Level Set Methods
CPSC 542D, Term 2, Winter 2006–2007
Assigned Monday, January 29. Due Friday March 2.

Homework #2

The questions are not in any particular order—you may be able to answer some of the ones that appear later before some of the ones that appear earlier.

1. **Motion by convection.** In $\mathbb{R}^2$, start with a square with sides of length $l = 1.0$ centered at $c = (1.0, 0.0)$. Using calls to the toolbox of level set methods, move the square under the velocity field
$$v(x,t) = \begin{bmatrix} 1 - 2x_2^2 \cos(2\pi t) \\ -x_2 \sin(\pi x_1) \end{bmatrix}.$$
Choose computational domain $\Omega = [-2, +2) \times [-1.5, +1.5]$, $\Delta x = 0.02$, periodic BC in x_1 and linear extrapolation BC in x_2. Simulate this motion for $t \in [0, 4]$ and display the motion as a 3×3 grid of subplots at $t_i = i/2$ for $i = \{0, 1, 2, \ldots, 8\}$. Perform the simulation twice:

   (a) Once with first order temporal and spatial approximations.

   (b) Once with third order temporal and spatial approximations.

   Submit any code you write and your two plots. How long do each of the two simulations take? In a paragraph or two, describe how you might evaluate their accuracy.

2. **Coding the details.** Consider a square with sides of length $l = 0.4$ centered at $c = (0.3, 0.5)$ in $\mathbb{R}^2$. We wish to move this square under the velocity field $v(x) = \begin{bmatrix} +1 & 0 \end{bmatrix}^T$ (convection to the right) using a dynamic implicit surface. Perform this task in each of two ways:

   (a) With a first order upwinded approximation of the spatial derivative
$$D_x^+ \phi(x_i) = \frac{\phi(x_{i+1}) - \phi(x_i)}{\Delta x} \qquad \text{or} \qquad D_x^- \phi(x_i) = \frac{\phi(x_i) - \phi(x_{i-1})}{\Delta x}$$
   and $\Delta t \propto \Delta x$.

   (b) With a first order centered approximation of the spatial derivative
$$D_x^0 \phi(x_i) = \frac{\phi(x_{i+1}) - \phi(x_{i-1})}{2\Delta x}$$
   and $\Delta t \propto \Delta x$.

   You may solve this problem either by coding a first order level set method from scratch or by modifying components of the toolbox.

   Use plots to demonstrate experimentally that method (a) appears to be stable, but (b) does not. In every case use a first order temporal approximation, a conservative CFL number of 0.5, a computational domain $\Omega = [0, 1]^2$, grid cell size $\Delta x = 0.02$ and BC of your choice.

3. **Motion in the normal direction.** Start with two circles of radius $r = 0.2$ centered at $c^{\pm} = (\pm 0.3, +0.5)$ in $\mathbb{R}^2$. Simulate motion in the (outward) normal direction at speed $s = +1.0$ for $t \in [0, \frac{1}{4}]$ in two different manners:

   - By an explicit surface representation. Choose a collection of twenty points on the surface of each circle, and connect the points with line segments. Move the points under the flow, maintaining connectivity.
   - By a dynamic implicit surface. Use the toolbox to simulate the motion of the surfaces on the domain $\Omega = [-1, +1] \times [0, +1]$ with $\Delta x = 0.02$ and linear extrapolated BC.

   For each of the two cases, provide a series of (sub)plots showing the motion, and your (commented) code.

   In one paragraph each, briefly answer the following questions.

   (a) How did you determine the direction of motion for the points in the explicit surface representation case? Is your scheme specific to this example, or can it be extended to other surfaces?

   (b) In each of the two cases, what happens to the surface(s) beyond $t = 0.1$?

   (c) What would be required to make the explicit surface representation algorithm generate the same topological behavior as the dynamic implicit surface algorithm?

4. **Order of Accuracy.** Some practice demonstrating consistency with Taylor series expansions.

   (a) We would like to approximate $D_x\varphi(x, t)$ for $x \in \mathbb{R}$ on a uniform grid with node spacing h. Let $\varphi_i = \varphi(x_i, t)$ and $x_i = ih$. Demonstrate that the following leftward finite difference approximation of $D_x\varphi(x, t)$ at x_i is at least $\mathcal{O}(h^2)$ accurate.

   $$\frac{3\varphi_i - 4\varphi_{i-1} + \varphi_{i-2}}{2h}. \tag{1}$$

   (b) Another possible finite difference approximation of $D_x\varphi(x, t)$ at x_i is

   $$\frac{\varphi_{i+1} - \varphi_{i-1}}{2h} = D_x\varphi_i + \frac{1}{6}h^2 D_x^3\varphi_i + \mathcal{O}(h^3).$$

   Give one or more reasons why this approximation might be better than (1). Give one or more reasons why this approximation might be worse than (1).

5. **Identify the Form of the Equation.** Separate each of the equations below into terms. For each term—except for the temporal derivative terms—describe what type of spatial term approximation from the toolbox would be used (if any). If more than one term is not a temporal derivative, specify how you would combine the terms. Specify any parameters needed by any term approximations. For example, if you identify a term as motion by convection, specify the velocity field.

   In each equation, ϕ is the unknown function of space and time. The variable κ is the mean curvature of the ϕ function. The remaining parameters may not have their usual meaning. Parameters and notation may not be consistent between equations. Some of the terms may not have a corresponding toolbox approximation.

   (a) $\phi_t + \|\nabla\phi\| \left(u(x,t)^T \frac{\nabla\phi}{\|\nabla\phi\|} + \kappa(x,t)v(x) \right) = 0.$

   (b) $\frac{\partial\phi}{\partial s} = b(x) \left\| \frac{\partial\phi}{\partial x} \right\|.$

   (c) $\phi_t + v(y)\|\phi_y\| + \rho\phi = 0.$

   (d) $D_t\phi + \kappa(x,t) = 0.$

   (e) $D_s\phi + \|\nabla\phi\|^2 = \sigma(x).$

   (f) $D_t\phi = 2\Delta\phi.$

   (g) $\max(\phi_t + f(x,t) \cdot \phi_x, \theta - \phi) = 0.$