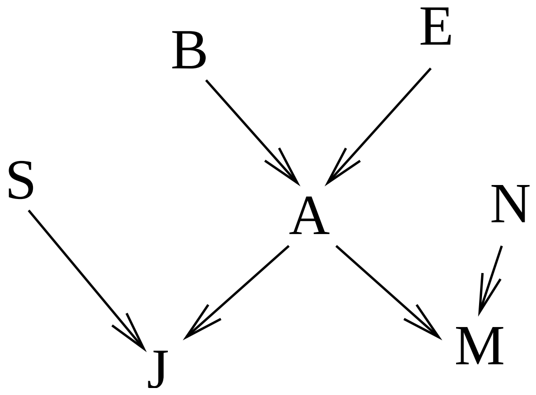
CS535c Fall 2004: Homework 2

Out Wed 22 Sep, Due Mon 27 Sep

1 I-maps for DAGs

[25 points, 3 per correct edge in the final Imap, plus 1 bonus if no errors.]

Consider the Bayes net G shown below. Draw the minimal I-map for $P(B, E, S, N, J, M)$ (i.e., marginalizing out the A variable), using the ordering B, E, S, N, J, M. You new model should have extra arcs, to compensate for the fact that A has been removed.



2 Independence in undirected Gaussian graphical models

[30 points, 10 per correct answer.]

Let x be a 4 dimensional Gaussian random variable with zero mean and covariance matrix Σ give by

$$\Sigma = \frac{1}{45} \begin{pmatrix} 21 & -9 & 6 & -9 \\ -9 & 21 & -9 & 6 \\ 6 & -9 & 21 & -9 \\ -9 & 6 & -9 & 21 \end{pmatrix}$$

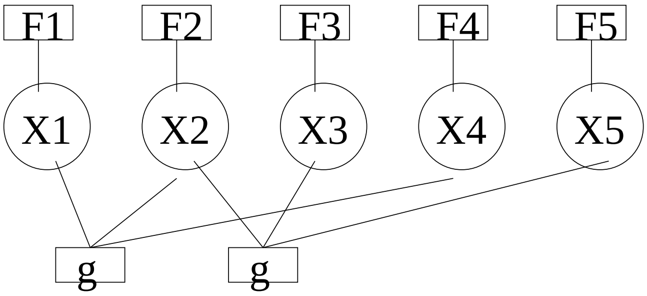
For each of the following assertions, say whether it is true or false.

- $x_1 \perp x_3$
- $x_1 \perp x_3 | x_2$
- $x_1 \perp x_3 | x_2, x_4$

3 Error correcting codes (Inference in factor graphs)

[45 points].

Consider the factor graph shown below, where circles represent binary random variables, and squares represent factors. X_1, X_2, X_3 are message bits that we are trying estimate; X_4 and X_5 are odd parity check bits, i.e., they are true if an odd number of their parents are true, otherwise false. In otherwords, $X_4 = X_1 \oplus X_2$, and $X_5 = X_2 \oplus X_3$, where $\oplus$ represents xor.



Suppose we receive noisy observations of all 5 bits; call these observations $Y_{1:5}$. Let $F(s,i) = p(Y_i|X_i = s)$ be the local evidence vector at node i. Thus $F(:,i) = [1, 0]$ means that X_i is observed to be in state $s = 1$, $F(:,i) = [0, 1]$ means that X_i is observed to be in state $s = 2$, and $F(:,i) = [0.5, 0.5]$ means that the observations about X_i are uninformative (uniform prior).

The joint distribution over all 5 bits is given by

$$P(X_{1:5}|y_{1:5}) = \frac{1}{Z} g(X_1, X_2, X_4) \times g(X_2, X_3, X_5) \times \prod_{i=1}^{5} f_i(X_i)$$

where g represents the parity check function and $f_i = F(:,i)$ is the local evidence. (Note that the evidence, $y_{1:5}$, has been "compiled" into the local evidence potentials f_i.)

By marginalizing out the parity check bits (i.e., computing $P(X_{1:3}|y_{1:5})$), we can decode the message. In general, there may not be a unique decoding (i.e., the posterior may have more than one mode).

Your task is to compute $P(X_{1:3}|y_{1:5})$ and Z for the 5 different evidence scenarios below Just fill in the numbers in the table; the first column (scenario) has been done for you. Then answer the questions below. Note: **You must vectorize your Matlab code; 2 point penalty for every unnecessary for loop!**.

Hint: in Matlab, the parity check factor can be represented as a $2 \times 2 \times 2$ table shown below. Since Matlab indexes arrays starting with 1, not 0, I use state 1 to mean true (1) and state 2 to mean 0 (false). Also, note that Matlab toggles indices from left to right (Fortran memory layout, the opposite of C). Hence

```
% xorTbl(x1,x2,x3) = P(x3|x1,x2)
% x1 x2 P(x3=1) P(x3=2)
% 1  1  0       1
% 0  1  1       0
% 1  0  1       0
% 0  0  0       1
xorTbl = reshape([0 1 1 0  1 0 0 1], [2 2 2]);
```

Other commands you might find useful: `ind2sub`, `repmat`, `warning off MATLAB:divideByZero`.

1. (5 points). Draw the Markov network corresponding to the factor graph above

2. (0 points). In scenario 1, the local evidence is as follows, where $F(1,i) = p(y_i|X_i = 1)$ (true) and $F(2,i) = p(y_i|X_i = 2)$ (false).

```
F = [0.5 0.5 0.5 0 0;
     0.5 0.5 0.5 1 1];
```

In other words, all the message bits are uncertain, but both parity checks are perfectly observed to be in state 2 (false). The value of Z and the distribution $P(X_{1:3}|y_{1:3})$ is shown in column 1 of the table. Please list your answers in the same order!

X_1	X_2	X_3	S1	S2	S3	S4	S5
1	1	1	0.5000	?	?	?	?
0	1	1	0.0000	?	?	?	?
1	0	1	0.0000	?	?	?	?
0	0	1	0.0000	?	?	?	?
1	1	0	0.0000	?	?	?	?
0	1	0	0.0000	?	?	?	?
1	0	0	0.0000	?	?	?	?
0	0	0	0.5000	?	?	?	?
Z			0.25	?	?	?	?

3. (4 points). What are the 2 most probable decode codewords in scenario 1? i.e., Compute $x_{1:3} = \arg\max P(X_{1:3}|y_{1:5})$. Explain why there are 2 modes in this distribution.

4. (9 points). In scenario 2, the local evidence is as follows:

```
F = [0.9 0.5 0.5 0 0;
     0.1 0.5 0.5 1 1];
```

   In other words, bit 1 is likely to be in state 1, bits 2 and 3 are uncertain; all parities are in state 2. Fill in column 2.

5. (9 points). In scenario 3, the local evidence is as follows:

```
F = [0.9 0.9 0.9 0 0;
     0.1 0.1 0.1 1 1];
```

   In other words, bits 1–3 are likely to be in state 1; all parities are in state 2. Fill in column 3.

6. (9 points). In scenario 4, the local evidence is as follows:

```
F = [0.9 0.9 0 0 0;
     0.1 0.1 1 1 1];
```

   In other words, bits 1–2 are likely to be in state 1, bit 3 is definitely in state 2; all parities are in state 2. Fill in column 4.

7. (9 points). In scenario 5, the local evidence is as follows:

```
F = [0.9 1 0 0 0;
     0.1 0 1 1 1];
```

   In other words, bit 1 is likely in state 1, bit 2 is definitely in state 1, bit 3 is definitely in state 2; and all parities are in state 2. Fill in column 5. Explain your answer.