CS340 Machine learning
Naïve Bayes classifiers

Document classification

- Let $Y \in \{1,\ldots,C\}$ be the class label and $x \in \{0,1\}^d$
- eg $Y \in$ {spam, urgent, normal},

  x_i = I(word i is present in message)
- Bag of words model

1 2 3 4 5 6 7
Words = {john, mary, sex, money, send, meeting, unk}

"John sent money to Mary after the meeting about money"

↓ Stop word removal

"john sent money mary after meeting about money"

↓ Tokenization

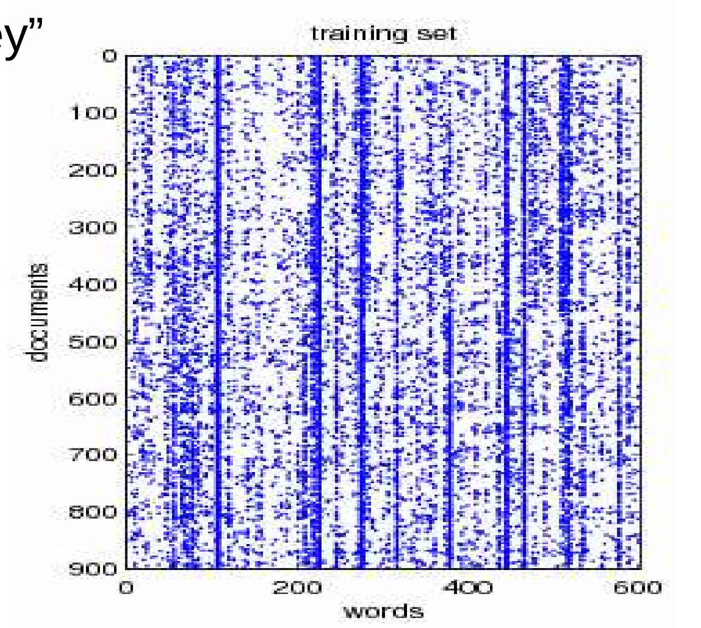
1 7 4 2 7 6 7 4

↓ Word counting

[1, 1, 0, 2, 0, 1,3]

↓ Thresholding (binarization)

[1, 1, 0, 1, 0, 1,1]



Bayes rule for classifiers

Class-conditional density

Class prior

Class posterior

$$p(y=c|x) = \frac{p(x|y=c)p(y=c)}{\sum_{c'} p(x|y=c')p(y=c')}$$

Normalization constant

Class conditional density p(x|y=c)

- What is the probability of generating a d-dimensional feature vector for each class c?
- Let us assume we generate each feature independently (naive Bayes assumption)

$$p(x|y=c) = \prod_{i=1}^{d} p(x_i|y=c)$$

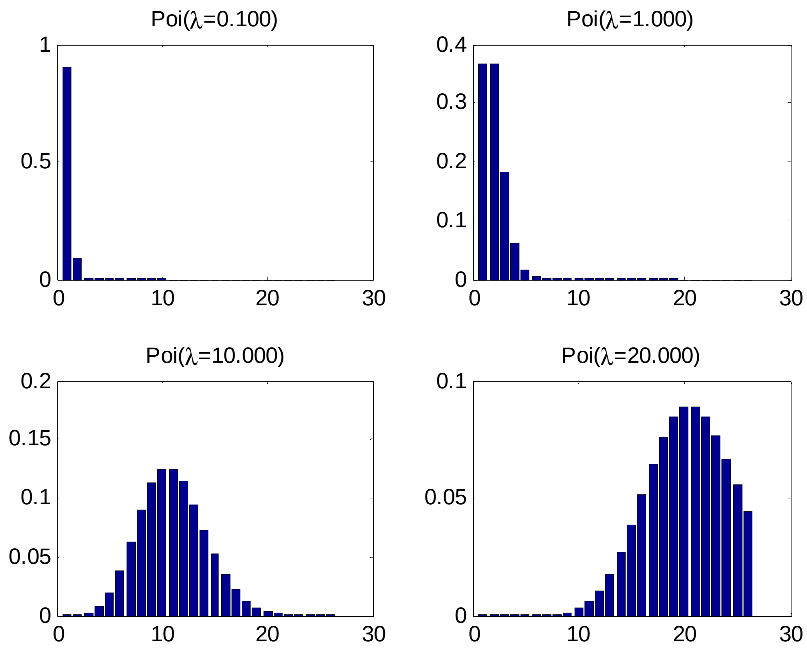
- E.g., prob of seeing "send" is assumed to be independent of seeing "money" given that we know this is a spam email
- Allows us to use 1 dimensional density models $p(x_i|y)$. Can combine features of different types.

Count features (multivariate Poisson)

- Suppose $X_i \in \{0,1,2,\ldots\}$ counts the number of times word i occurs.
- A suitable class-conditional density is

$$X_i | y = c \sim Poi(\lambda_{ic})$$

- The likelihood is $$p(x|y=c) \propto \prod_{i=1}^{d} e^{-\lambda_{ic}} \lambda_{ic}^{x_i}$$



Count features (multinomial model)

- Let $(X_1,\ldots X_d) \mid y=c, N \sim \text{Mult}(\theta_c, N)$

$$P(x_1, \ldots, x_d | \theta_c, N) = \binom{N}{x_1 \; \ldots \; x_d} \prod_{i=1}^{d} \theta_{ic}^{x_i}$$

X_i's no longer conditionally independent since $\sum_i x_i = N$

$$= \frac{N!}{x_1! x_2! \ldots x_d!} \prod_{i=1}^{d} \theta_{ic}^{x_i}$$

We also require $\sum_i \theta_i = 1$.

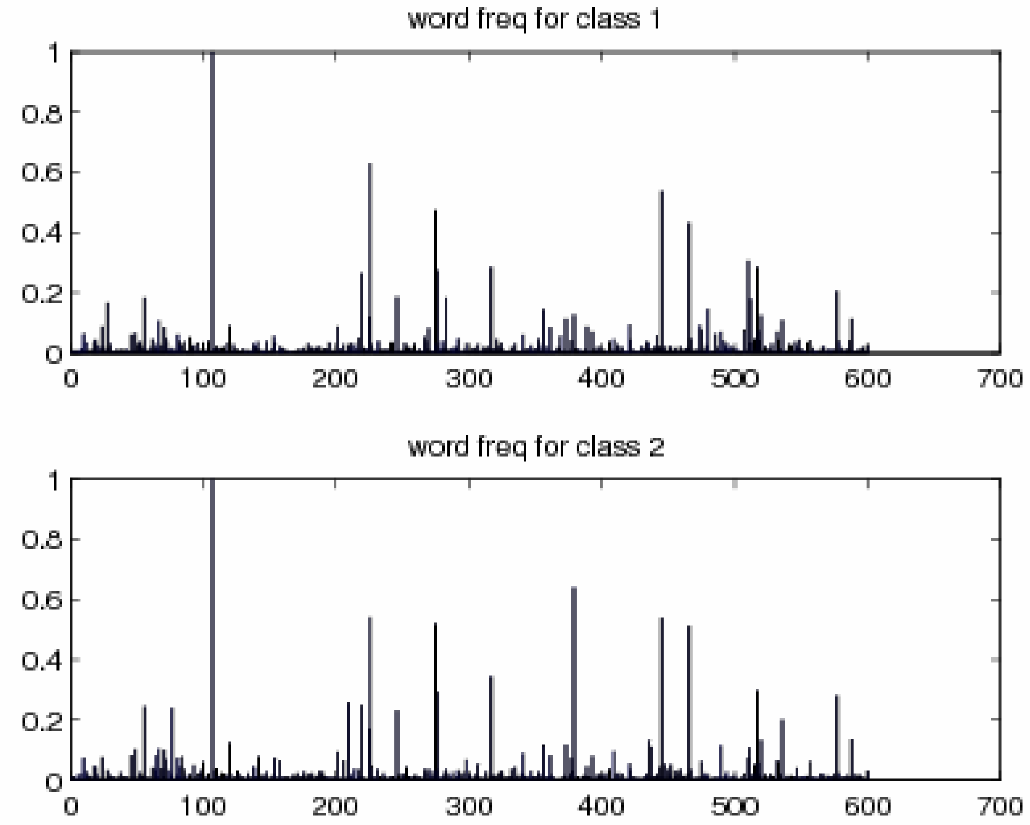
$$= (\sum_i x_i)! \prod_i \frac{\theta_{ic}^{x_i}}{x_i!}$$

where $N=\sum_i x_i$ is the number of words in the document (assumed independent of Y=c).

Binary features (multivariate Bernoulli)

- Let $X_i|y=c \sim \text{Ber}(\theta_{ic})$ so $p(X_i=1|y=c) = \theta_{ic}$

$$p(x|y=c) = \prod_{i=1}^{d} \theta_{ic}^{I(x_i=1)} (1-\theta_{ic})^{I(x_i=0)}$$



Which class-conditional density?

- For document classification, the multinomial model is found to work best. However, we will mostly focus on the multivariate Bernoulli (binary features) model, for simplicity.
- We can easily handle features of different types, eg $x_1 \in \{0,1\}$, $x_2 \in R$, $x_3 \in R^+$, $x_4 \in \{0,1,2,\dots\}$
- We can use mixtures of Gaussians/ Gammas/ Bernoullis etc. to get more accurate models (see later).

Class prior

- Let $(Y_1,..,Y_C) \sim \text{Mult}(\pi, 1)$ be the class prior.

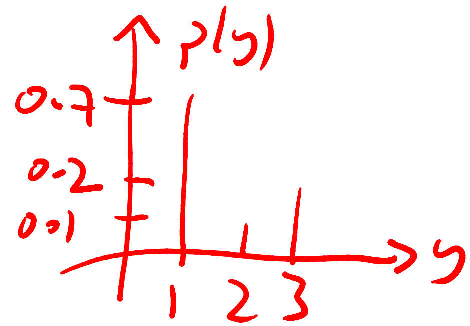
$$P(y_1,\ldots,y_C|\pi) = \prod_{c=1}^{C} \pi_c^{I(y_c=1)} \qquad \sum_{c=1}^{C} \pi_c = 1$$

- Since $\sum_c Y_c=1$, only one bit can be on. This is called a 1-of-C encoding. We can write Y=c instead.

$$Y=2 \equiv (Y_1,Y_2,Y_3) = (0,1,0)$$

$$P(y|\pi) = \prod_{c=1}^{C} \pi_c^{I(y=c)} = \pi_y$$

- e.g., p(spam)=0.7, p(urgent)=0.1, p(normal)=0.2



Class posterior

- Bayes rule

$$p(y=c|x) = \frac{p(y=c)p(x|y=c)}{p(x)} = \frac{\pi_c \overbrace{\prod_{i=1}^{d} \theta_{ic}^{I(x_i=1)} (1-\theta_{ic})^{I(x_i=0)}}^{f_c}}{p(x)}$$

- Since numerator and denominator are very small number, use logs to avoid underflow

$$\log p(y=c,x) = \log \pi_c + \sum_{i=1}^{d} I(x_i=1)\log\theta_{ic} + I(x_i=0)\log(1-\theta_{ic}) - \log p(x)$$

- How compute the normalization constant?

$$\log p(x) = \log[\sum_c p(y=c,x)] = \log[\sum_c \pi_c f_c]$$

Log-sum-exp trick

- Define

$$\begin{aligned}
\log p(x) &= \log[\sum_c \pi_c f_c] \\
b_c &= \log \pi_c + \log f_c \\
\log p(x) &= \log \sum_c e^{b_c} = \log \left[(\sum_c e^{b_c}) e^{-B} e^B \right] \\
&= \log \left[(\sum_c e^{b_c - B}) e^B \right] = \left[\log(\sum_c e^{b_c - B}) \right] + B \\
B &= \max_c b_c
\end{aligned}$$

$$\log(e^{-120} + e^{-121}) = \log\left(e^{-120}(e^0 + e^{-1})\right) = \log(e^0 + e^{-1}) - 120$$

- In Matlab, use Minka's function S = logsumexp(b)

logjoint = log(prior) + counts * log(theta) + (1-counts) * log(1-theta); $\log p(y=c, x)$
logpost = logjoint – logsumexp(logjoint) $\log p(y=c|x)$

Missing features

- Suppose the value of x_1 is unknown
- We can simply drop the term $p(x_1|y=c)$.

$$
\begin{aligned}
p(y=c|x_{2:d}) &\propto p(y=c, x_{2:d}) \\
&= \int p(y=c, x_1, x_{2:d}) dx_1 \\
&= p(y=c)[\int p(x_1|y=c)dx_1] \prod_{i=2}^{d} p(x_i|y=c) \\
&= p(y=c) \prod_{i=2}^{d} p(x_i|y=c)
\end{aligned}
$$

- This is a big advantage of generative classifiers (which specify $p(x|y=c)$) over discriminative classifiers (that learn $p(y=c|x)$ directly).

Parameter estimation

- So far we have assumed that the parameters of p(x|y=c) and p(y=c) are known.
- To estimate p(y=c), we can use MLE or MAP or fully Bayesian estimation of a multinomial, eg

$$\hat{\pi}_c^{MAP} = \frac{N_c + \alpha_c - 1}{\sum_{c'} (N_c + \alpha_{c'} - 1)}$$

- We can then use the plug-in approximation

$$p(y|D) \approx \prod_c \hat{\pi}_c^{I(y=c)}$$

or the posterior predictive

$$p(y|D) = \prod_c \overline{\pi}_c^{I(y=c)}$$

Posterior predictive for a multinomial

- Recall that, for the Dirichlet-multinomial model, the posterior predictive is equivalent to plugging in the posterior mean parameters, since

$$
\begin{aligned}
p(y=c|D) &= \int p(y=c|\pi_c)p(\pi_c|D)d\pi_c \\
&= \int \pi_c Dir(\pi|\alpha'_{c1},\ldots,\alpha'_{cK})d\pi_c \\
&= \overline{\pi}_c = \frac{N_c+\alpha_c}{N+\alpha}
\end{aligned}
$$

MLE for Bernoulli features

- We will assume the params for p(x|y=c) are independent for each class.
- Since we treat each feature separately, we just count how many times word j occurred in documents of class c, and divide by the number of documents of class c

$$\hat{\theta}_{jc} = \frac{\sum_{i:y_i=c} \sum_{w \in i} I(w=j)}{\sum_{i:y_i=c} 1} = \frac{N_{jc}}{N_c}$$

- We can easily add priors to regularize this.

Sum over documents i which belong to class c

Sum over words w in document i

Class conditional densities

- At test time, we can either use a plug-in approximation

$$p(\mathbf{x}|y=c,D) \approx \prod_j \hat{\theta}_{jc}^{I(x_j=1)}(1-\hat{\theta}_{jc})^{I(x_j=0)}$$

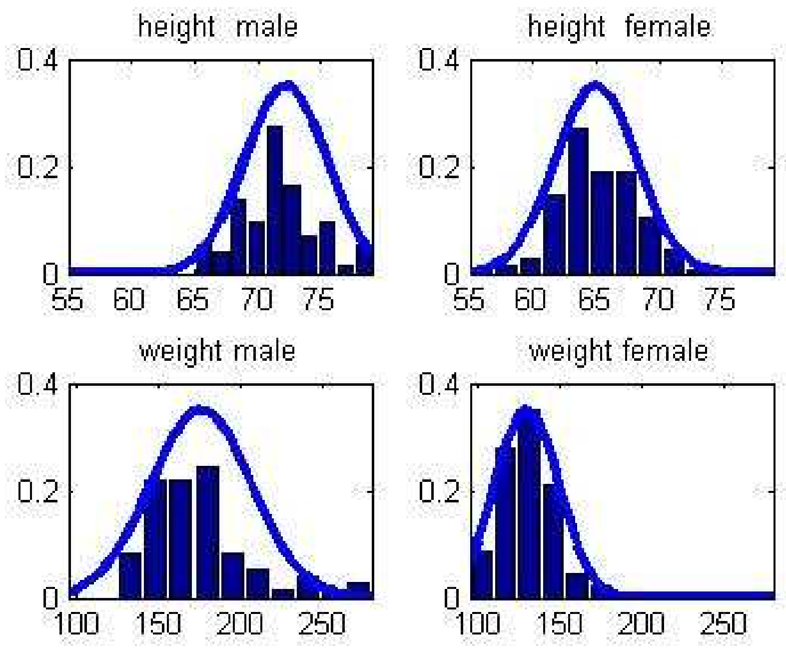
or the exact posterior predictive

$$p(\mathbf{x}|y=c,D) = \prod_j \overline{\theta}_{jc}^{I(x_j=1)}(1-\overline{\theta}_{jc})^{I(x_j=0)}$$

Naïve Bayes with real-valued features

- If $X_j \in R$, we can use Gaussian class conditional densities $X_j|y=c \sim N(\mu_{jc}, \sigma_{jc})$

$$p(x|y=c) = \prod_{j=1}^{d} \frac{1}{\sqrt{2\pi\sigma_{jc}^2}} \exp(-\frac{1}{2\sigma_{jc}^2}(x_j - \mu_{jc})^2)$$



Plug-in approximation

- We can compute MLEs for each feature j and class c separately

$$\begin{aligned}
\hat{\theta}_{jc} &= (\hat{\mu}_{jc}, \hat{\sigma}^2_{jc}) \\
\hat{\mu}_{jc} &= \frac{1}{n_c} \sum_{i:y_i=c} x_{ij} = \overline{x}_{jc} \\
\hat{\sigma}^2_{jc} &= \frac{1}{n_c} \sum_{i:y_i=c} (x_{ij} - \overline{x}_{jc})^2
\end{aligned}$$

- Then we can use a plug-in approximation

$$\begin{aligned}
p(y=c|x_{1:d}, D) &\propto p(y=c|D) \prod_{j=1}^{d} p(x_j|y=c, D) \\
&\approx p(y=c|\hat{\pi}) \prod_{j=1}^{d} p(x_j|y=c, \hat{\theta}_{jc}) \\
&= \hat{\pi}_c \prod_{j} \mathcal{N}(x_j|\hat{\mu}_{jc}, \hat{\sigma}^2_{jc})
\end{aligned}$$

Fully Bayesian solution

- If we use conjugate priors, it is simple to derive a fully Bayesian solution: we just update the hyper-parameters for each feature j and class c, and then use the predictive distribution, which is a student T

$$
\begin{aligned}
p(y=c|x_{1:d}, D) &\propto p(y=c|D) \prod_{j=1}^{d} p(x_j|y=c, D) \\
&= \overline{\pi}_c \prod_j t_{2\alpha_{jcn}} \left(x_j | \mu_{jcn}, \frac{\beta_{jcn}(\kappa_{jcn}+1)}{\alpha_{jcn}\kappa_{jcn}} \right)
\end{aligned}
$$