i=1}^n \overline{\mathcal{E}_i}\right] \approx 1$.

In this chapter we consider the weaker goal of showing that $\Pr\left[\bigwedge_{i=1}^n \overline{\mathcal{E}_i}\right] > 0$. There are two cases in which this goal is particularly simple.

- **Mutually independent events.** Suppose that the events are mutually independent, and that $\Pr[\mathcal{E}_i] < 1$ for every i. Then

$$\Pr\left[\bigwedge_{i=1}^n \overline{\mathcal{E}_i}\right] = \prod_{i=1}^n \Pr\left[\overline{\mathcal{E}_i}\right] > 0. \tag{28.1.1}$$

  For example, suppose we pick n digits at random and let $\mathcal{E}_i =$ "i^{th} digit is non-zero". Then $\bigwedge_{i=1}^n \overline{\mathcal{E}_i}$ is the event that all digits are zeros. This event does indeed happen with positive probability, due to (28.1.1).

- **Union bound works.** Suppose that $\sum_{i=1}^n \Pr[\mathcal{E}_i] < 1$. Then, by a union bound (see (A.3.1)),

$$\Pr\left[\bigwedge_{i=1}^n \overline{\mathcal{E}_i}\right] = 1 - \Pr\left[\bigvee_{i=1}^n \mathcal{E}_i\right] \geq 1 - \sum_{i=1}^n \Pr[\mathcal{E}_i] > 0.$$

If neither of these scenarios applies, then there are not many general-purpose techniques to try. The Lovász Local Lemma (LLL) is one of the few, and it has intriguing applications for a range of problems.

Roughly speaking, the LLL is applicable in scenarios where the $\mathcal{E}_i$'s are not mutually independent, but they can have some sort of limited dependencies. Formally, a ***dependency graph*** for events $\mathcal{E}_1, \ldots, \mathcal{E}_n$

is defined as follows. The vertex set is $[n]$. The neighbors of vertex i (excluding i itself) are denoted $\Gamma(i)$, and we also define $\Gamma^+(i) = \Gamma(i) \cup \{i\}$. The key requirement is that event $\mathcal{E}_i$ is independent from $\{ \mathcal{E}_j : j \notin \Gamma^+(i) \}$, the events that are not neighbors of i. This last condition is stated more precisely as

$$\Pr[\mathcal{E}_i] \;=\; \Pr\left[\mathcal{E}_i \mid \bigcap_{j\in J} \mathcal{E}_j\right] \qquad \text{for all } J \subseteq [n] \setminus \Gamma^+(i).$$

So, regardless of whether some of the events outside $\Gamma^+(i)$ occur, the probability of $\mathcal{E}_i$ occurring is unaffected.

Theorem 28.1.1 (The Symmetric LLL). Suppose that there is a dependency graph of maximum degree d. If $\Pr[\mathcal{E}_i] \le p$ for every i and

$$p \cdot e \cdot (d+1) \;\le\; 1 \tag{SLL}$$

then $\Pr\left[\bigcap_{i=1}^n \overline{\mathcal{E}_i}\right] \ge \left(\frac{d}{d+1}\right)^n > 0$.

References: (Motwani and Raghavan, 1995, Corollary 5.12), (Alon and Spencer, 2000, Corollary 5.1.2), (Mitzenmacher and Upfal, 2005, Theorem 6.11), Wikipedia.

Remark 28.1.2.

- Since $\left(\frac{d}{d+1}\right)^n \approx \exp(-n/d)$, due to Fact A.2.5, the LLL only gives an exponentially small probability of avoiding $\mathcal{E}_1, \ldots, \mathcal{E}_n$.
- The condition (SLL) can be improved to $ped \le 1$. See, e.g., Harvey & Vondrák, Lemma 9.

28.2 Application: k-SAT

Instead, we will illustrate the LLL by considering a concrete application of it in showing satisfiability of k-CNF Boolean formulas. Recall that a k-CNF formula is a Boolean formula, involving any finite number of variables, where the formula is a *conjunction* ("and") of any number of clauses, each of which is a *disjunction* ("or") of exactly k *literals* (a variable or its negation). Let us assume that, for each clause, the variables appearing in it are all distinct.

For example, here is a 3-CNF formula with three variables and eight clauses.

$$\begin{aligned}\phi(a,b,c) \;=\; & (a \cup b \cup c) \cap (a \cup b \cup \overline{c}) \cap (a \cup \overline{b} \cup c) \cap (a \cup \overline{b} \cup \overline{c}) \cap \\ & (\overline{a} \cup b \cup c) \cap (\overline{a} \cup b \cup \overline{c}) \cap (\overline{a} \cup \overline{b} \cup c) \cap (\overline{a} \cup \overline{b} \cup \overline{c})\end{aligned}$$

This formula is obviously unsatisfiable. One can easily generalize this construction to get an unsatisfiable k-CNF formula with k variables and 2^k clauses. Our next theorem says: the reason this formula is unsatisfiable is that we allowed each variable to appear in too many clauses.

Theorem 28.2.1. Let ϕ be a k-CNF formula where each variable appears in at most $2^k/ek$ clauses. Then ϕ is satisfiable.

Proof. Consider the probability space in which each variable is independently set to true or false with equal probability. A clause is not satisfied if every literal appearing in that clause is false. (For example, $\overline{a} \cup b \cup \overline{c}$ is unsatisfied if a is true, b is false, and c is true.) This happens with probability 2^{-k}, since the clause involves k distinct variables.

Let $\mathcal{E}_i$ be the event that the i^{th} clause is unsatisfied. We have just argued that

$$\Pr[\mathcal{E}_i] \;=\; 2^{-k} \;=:\; p.$$

Consider the graph defined on $[n]$ in which there is an edge $\{i, j\}$ if some variable appears in both clause i and clause j. It is easy to see that this is a dependency graph: whether clause i is satisfied is independent from the clauses sharing no variables with clause i.

Each variable in clause i appears in at most $2^k/ek - 1$ other clauses. The number of neighbors of clause i is at most k times larger, since it contains k variables. That is,

$$|\Gamma(i)| \;\leq\; 2^k/e - 1 \;=:\; d.$$

Since $pe(d+1) \leq 1$, condition (SLL) is satisfied and $\Pr\left[\bigcap_{i=1}^n \overline{\mathcal{E}_i}\right] > 0$. So, if we pick values for the variables at random, there is positive probability of satisfying ϕ. This shows that ϕ is satisfiable. □

28.3 Symmetric LLL: a proof sketch

In this section we give a sketchy proof of the symmetric LLL that tries to explain the sequence of ideas that leads to the proof. If this sketch is not to your taste, a correct and concise proof is given in Section 28.4.1. Instead of assuming (SLL), it will be convenient to assume the strengthened hypothesis

$$4pd \;\leq\; 1. \tag{28.3.1}$$

To make the notation more meaningful, let B_i denote the "bad" event $\mathcal{E}_i$ and let G_i be the "good" event $\overline{B_i}$. Note that independence from B_i is equivalent to independence from G_i. For any set $S \subseteq [n]$, let $G_S = \cap_{i \in S} G_i$.

The objective of the proof is to show that $\Pr\left[G_{[n]}\right] > 0$, i.e., with positive probability, all good events occur simultaneously. Note that the union bound gives the easy lower bound

$$\Pr\left[G_S\right] \;=\; 1 - \Pr\left[\cup_{i \in S} B_i\right] \;\geq\; 1 - \sum_{i \in S} \Pr\left[B_i\right] \;\geq\; 1 - |S|p, \tag{28.3.2}$$

but in our scenario, this is too weak — we could have $|S|p \gg 1$. Instead, a natural idea is to use the chain rule (Fact A.3.5) to break apart this large conjunction:

$$\Pr\left[G_{[n]}\right] \;=\; \prod_{k=1}^{n} \Pr\left[G_k \;\middle|\; G_{[k-1]}\right]. \tag{28.3.3}$$

We just need to show that each factor in this product is strictly positive.

Idea 1: Rather than showing that each factor is positive, it turns out to be convenient to negate the event and prove an upper bound. Specifically, we want to show that

$$\Pr\left[B_k \;\middle|\; G_S\right] \;\leq\; \alpha p \qquad \forall S \subseteq [n] \tag{Hope}$$

for some α to be chosen later. In order for this conditional probability to be well defined, we need that $\Pr\left[G_S\right] > 0$. Let's ignore that for now as our whole purpose is to prove the stronger fact that $\Pr\left[G_{[n]}\right] > 0$. Trivially,

$$\Pr\left[B_k \;\middle|\; G_S\right] \;=\; \Pr\left[B_k \;\middle|\; G_{S \cap \Gamma(k)} \cap G_{S \setminus \Gamma(k)}\right].$$

Idea 2: We would like to somehow use the fact that B_k is independent of $G_{S \setminus \Gamma(k)}$ (due to the dependency graph). But it is difficult to use that fact due to the additional conditioning on $G_{S \cap \Gamma(k)}$. So as a first step we use the definition of conditional probability to write

$$\Pr\left[B_k \;\middle|\; G_S\right] \;=\; \frac{\Pr\left[B_k \cap G_{S \cap \Gamma(k)} \;\middle|\; G_{S \setminus \Gamma(k)}\right]}{\Pr\left[G_{S \cap \Gamma(k)} \;\middle|\; G_{S \setminus \Gamma(k)}\right]}.$$

Idea 3: The next idea is to drop the "$\cap G_{S\cap\Gamma(k)}$" event yielding the following upper bound. We would hope that still this gives a good bound since $\Gamma(k)$ is small and $G_{S\cap\Gamma(k)}$ is a very likely event.

$$\Pr\left[B_k \mid G_S\right] \leq \frac{\Pr\left[B_k \mid G_{S\setminus\Gamma(k)}\right]}{\Pr\left[G_{S\cap\Gamma(k)} \mid G_{S\setminus\Gamma(k)}\right]} = \frac{\Pr\left[B_k\right]}{\Pr\left[G_{S\cap\Gamma(k)} \mid G_{S\setminus\Gamma(k)}\right]} \leq \frac{p}{\Pr\left[G_{S\cap\Gamma(k)} \mid G_{S\setminus\Gamma(k)}\right]}$$

The equality here uses our second idea, that B_k is independent of the events outside of $\Gamma(k)$.

Now, to prove (Hope), it suffices to prove that $\Pr\left[G_{S\cap\Gamma(k)} \mid G_{S\setminus\Gamma(k)}\right] \geq 1/\alpha$. The good news is that the conjunction $G_{S\cap\Gamma(k)}$ involves few events, so we are in good shape to use the union bound as in (28.3.2):

$$\Pr\left[G_{S\cap\Gamma(k)} \mid G_{S\setminus\Gamma(k)}\right] \geq 1 - \sum_{i\in S\cap\Gamma(k)} \Pr\left[B_i \mid G_{S\setminus\Gamma(k)}\right]. \tag{28.3.4}$$

If $S\cap\Gamma(k) = \emptyset$ then this quantity is 1 as the sum is empty. Otherwise, $|S\setminus\Gamma(k)| < |S|$, so we can use induction on $|S|$. We have

$$\begin{aligned}
\Pr\left[G_{S\cap\Gamma(k)} \mid G_{S\setminus\Gamma(k)}\right] &\geq 1 - \sum_{i\in\Gamma(k)} \Pr\left[B_i \mid G_{S\setminus\Gamma(k)}\right] && \text{(by (28.3.4))}\\
&\geq 1 - d\alpha p && \text{(inductively using (Hope), and } |\Gamma(k)|\leq d)\\
&\geq 1 - \alpha/4 && \text{(by the strengthened hypothesis (28.3.1))}\\
&= 1/\alpha
\end{aligned}$$

if we now choose $\alpha = 2$. This shows that (Hope) is satisfied.

28.4 The General LLL

The Symmetric Local Lemma is useful, but often somewhat restrictive. It works best when all events are equally probable, and when all neighborhood sizes are the same. We might want to consider scenarios in which some events are quite likely and some are quite rare. The likely events would need to depend on few other events, and the rare events could perhaps depend on many other events. There is a general form of the local lemma that can handle such scenarios.

Theorem 28.4.1 (General LLL). Suppose that there is a dependency graph and an $x\in(0,1)^n$ satisfying

$$\Pr\left[\mathcal{E}_i\right] \leq x_i \cdot \prod_{j\in\Gamma(i)} (1-x_j) \qquad \forall i. \tag{GLL}$$

Then $\Pr\left[\bigcap_{i=1}^n \overline{\mathcal{E}_i}\right] \geq \prod_{i=1}^n (1-x_i) > 0$.

References: (Motwani and Raghavan, 1995, Theorem 5.11), (Alon and Spencer, 2000, Lemma 5.1.1), Wikipedia.

This form of the local lemma is confusing at first because it's not obvious what these x_i values should be. In order to satisfy (GLL), on the right-hand side we want x_i to be big and each x_j to be small. Due to that tension, care is needed in finding the right x_i.

28.4.1 A Concise Proof of Theorem 28.4.1

We simultaneously prove by induction on $|S|$ that, for all $S\subseteq[n]$,

$$\Pr\left[G_S\right] > 0 \tag{28.4.1a}$$
$$\Pr\left[B_k \mid G_S\right] \leq x_k. \tag{28.4.1b}$$

In the case $S = \emptyset$, (28.4.1a) is trivial and (28.4.1b) follows directly from (GLL).

By relabeling, we may assume that $S = \{1, \ldots, s\}$, so

$$\Pr[G_S] = \prod_{j=1}^{s} \Pr[G_j \mid G_{\{1,\ldots,j-1\}}] = \prod_{j=1}^{s} \left(1 - \Pr[B_j \mid G_{\{1,\ldots,j-1\}}]\right) \geq \prod_{j=1}^{s} (1 - x_j),$$

where we inductively use (28.4.1a) to ensure that the conditional probabilities are well-defined, and we inductively use (28.4.1b) to provide the inequality. This proves (28.4.1a).

Next consider (28.4.1b). If $S \cap \Gamma(k) = \emptyset$ then $\Pr[B_k \mid G_S] = \Pr[B_k]$, so (28.4.1b) follows directly from (GLL). Otherwise, relabeling so that $S \cap \Gamma(k) = \{1, \ldots, t\}$, we have

$$\begin{aligned}
\Pr[B_k \mid G_S] &= \Pr[B_k \mid G_{S\cap\Gamma(k)} \cap G_{S\setminus\Gamma(k)}] = \frac{\Pr[B_k \cap G_{S\cap\Gamma(k)} \mid G_{S\setminus\Gamma(k)}]}{\Pr[G_{S\cap\Gamma(k)} \mid G_{S\setminus\Gamma(k)}]} \\
&\leq \frac{\Pr[B_k \mid G_{S\setminus\Gamma(k)}]}{\Pr[G_{S\cap\Gamma(k)} \mid G_{S\setminus\Gamma(k)}]} = \frac{\Pr[B_k]}{\prod_{j=1}^{t} \Pr[G_j \mid G_{(S\setminus\Gamma(k))\cup\{1,\ldots,j-1\}}]} \\
&\leq \frac{x_k \cdot \prod_{i\in\Gamma(k)}(1-x_i)}{\prod_{j=1}^{t}\left(1 - \Pr[B_j \mid G_{(S\setminus\Gamma(k))\cup\{1,\ldots,j-1\}}]\right)} \leq \frac{x_k \cdot \prod_{i\in\Gamma(k)}(1-x_i)}{\prod_{j=1}^{t}(1-x_j)}.
\end{aligned}$$

The second inequality uses (GLL) and the third uses induction. The last expression is clearly at most x_k, proving (28.4.1b).

28.4.2 General implies Symmetric

Finally, let us conclude by noting that Theorem 28.4.1 implies Theorem 28.1.1. To see this, consider an instance satisfying (SLL). Set $x_i = 1/(d+1)$. The RHS of (GLL) is

$$\begin{aligned}
x_i \cdot \prod_{j\in\Gamma(i)} (1 - x_i) &\geq \frac{1}{d+1} \cdot \left(1 - \frac{1}{d+1}\right)^d \\
&\geq \frac{1}{e(d+1)} \qquad \text{(by calculus)} \\
&\geq p \qquad \text{(by (SLL))}.
\end{aligned}$$

This shows that (GLL) is satisfied, so Theorem 28.4.1 implies

$$\Pr\left[\bigcap_{i=1}^{n} \overline{\mathcal{E}_i}\right] \geq \prod_{i=1}^{n} (1 - x_i) = \left(\frac{d}{d+1}\right)^n,$$

which is the conclusion of Theorem 28.1.1.

28.5 An Algorithmic Local Lemma

As stated above, the LLL asserts the existence of a point in a probability space that simultaneously avoids certain bad events, assuming that their probabilities and dependencies are small enough. However, it does *not* suggest an efficient method to *find* such a point. This naturally leads to a research question:

is there an algorithmic form of the LLL?

This questioned had been studied for decades, culminating in a significant breakthrough by Robin Moser in 2009.

Suppose we wanted to actually find an point in the probability space avoiding the bad events. If we repeatedly picked independent samples from the underlying probability distribution, the expected time to find a point avoiding the bad events could be exponential in n. Usually we would like to find this point in time $\text{poly}(n)$. This can be done for essentially all known applications of the local lemma.

Today we will discuss an algorithmic LLL which is not quite that general. Instead, we will focus on the application of the LLL to showing that SAT formulae are satisfiable, as discussed last time.

Theorem 28.5.1. There is a universal constant $\alpha \geq 1$ such that the following is true. Let ϕ be a k-CNF formula where each variable appears in at most $T := 2^{k-\alpha}/k$ clauses. Then ϕ is satisfiable. Moreover, there is a randomized, polynomial time algorithm to find a satisfying assignment.

The theorem is stronger when α is small. The proof that we will present can be optimized to get $\alpha = 3$. The existential result from last time achieves $\alpha = \lg_2(e) \approx 1.44$, which is essentially optimal. So today's result is weaker only by a small constant factor.

The algorithm proving the theorem is extremely simple, and algorithms of this sort were certainly considered decades ago. But, they were not analyzed until 2009, when the audacious and brilliant graduate student Robin Moser at ETH made a tremendous breakthrough. Related ideas were independently discovered by Pascal Schweitzer, also a graduate student.

References: Moser's PhD Thesis Section 2.5, Schweitzer's paper.

28.5.1 The Algorithm

Moser's algorithm is given in Algorithm 28.1.

Algorithm 28.1 Algorithm to find a satisfying assignment for a k-SAT formula ϕ, under the conditions of Theorem 28.5.1.

```
1:  function SOLVE(φ)
2:      Randomly pick {0,1} values for each variable in φ        ▷ This uses n random bits
3:      while there is an unsatisfied clause C
4:          FIX(C)
5:      return the variable assignment
6:  end function
7:  function FIX(C)
8:      Set each variable in C to 0 or 1 randomly and independently    ▷ This uses k random bits
9:      while there is an unsatisfied clause D sharing some variable with C
10:         FIX(D)                                                  ▷ Possibly D = C
11: end function
```

Notation. Let n be the number of variables and m be the number of clauses in ϕ. Let $T := 2^{k-\alpha}/k$ be the maximum number of occurrences of each variable. Each clause contains k variables, each of which can appear in only $T-1$ other clauses. So each clause shares a variable with less than $R := kT = 2^{k-\alpha}$ other clauses.

Claim 28.5.2. Consider any call to FIX that terminates. Let V be the set of variables that are assigned a different value after the call than before the call. Then every clause containing a variable in V is satisfied after the call.

Proof. Consider some clause D that contains a variable in V but is not satisfied after the call terminates. Consider the last time that any variable x in D was resampled. This must have happened during some call to FIX(E), for some clause E that also contains x. But FIX(E) would not terminate until D was satisfied, which is a contradiction. □

Corollary 28.5.3. Consider any call to FIX that terminates. Every clause that was satisfied *before* the call is still satisfied *after* the call completes.

Proof. If none of its variables changed, it is still satisfied. If at least one of its variables changed, Claim 28.5.2 applies. □

Corollary 28.5.4. Assume that C is violated, and consider any call to FIX(C) that terminates. The number of satisfied clauses before the call is strictly more than the number of satisfied clauses after the call.

Proof. By Corollary 28.5.3, the number of satisfied clauses cannot decrease. Furthermore, clause C must certainly be satisfied afterwards in order for FIX(C) to terminate. □

Corollary 28.5.5. SOLVE calls FIX at most m times. If the algorithm terminates, the output is a satisfying assignment.

Proof. By Corollary 28.5.4, every call from SOLVE to FIX(C) that terminates increases the number of satisfied clauses by at least one. The first claim follows since there are m clauses. The second claim is obvious: the only way that SOLVE can terminate is that all clauses are simultaneously satisfied. □

So it remains to show that, with high probability, every call to FIX terminates. The analysis of the algorithm is quite unusual, and counterintuitive the first time one sees it.

28.5.2 Incompressibility

From experience with software tools like `gzip`, one is likely familiar with the statement that "random information cannot be compressed". Let us now formalize that fact by stating that *some* strings can be compressed, but only a small fraction of them.

Claim 28.5.6. Let $x \in \{0,1\}^\ell$ be a uniformly random bit string of length ℓ. The probability that x can be compressed by $\log(1/\delta)$ bits is at most δ.

Proof. Consider any deterministic algorithm for encoding all bit strings of length ℓ into bit strings of arbitrary length. The number of bit strings that are encoded into $\ell - b$ bits is at most $2^{\ell-b}$. So, a random bit string has probability 2^{-b} of being encoded into $\ell - b$ bits. □

One can view this as a simple special case of the Kraft inequality.

28.5.3 Analysis of Algorithm 28.1

Theorem 28.5.7. Let $s = m \cdot (\log m + c) + \log(1/\delta)$ where c is a sufficiently large constant. Then the probability that the algorithm makes more than s calls to Fix (including both the top-level and recursive calls) is at most δ.

The proof proceeds by considering the interactions between two agents: the "CPU" and the "Debugger". The CPU runs the algorithm, periodically sending messages to the Debugger (we describe these messages in more detail below). However, if Fix gets called more than s times the CPU interrupts the execution and halts the algorithm.

The CPU needs n bits of randomness to generate the initial assignment in Solve, and needs k bits to regenerate variables in each call to Fix. Since the CPU will not execute Fix more than s times, it might as well generate all its random bits at the very start of the algorithm. So the first step performed by the CPU is to generate a random bitstring x of length $n + sk$ to provide all the randomness used in executing the algorithm.

The messages sent from the CPU to the Debugger are as follows.

- Every time the CPU runs Fix(C), it sends a message containing the identity of the clause C, and an extra bit indicating whether this is a top-level Fix (i.e., a call from Solve) or a recursive Fix.
- Every time Fix(C) finishes the CPU sends a message stating "recursive call finished".
- If Fix gets called s times, the CPU sends a message to the Debugger containing the current $\{0,1\}$ assignment of all n variables.

Because the Debugger is notified when every call to Fix starts or finishes, it always knows which clause is currently being processed by Fix. A crucial detail is to figure out how many bits of communication are required to send these messages.

- For a top-level Fix, $\log m + O(1)$ bits suffice because there are only m clauses in ϕ.
- For a recursive Fix, $\log R + O(1)$ bits suffice because the Debugger already knows what clause is currently being fixed, and that clause shares variables with only R other clauses, so only R possible clauses could be passed to the next call to Fix.
- When each call to Fix(C) finishes, the corresponding message takes $O(1)$ bits.
- When Fix gets called s times, the corresponding message takes $n + O(1)$ bits.

The main point of the proof is to show that, if Fix gets called s times, then these messages reveal the random string x to the Debugger.

Since each clause is a *disjunction* (an "or" of k literals), there is *exactly one* assignment to those variables that does not satisfy the clause. So, whenever the CPU tells the Debugger that it is calling Fix(C), the Debugger knows exactly what the current assignment to C is. So, starting from the assignment that the Debugger received in the final message, it can work backwards and figure out what the previous assignment was before calling Fix. Repeating this process, it can figure out how the variables were set in each call to Fix, and also what the initial assignment was. Thus the Debugger can reconstruct the random string x.

The total number of bits sent by the CPU are

- $m(\log m + O(1))$ bits for all the messages sent when SOLVE calls FIX.
- $s \cdot (\log R + O(1))$ for all the messages sent in the $\leq s$ recursive calls.
- $n + O(1)$ bits to send the final assignment.

Let's ignore the $O(1)$ terms, which can be eliminated by choosing the constant c and α appropriately. Then x has been compressed from $n + sk$ bits to

$$m \log m \ + \ s \log R \ + \ n \quad \text{bits.}$$

This is an overall shrinking of

$$\begin{aligned}
& \big(n + sk\big) \ - \ \big(m \log m + s \log R + n\big) \\
= \ & s(k - \log R) \ - \ m \log m \\
= \ & s\alpha \ - \ m \log m && \text{(since } R = 2^{k-\alpha}\text{)} \\
= \ & \big(m(\log m + c) + \log(1/\delta)\big)\alpha \ - \ m \log m && \text{(definition of } s\text{)} \\
\geq \ & \log(1/\delta)
\end{aligned}$$

bits, assuming that c and α are sufficiently big constants.

We have argued that, if FIX gets called s times, then x can be compressed by $\log(1/\delta)$ bits. By Claim 28.5.6, this is possible with probability at most δ.

28.6 Exercises

Exercise 28.1 **Local lemma for 0-1 matrices.** Let M be a matrix with m rows, n columns such that

- every entry $M_{i,j} \in \{0, 1\}$,
- every row sums to r (i.e., $\sum_{j=1}^{n} M_{i,j} = r$ for all i),
- every column sums to c (i.e., $\sum_{i=1}^{m} M_{i,j} = c$ for all j.)

Show that there exists a vector $Y \in \{0, 1\}^n$ such that, letting $Z = M \cdot Y$, we have

$$\begin{aligned}
\max_i Z_i \ &\leq \ (r/2) + O\big(\sqrt{r \log(rc)}\big) \\
\min_i Z_i \ &\geq \ (r/2) - O\big(\sqrt{r \log(rc)}\big).
\end{aligned}$$

Part I

Back matter

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Appendix B

Mathematical Background

B.1 Miscellaneous Facts

Fact B.1.1 (Stirling's Approximation).

$$e\Big(\frac{n}{e}\Big)^n \;<\; n! \;<\; en\Big(\frac{n}{e}\Big)^n$$

References: (Lehman et al., 2018, Theorem 14.5.1), Wikipedia.

Fact B.1.2 (Bounds on binomial coefficients). For any integers n, k with $1 \le k \le n$,

$$\Big(\frac{n}{k}\Big)^k \;\le\; \binom{n}{k} \;\le\; \sum_{i=0}^{k}\binom{n}{i} \;\le\; \Big(\frac{ne}{k}\Big)^k.$$

References: (Cormen et al., 2001, page 1186), (Vershynin, 2018, Exercise 0.0.5), (Motwani and Raghavan, 1995, Proposition B.2), (Shalev-Shwartz and Ben-David, 2014, Lemma A.5), Wikipedia.

Fact B.1.3. For $0 \le \epsilon \le 1$, we have $(1+\epsilon)^2 \le 1 + 3\epsilon$.

Proof. Expand $(1+\epsilon)^2$ as $1 + 2\epsilon + \epsilon^2$, and use $\epsilon^2 \le \epsilon$ when $\epsilon \le 1$. □

B.2 Geometry and norms

Norms. For a vector $x \in \mathbb{R}^d$, its ℓ_p norm is defined to be

$$\begin{aligned} \|x\|_p &= \Big(\sum_{i=1}^{d}|x_i|^p\Big)^{1/p} \qquad \forall p \in [1,\infty) \\ \|x\|_\infty &= \max_{1\le i\le d}|x_i| \end{aligned}$$

Every norm $\|\cdot\|$ satisfies the

$$\text{Triangle inequality:} \qquad \|a+b\| \le \|a\| + \|b\|\,.$$

The ℓ_2 norm is also called the ***Euclidean norm***. One useful identity it satisfies is

$$\|a-b\|_2^2 \;=\; \|a\|_2^2 - 2a^\mathsf{T} b + \|b\|_2^2 \qquad \forall a,b \in \mathbb{R}^n. \tag{B.2.1}$$

This identity might not have a canonical name, but some suggestions I have heard include the *generalized Pythagoras identity*, the *law of cosines*, or simply *completing the square*.

References: This is immediate from the bilinearity of an inner product. See, e.g., Apostol "Calculus, Volume II", page 17, or Wikipedia.

Let

$$B(p,r) \;=\; \{\, x \,:\, \|p-x\| \le r \,\} \tag{B.2.2}$$

denote the Euclidean ball in $\mathbb{R}^d$ around p of radius r. If d is even, the ***volume*** of $B(p,r)$ is

$$\operatorname{vol} B(p,r) \;=\; \frac{\pi^{d/2} r^d}{(d/2)!}.$$

References: (Blum et al., 2018, Section 2.4.1), Wikipedia.

By Stirling's formula, we have the bound

$$\operatorname{vol} B(p,r) \;\le\; \frac{\pi^{d/2} r^d}{(d/2e)^{d/2}} \;=\; \frac{(2e\pi)^{d/2} r^d}{d^{d/2}}. \tag{B.2.3}$$

The following fact allows us to compare norms.

Fact B.2.1. For all $x \in \mathbb{R}^d$,

$$\|x\|_p \;\le\; \|x\|_r \;\le\; d^{1/r-1/p} \cdot \|x\|_p \qquad \forall 1 \le r \le p \le \infty.$$

In particular, the most useful cases are

$$\begin{aligned} \|x\|_1 \;&\ge\; \|x\|_2 \;\ge\; \|x\|_\infty \\ \frac{1}{\sqrt{d}} \|x\|_1 \;&\le\; \|x\|_2 \;\le\; \sqrt{d} \cdot \|x\|_\infty . \end{aligned}$$

References: Wikipedia.

Exercises

Exercise B.1. Although every norm satisfies the triangle inequality, the *squared* Euclidean norm does not. However, it does satisfy an *approximate* triangle inequality. Prove that, for any vectors u, v, we have

$$\|u-v\|_2^2 \;\le\; 2 \cdot (\|u\|_2^2 + \|v\|_2^2).$$

B.3 Facts from Convex Analysis

B.3.1 One-dimensional functions

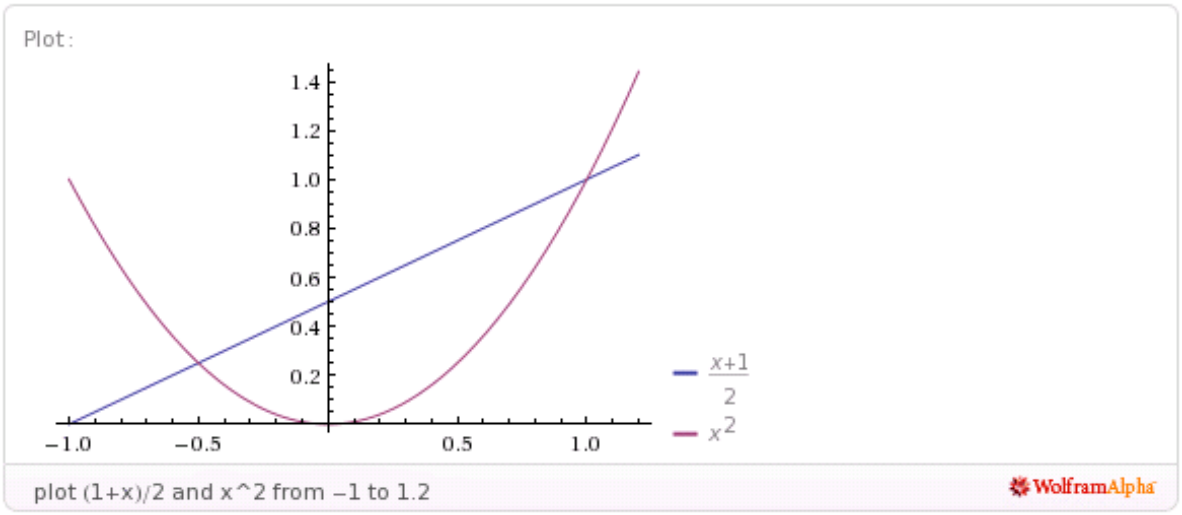
Let $f : S \to \mathbb{R}$ be a function defined on an interval $S \subseteq \mathbb{R}$.

Definition B.3.1. We say that f is convex on S if

$$f(\lambda x + (1-\lambda)y) \;\leq\; \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in S$ and all $\lambda \in [0,1]$.

Geometrically, this says that the secant line between the points $(x, f(x))$ and $(y, f(y))$ lies above the function f. The following example illustrates that $f(x) = x^2$ is convex.



An equivalent statement of this definition is as follows.

Fact B.3.2. Suppose $f : S \to \mathbb{R}$ is convex. Then, for any points $x, y \in S$ with $x < y$, we have

$$f(z) \;\leq\; \frac{f(y) - f(x)}{y - x} \cdot (z - x) + f(x)$$

for all $z \in [x, y]$.

For sufficiently nice functions, one can easily determine convexity by looking at its second derivative.

Fact B.3.3. Suppose $f : S \to \mathbb{R}$ is twice differentiable. Then f is convex if and only if the second derivative f'' is non-negative on the interior of S.

In our example above we used $f(x) = x^2$. Its second derivative is $f''(x) = 2$, which is non-negative, so f is convex.

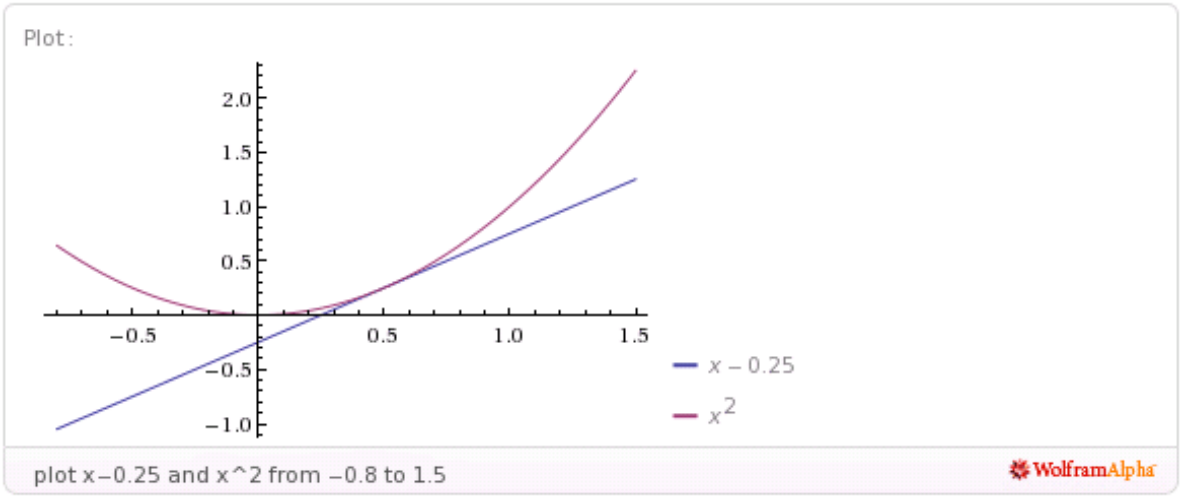
The next fact says that, for convex functions, the linear approximation at any point lies beneath the function.

Fact B.3.4 (Subgradient Inequality). Suppose $f : S \to \mathbb{R}$ is convex, and f is differentiable at a point $x \in S$. Then

$$f(y) \;\geq\; f(x) + f'(x)(y - x),$$

for any point $y \in S$.

The following example illustrates this for $f(x) = x^2$ at the point $x = 0.5$.



B.3.2 Various Inequalities from Convexity

In the analysis of randomized algorithms, we frequently use various inequalities to simplify expressions or make them easier to work with. These inequalities usually follow by convexity arguments and basic calculus. For example, let us revisit the following familiar fact.

Fact A.2.5 (Approximating e^x near zero). For all real numbers x,

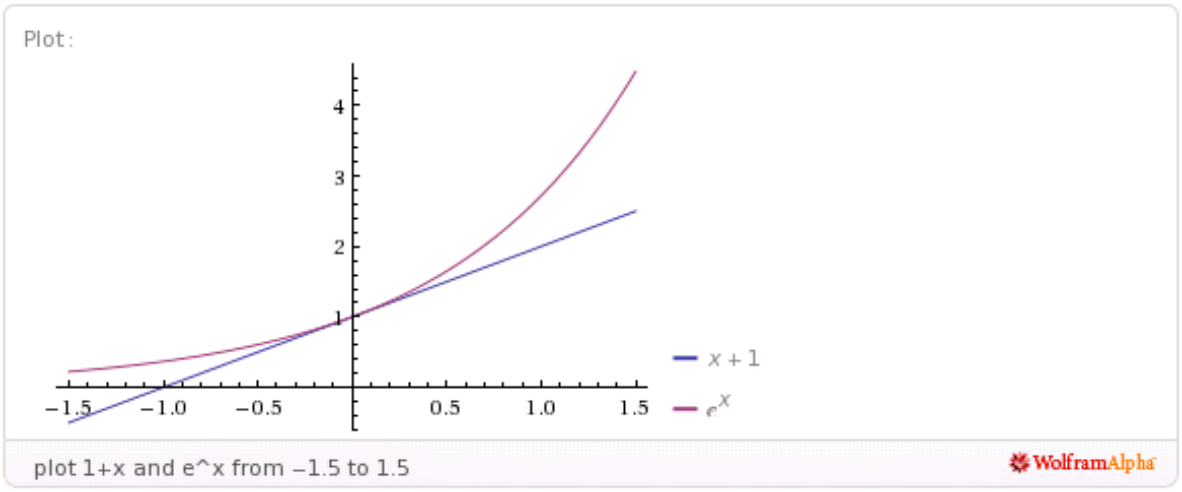
$$1 + x \leq e^x.$$

Moreover, for x close to zero, we have $1 + x \approx e^x$.

Proof. Convexity of e^x follows from Fact B.3.3 since its second derivative is e^x, which is non-negative on $\mathbb{R}$. Applying Fact B.3.4 at the point $x = 0$, we obtain

$$f(y) \;\geq\; f'(x) \cdot (y - x) + f(x) \;=\; y + 1.$$

□



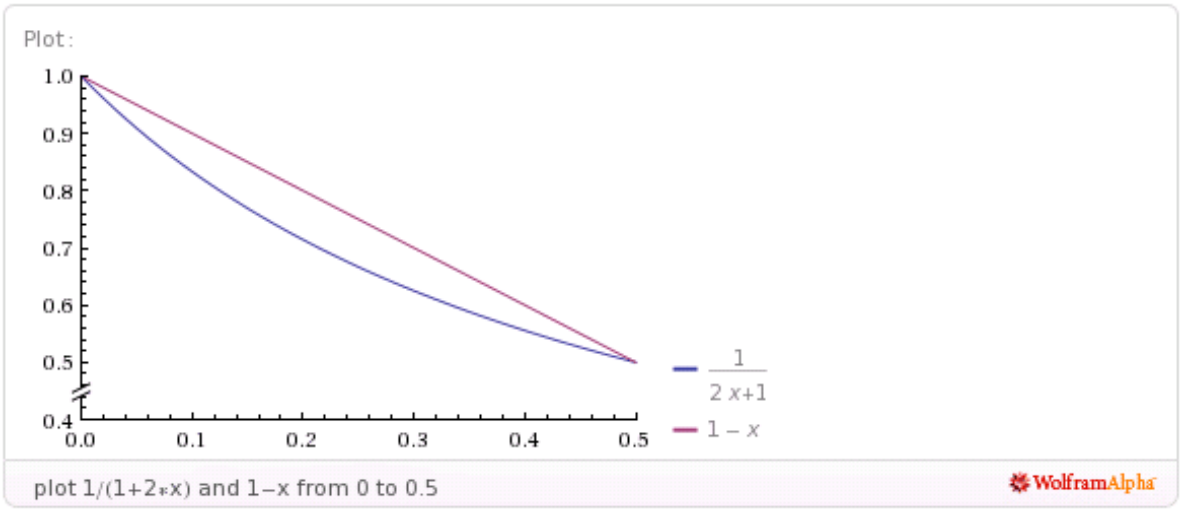
Fact B.3.5.

$$\frac{1}{1+2z} \;\leq\; 1 - z \qquad \forall z \in [0, 1/2].$$

Proof. Convexity of $f(z) = 1/(1+2z)$ on the set $S = (0,\infty)$ follows from Fact B.3.3 since its second derivative is $8/(1+2z)^3$, which is non-negative on S. Applying Fact B.3.2 at $x = 0$ and $y = 1/2$, we obtain

$$\begin{aligned} f(z) &\le \frac{f(y)-f(x)}{y-x}\cdot(z-x)+f(x) \\ &= \frac{1/2-1}{1/2}\cdot(z-0)+1 \;=\; 1-z \qquad \forall z\in[0,1/2]. \end{aligned}$$

□

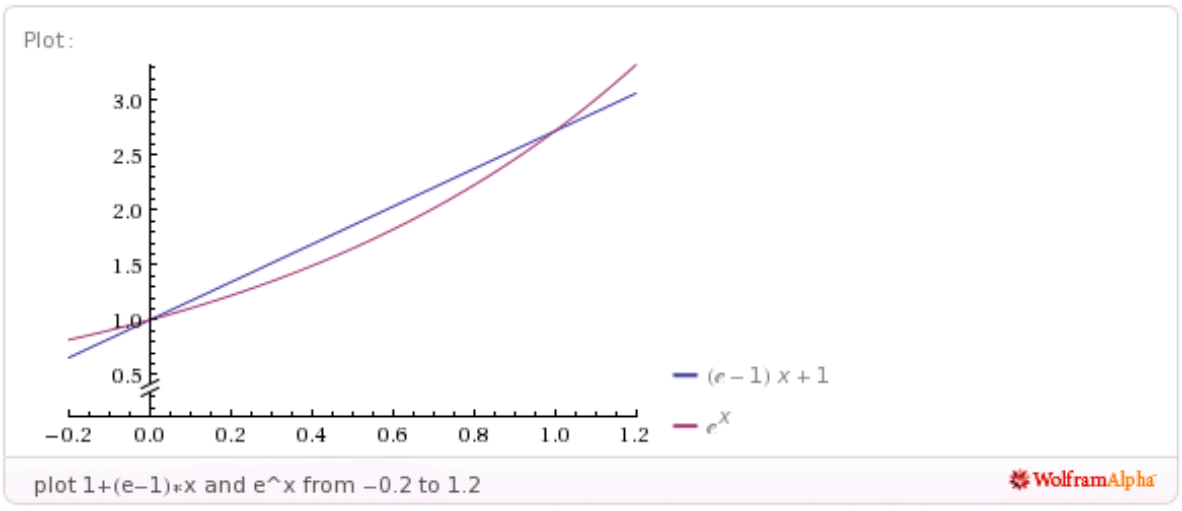


Fact B.3.6. Fix any $c > 0$. Then $c^z \le 1 + (c-1)z$ for all $z \in [0,1]$.

Proof. The second derivative of c^z is $c^z \cdot \ln^2(c)$, which is non-negative, so c^z is convex. Applying Fact B.3.2 at the points $x = 0$ and $y = 1$, we obtain

$$c^z \;\le\; \frac{c^y - c^x}{y-x}\cdot(z-x)+c^x \;=\; (c-1)\cdot z + 1.$$

□



Exercises

Exercise B.2. Prove that

$$\begin{aligned} \ln(1+z) &\le z && \text{for } z > -1 \\ \ln(1+z) &\ge \ln(2)z \ge z/2 && \text{for } z \in [0,1] \end{aligned}$$

B.3.3 Multi-dimensional functions

Definition B.3.7 (Subgradient). Let $f : \mathcal{X} \to \mathbb{R}^n$ be a function. Recall that a ***subgradient*** of f at x is any vector g satisfying:

$$f(y) \ge f(x) + \langle\, g,\, y - x \,\rangle \qquad \forall y \in \mathcal{X}. \tag{B.3.1}$$

References: (Shalev-Shwartz and Ben-David, 2014, Definition 14.4).

Fact B.3.8 (Lipschitz equivalence). Let $\mathcal{X}$ be convex and open. Let $f : \mathcal{X} \to \mathbb{R}$ be convex. The following conditions are equivalent.

- $f : \mathcal{X} \to \mathbb{R}$ is L-Lipschitz:

$$|f(x) - f(y)| \le L \|x - y\|_2 \qquad \forall x, y \in \mathcal{X}. \tag{B.3.2}$$

- f has bounded subgradients:

$$\|g\|_2 \le L \qquad \forall w \in X,\ g \in \partial f(w). \tag{B.3.3}$$

References: (Shalev-Shwartz and Ben-David, 2014, Definition 12.6 and Lemma 14.7).

Fact B.3.9. Let $f_1, \ldots, f_n$ be convex functions, and let $\alpha_1, \ldots, \alpha_n \ge 0$. Then

$$\partial\big(\alpha_1 f_1(x) + \cdots + \alpha_n f_n(x)\big) = \alpha_1 \partial f_1(x) + \cdots \alpha_n \partial f_n(x).$$

The sum on the right-hand side is a Minkowski sum.

References: (Hiriart-Urruty and Lemaréchal, 2001, Theorem D.4.1.1).

Fact B.3.10 (Projection decreases Euclidean distance). $\|\Pi_{\mathcal{X}}(y) - x\|_2 \le \|y - x\|_2$ for all $x \in \mathcal{X}$.

References: (Shalev-Shwartz and Ben-David, 2014, Lemma 14.9).

B.4 Probability

B.4.1 Expectation

For non-negative integer-valued random variables, the expectation has the formula Fact A.3.10. The following fact gives the analogous formula for arbitrary random variables.

Fact B.4.1. Let X be a non-negative random variable. Then

$$\mathrm{E}[\,X\,] = \int_0^\infty \Pr[\,X \ge x\,]\, dx = \int_0^\infty (1 - F(x))\, dx,$$

where F is the CDF of X.

References: For the special case of continuous random variables, see (Grimmett and Stirzaker, 2001, Lemma 4.3.4) or (Mitzenmacher and Upfal, 2005, Lemma 8.1). For arbitrary random variables, see (Vershynin, 2018, Lemma 1.2.1), (Durrett, 2019, Exercise 1.7.2) and (Klenke, 2008, Theorem 4.26).

Fact B.4.2 (Jensen's inequality). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and X a random vector.

$$\begin{aligned} \text{If } f \text{ is convex:} \quad & f(\mathrm{E}[X]) \leq \mathrm{E}[f(X)] \\ \text{If } f \text{ is concave:} \quad & f(\mathrm{E}[X]) \geq \mathrm{E}[f(X)] \end{aligned}$$

The following special case is useful. Let $x_1, \ldots, x_n \in \mathbb{R}^n$. Let $\lambda_1, \ldots, \lambda_n \in [0,1]$ satisfy $\sum_{i=1}^n \lambda_i = 1$. Then

$$\text{If } f \text{ is convex:} \quad f\Big(\sum_{i=1}^n \lambda_i x_i\Big) \leq \sum_{i=1}^n \lambda_i f(x_i).$$

References: (Cormen et al., 2001, Equation (C.26)), (Vershynin, 2018, page 7), (Mitzenmacher and Upfal, 2005, Lemma 2.4), (Klenke, 2008, Theorem 7.9), (Durrett, 2019, Theorem 1.5.1), (Grimmett and Stirzaker, 2001, Exercise 5.6.1), Wikipedia.

B.4.2 Variance

Definition B.4.3. The ***variance*** of a random variable X is defined to be

$$\mathrm{Var}[X] = \mathrm{E}\left[(X - \mathrm{E}[X])^2\right].$$

References: (Lehman et al., 2018, Definition 20.2.2), (Cormen et al., 2001, Equation (C.27)), (Motwani and Raghavan, 1995, page 443), (Mitzenmacher and Upfal, 2005, Definition 3.2), (Grimmett and Stirzaker, 2001, Definition 3.3.5), (Durrett, 2019, page 29).

Fact B.4.4. Variance satisfies the following identity.

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2. \tag{B.4.1}$$

Consequently,

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] \qquad \text{if} \qquad \mathrm{E}[X] = 0. \tag{B.4.2}$$

References: (Lehman et al., 2018, Definition 20.3.1), (Anderson et al., 2017, Fact 3.48), (Cormen et al., 2001, Equation (C.27)), (Motwani and Raghavan, 1995, Proposition C.8), (Mitzenmacher and Upfal, 2005, Definition 3.2), (Grimmett and Stirzaker, 2001, page 51), (Durrett, 2019, Equation (1.6.2)), (Klenke, 2008, Definition 5.1(iii)).

Proof.

$$\begin{aligned} \mathrm{Var}[X] &= \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] && \text{(by definition)} \\ &= \mathrm{E}\left[X^2 - 2X\,\mathrm{E}[X] + \mathrm{E}[X]^2\right] && \text{(expand the quadratic)} \\ &= \mathrm{E}\left[X^2\right] - 2\,\mathrm{E}[X]\,\mathrm{E}[X] + \mathrm{E}[X]^2 && \text{(linearity of expectation)} \\ &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2. && \text{(simplifying).} \end{aligned}$$

This proves (B.4.1). The equation (B.4.2) is immediate. □

Fact B.4.5. Let $G_1, \ldots, G_d$ be *pairwise* independent random variables with finite variance. Let $\sigma_1, \ldots, \sigma_d \in \mathbb{R}$ be arbitrary. Then $\mathrm{Var}\left[\sum_{i=1}^d \sigma_i G_i\right] = \sum_{i=1}^d \sigma_i^2 \mathrm{Var}[G_i]$.

References: (Lehman et al., 2018, Lemma 20.3.4 and Lemma 20.3.8), (Anderson et al., 2017, Facts 3.52 and 8.11), (Cormen et al., 2001, page 1200), (Mitzenmacher and Upfal, 2005, Theorem 3.5 and Exercise 3.4), (Grimmett and Stirzaker, 2001, Theorem 3.3.11).

B.4.3 Gaussian random variables

One of the most important continuous distributions is the ***Gaussian distribution***. It is also called the ***Normal distribution***. This distribution has two real parameters, its mean (denoted μ) and its variance (denoted σ^2). The distribution with those parameters is denoted $N(\mu, \sigma^2)$.

References: (Anderson et al., 2017, Definition 3.60), (Grimmett and Stirzaker, 2001, Definition 4.4.4), Wikipedia.

One useful property is that sums of Gaussians are also Gaussian.

Fact B.4.6. Let $g_1, \ldots, g_d$ be independent random variables where g_i has distribution $N(0,1)$. Then, for any scalars $\sigma_1, \ldots, \sigma_d$, the sum $\sum_{i=1}^d \sigma_i g_i$ has distribution $N(0, \sum_{i=1}^d \sigma_i^2) = N(0, \|\sigma\|_2^2)$.

References: (Anderson et al., 2017, Example 7.8 and Example 8.20), (Grimmett and Stirzaker, 2001, Example 4.8.3), (Durrett, 2019, Theorem 2.1.20 and Corollary 3.3.13), Wikipedia.

Remark B.4.7. Fact B.4.6 can be viewed in the more abstract context of ***stable random variables***. The Gaussian distribution is 2-stable. More generally, if X is an α-stable RV, and $X_1, \ldots, X_d$ are independent copies of X, then $\sum_{i=1}^d \sigma_i X_i$ has the distribution $\|\sigma\|_\alpha X$. See Equation (1.8) in "Stable Distributions: Models for Heavy Tailed Data" with $\beta = 0$, $\gamma_i = \sigma_i$ and $\delta = 0$.

References: (Durrett, 2019, Section 3.8), Wikipedia.

A useful fact about the Gaussian distribution is the following bound on its right tail.

Fact B.4.8 (Gaussian tail bound). Let X have the distribution $N(0,1)$. Let $x > 0$. Then

$$\frac{1}{\sqrt{2\pi}}(x^{-1} - x^{-3}) \exp(-x^2/2) \;\le\; \Pr[\, X \ge x \,] \;\le\; \frac{1}{\sqrt{2\pi}} x^{-1} \exp(-x^2/2).$$

References: (Vershynin, 2018, Proposition 2.1.2), (Durrett, 2019, Theorem 1.2.6), (Feller, 1968, Lemma VII.1.2), (Wainwright, 2019, Exercise 2.2).

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