Reasoning Under Uncertainty: Variable Elimination for Bayes Nets

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UBC CS 322 – Uncertainty 6

March 22, 2013

Textbook §6.4, 6.4.1

Announcements (1)

- Assignment 4 due Wednesday, April 3rd

- Final exam
  - Thursday, April 18th, 8:30 – 11am in PHRM 1101
  - Same general format as midterm (~60% short questions)
    - List of short questions will be on Connect
  - Practice final to be provided
  - More emphasis on material after midterm
  - How to study?
    - Practice exercises, assignments, short questions, lecture notes, text, problems in text, …
    - Use TA and my office hours (extra office hours TBA if needed)
    - Review sessions: last class plus more TBA if needed
    - Submit topics you want reviewed on Connect

Announcements (2)

- Teaching Evaluations are online
  - You should have received a message about them
  - Secure, confidential, mobile access
- Your feedback is important!
  - Allows us to assess and improve the course material
  - I use it to assess and improve my teaching methods
  - The department as a whole uses it to shape the curriculum
  - Teaching evaluation results are important for instructors
    - Appointment, reappointment, tenure, promotion and merit, salary
  - UBC takes them very seriously (now)
  - Evaluations close at 11:59PM on April 9, 2013.
    - Before exam, but instructors can't see results until *after* we submit grades
  - Please do it!
- Take a few minutes and visit https://eval.olt.ubc.ca/science

Lecture Overview

- Recap Observations and Inference
- Inference in General Bayesian Networks
  - Factors:
    - Assigning Variables
    - Summing out Variables
    - Multiplication of Factors
  - The variable elimination algorithm
  - Example trace of variable elimination

Learning Goals For Previous Class

- Build a Bayesian Network for a given domain
- Understand basics of Markov Chains and Hidden Markov Models
- Classify the types of inference:
  - Diagnostic, Predictive, Mixed, Intercausal

Assignment 4: Q1, Q2, Q3 and Q4 NOW.

Q5: variable elimination (VE) this class.

Inference in Bayesian Networks

Given:

- A Bayesian Network BN, and
- Observations of a subset of its variables E: E=e
- A subset of its variables Y that is queried

Compute: The conditional probability P(Y|E=e)

How: Run the Variable Elimination (VE) algorithm

N.B. We can already do all this: See lecture “Uncertainty2” Inference by Enumeration topic.

The BN represents the JPD. Could just multiply out the BN to get full JPD and then do Inference by Enumeration BUT that’s extremely inefficient - does not scale.

Inference in Bayesian Networks

Given:

- A Bayesian Network BN, and
- Observations of a subset of its variables E: E=e
- A subset of its variables Y that is queried

Compute: The conditional probability P(Y|E=e)

How: Run the Variable Elimination (VE) algorithm

The VE algorithm manipulates conditional probabilities in the form of "factors". So first we have to introduce factors and the operations we can perform on them.

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Factors

- A factor is a function from a tuple of random variables to the real numbers R
- We write a factor on variables $X_1, \ldots, X_j$ as $f(X_1, \ldots, X_j)$

- P(Z|X,Y) is a factor f (X,Y,Z)
  - Factors do not have to sum to one
  - P(Z|X,Y) is a set of probability distributions: one for each combination of values of X and Y

- P(Z=f|X,Y) is a factor f(X,Y)

$f(X, Y)_{Z=f}$

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

Operation 1: assigning a variable

- We can make new factors out of an existing factor
- Our first operation:
  we can assign some or all of the variables of a factor.

f(X,Y,Z):

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
~~f~~	~~t~~	~~t~~	~~0.4~~
~~f~~	~~t~~	~~f~~	~~0.6~~
~~f~~	~~f~~	~~t~~	~~0.3~~
~~f~~	~~f~~	~~f~~	~~0.7~~

What is the result of assigning X= t ?

$f(X=t,Y,Z) = f(X, Y, Z)_{X = t}$

Y	Z	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

Factor of Y,Z

More examples of assignment

f(X,Y,Z):

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

f(X=t,Y,Z)

Factor of Y,Z

Y	Z	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f(X=t,Y,Z=f):

Y	val
t	0.9
f	0.8

Factor of Y

f(X=t,Y=f,Z=f): 0.8

Number

Operation 2: Summing out a variable

- Our second operation on factors:
  we can marginalize out (or sum out) a variable
  - Exactly as before. Only difference: factors don't have to sum to 1
  - Marginalizing out a variable X from a factor f($X_1, \ldots, X_n$) yields a new factor defined on $\{X_1, \ldots, X_n\} \setminus \{X\}$

$$\left(\sum_{X_1} f\right)(X_2, \ldots, X_j) = \sum_{x \in dom(X_1)} f(X_1 = x, X_2, \ldots, X_j)$$

f_3=

B	A	C	val
t	t	t	0.03
t	t	f	0.07
f	t	t	0.54
f	t	f	0.36
t	f	t	0.06
t	f	f	0.14
f	f	t	0.48
f	f	f	0.32

$(\sum_B f_3)(A,C)$

A	C	val
t	t	0.57
t	f	0.43
f	t	0.54
f	f	0.46

Operation 3: multiplying factors

$f_1(A,B)$:

A	B	Val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

$f_2(B,C)$:

B	C	Val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1(A,B) \times f_2(B,C)$:

A	B	C	val
t	t	t	0.03
t	t	f	0.1x0.7
t	f	t	0.9x0.6
t	f	f	…
f	t	t	
f	t	f	
f	f	t	
f	f	f	

Operation 3: multiplying factors

- The product of factor $f_1(A, B)$ and $f_2(B, C)$, where B is the variable in common, is the factor $(f_1 \times f_2)(A, B, C)$ defined by

$$(f_1 \times f_2)(A, B, C) = f_1(A, B) f_2(B, C)$$

- Note: A, B, and C can be sets of variables
  - The domain of $f_1 \times f_2$ is $A \cup B \cup C$

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General Inference in Bayesian Networks

Given

- A Bayesian Network BN, and
- Observations of a subset of its variables E: E=e
- A subset of its variables Y that is queried

Compute the conditional probability P(Y=y|E=e)

Definition of conditional probability

Marginalization over Y:
$P(E{=}e) = \Sigma_{y' \in dom(Y)} P(E{=}e, Y{=}y')$

$$P(Y = y \mid E = e) = \frac{P(Y = y, E = e)}{P(E = e)} = \frac{P(Y = y, E = e)}{\sum_{y' \in dom(Y)} P(Y = y', E = e)}$$

All we need to compute is the
joint probability of the query variable(s) and the evidence!

Variable Elimination: Intro (1)

- We can express the joint probability as a factor
  - Query: Y; Observed: $E_1, \ldots, E_j$; Other variables not involved in the query: $Z_1, \ldots, Z_k$
  - $f(Y, E_1, \ldots, E_j, Z_1, \ldots, Z_k)$
- We can compute $P(Y, E_1=e_1, \ldots, E_j=e_j)$ by
  - Assigning $E_1=e_1, \ldots, E_j=e_j$
  - Marginalizing out variables $Z_1, \ldots, Z_k$, one at a time
    - the order in which we do this is called our elimination ordering

$$P(Y, E_1 = e_1, \ldots, E_j = e_j) = \sum_{Z_k} \cdots \sum_{Z_1} f(Y, E_1, .., E_j, Z_1, .., Z_k)_{E_1 = e_1, \ldots, E_j = e_j}$$

- Are we done?
  - No. This would still represent the whole JPD (as a single factor)
  - We need to exploit the compactness of Bayesian networks

Variable Elimination: Intro (2)

- Recall the joint probability distribution of a Bayesian network
  - $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i|X_1,\ldots,X_{i-1})$
    $= \prod_{i=1}^{n} P(X_i|pa(X_i))$
- We will have a factor f_i for each conditional probability:
  - For each variable X_i, there is a factor f_i with domain $\{X_i\} \cup pa(X_i)$:
    $f_i(\{X_i\} \cup pa(X_i)) = P(X_i|pa(X_i))$

$$P(Y, E_1 = e_1, \ldots, E_j = e_j) = \sum_{Z_k} \cdots \sum_{Z_1} f(Y, E_1, \ldots, E_j, Z_1, \ldots, Z_k)_{E_1=e_1,\ldots,E_j=e_j}$$

$$= \sum_{Z_k} \cdots \sum_{Z_1} \prod_{i=1}^{n} (f_i)_{E_1=e_1,\ldots,E_j=e_j}$$

Computing sums of products

- Inference in Bayesian networks thus reduces to computing the sums of products
  - Example: it takes 9 multiplications to evaluate the expression ab + ac + ad + aeh + afh + agh.
  - How can this expression be evaluated more efficiently?
    - Factor out the a and then the h giving a(b + c + d + h(e + f + g))
    - This takes only 2 multiplications (same number of additions as above)
- Similarly, how can we compute $\sum_{Z_k} \prod_{i=1}^{n} f_i$ efficiently?
  - Factor out those terms that don't involve Z_k, e.g.:

$$\sum_{Z_k} f_1(Z_k) f_2(Y) f_3(Z_k, Y) f_4(X, Y)$$

$$= f_2(Y) f_4(X, Y) \left(\sum_{Z_k} f_1(Z_k) f_3(Z_k, Y) \right)$$

Summing out a variable efficiently

- To sum out a variable Z from a product $f_1 \times \ldots \times f_k$ of factors:
  - Partition the factors into
    - those that don't contain Z say $f_1 \times \ldots \times f_i$
    - those that contain Z say $f_{i+1} \times \ldots \times f_k$
- We know:

$$\sum_Z f_1 \times \ldots \times f_k = f_1 \times \ldots \times f_i \times \left(\sum_Z f_{i+1} \times \ldots \times f_k \right)$$

New factor! Let's call it f'

- We thus have $\sum_Z f_1 \times \ldots \times f_k = f_1 \times \ldots \times f_i \times f'$
- Store f' explicitly, and discard $f_{i+1} \ldots f_k$
- Now we've summed out Z

The variable elimination algorithm

To compute P(Y=y|E=e):

1. Construct a factor for each conditional probability
2. Assign the observed variables E to their observed values
3. Decompose the sum
4. Sum out all variables $Z_1, \ldots, Z_k$ not involved in the query
5. Multiply the remaining factors (which only involve Y)
6. Normalize by dividing the resulting factor f(Y) by $\sum_{y \in dom(Y)} f(Y)$

See the algorithm VE_BN in the P&M text, Section 6.4.1, Figure 6.8, p. 254.

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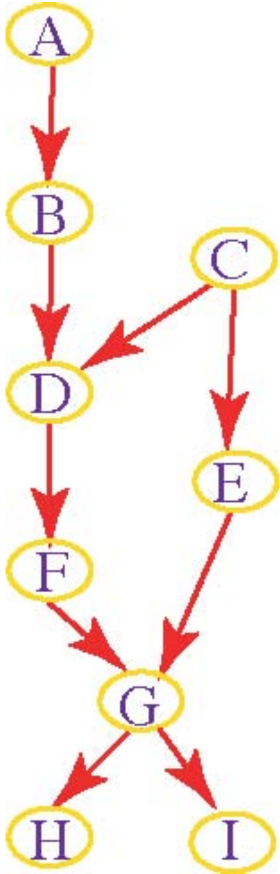
Variable elimination example: compute $P(G|H=h_1)$
Step 1: construct a factor for each cond. probability

$P(G,H) = \sum_{A,B,C,D,E,F,I} P(A,B,C,D,E,F,G,H,I) =$

$= \sum_{A,B,C,D,E,F,I} P(A)P(B|A)P(C)P(D|B,C)P(E|C)P(F|D)P(G|F,E)P(H|G)P(I|G)$

$= \sum_{A,B,C,D,E,F,I} f_0(A)\ f_1(B,A)\ f_2(C)\ f_3(D,B,C)\ f_4(E,C)\ f_5(F, D)\ f_6(G,F,E)\ f_7(H,G)\ f_8(I,G)$

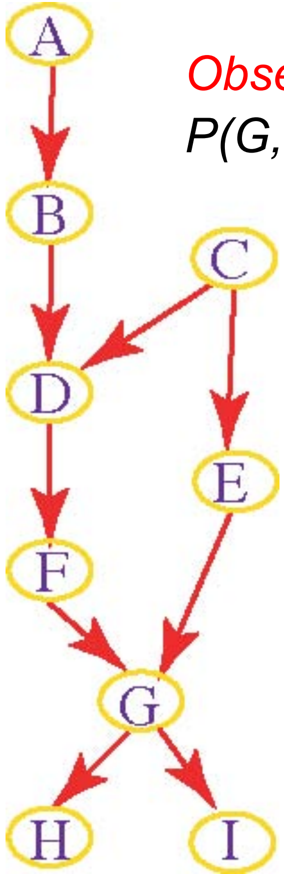


Variable elimination example: compute $P(G|H=h_1)$
Step 2: assign observed variables their observed value

$P(G,H) = \sum_{A,B,C,D,E,F,I} P(A,B,C,D,E,F,G,H,I) =$

$= \sum_{A,B,C,D,E,F,I} P(A)P(B|A)P(C)P(D|B,C)P(E|C)P(F|D)P(G|F,E)P(H|G)P(I|G)$

$= \sum_{A,B,C,D,E,F,I} f_0(A)\ f_1(B,A)\ f_2(C)\ f_3(D,B,C)\ f_4(E,C)\ f_5(F,D)\ f_6(G,F,E)\ f_7(H,G)\ f_8(I,G)$



Observe $H=h_1$:

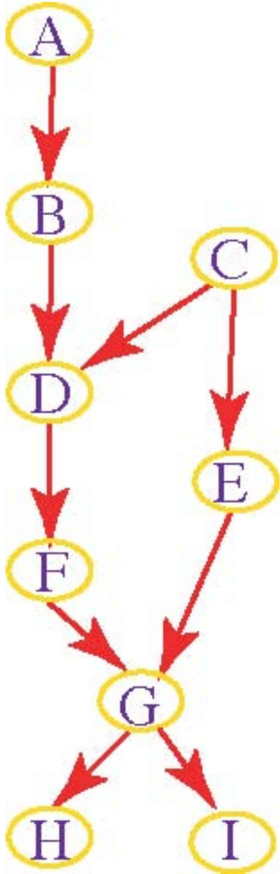
$P(G,H=h_1)=\sum_{A,B,C,D,E,F,I} f_0(A)\ f_1(B,A)\ f_2(C)\ f_3(D,B,C)\ f_4(E,C)$
$f_5(F,D)\ f_6(G,F,E)\ f_9(G)\ f_8(I,G)$

Assigning the variable $H=h_1$:
$f_7(H,G)_{H=h1} = f_9(G)$

Variable elimination example: compute $P(G|H=h_1)$
Step 3: decompose sum

$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$

$= \quad \sum_F \sum_D \quad \sum_B \sum_I \quad \sum_E \quad \sum_C \quad \sum_A$

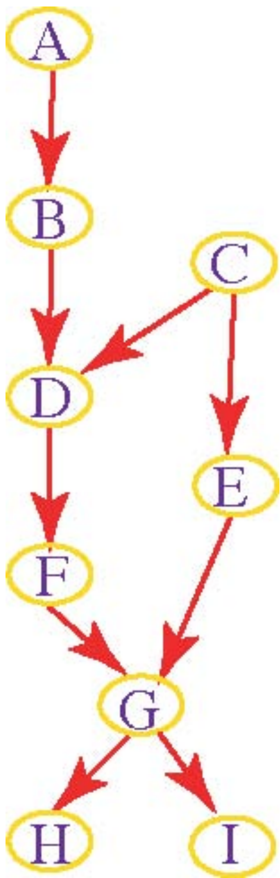


Elimination ordering: A, C, E, I, B, D, F

Variable elimination example: compute $P(G|H=h_1)$
Step 3: decompose sum

$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C) \sum_A f_0(A)\, f_1(B,A)$



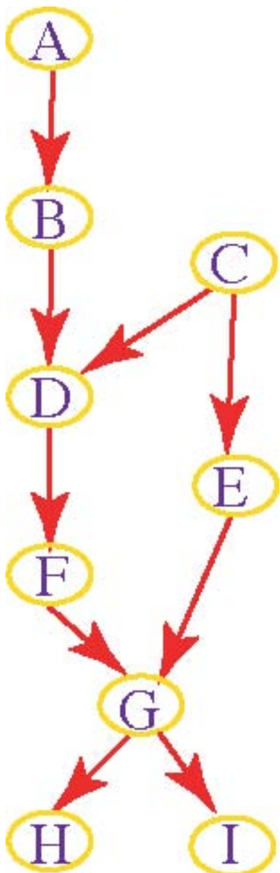
Elimination ordering: A, C, E, I, B, D, F

Variable elimination example: compute $P(G|H=h_1)$
Step 4: sum out non- query variables (one at a time)

$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F,D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$

$= f_9(G) \sum_F \sum_D f_5(F,D) \sum_B \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C) \sum_A f_0(A)\, f_1(B,A)$

$= f_9(G) \sum_F \sum_D f_5(F,D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C)$



Summing out A: $\sum_A f_0(A)\, f_1(B,A) = f_{10}(B)$
This new factor does not depend on C, E, or I, so we can push it outside of those sums.

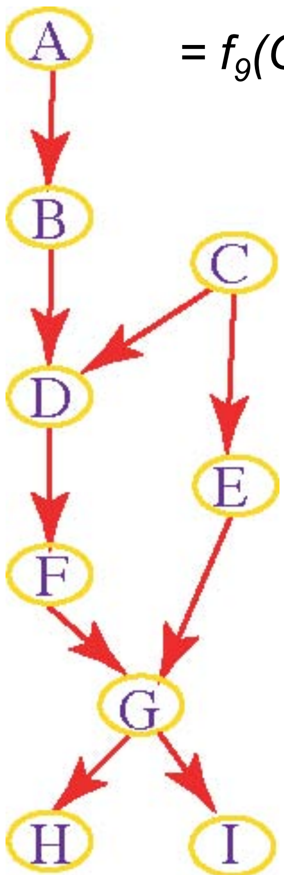
Elimination ordering: A, C, E, I, B, D, F

Variable elimination example: compute $P(G|H=h_1)$
Step 4: sum out non- query variables (one at a time)

$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C) \sum_A f_0(A)\, f_1(B,A)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C)$



$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E)\, f_{11}(D,B,E)$

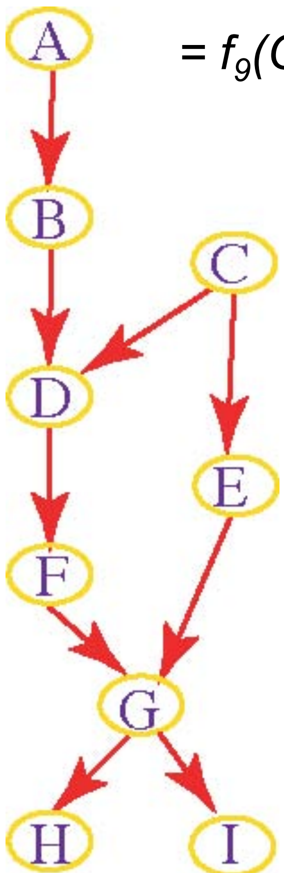
Elimination ordering: A, C, E, I, B, D, F

Variable elimination example: compute $P(G|H=h_1)$
Step 4: sum out non- query variables (one at a time)

$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C) \sum_A f_0(A)\, f_1(B,A)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C)$



$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E)\, f_{11}(D,B,E)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{12}(G,F,D,B) \sum_I f_8(I,G)$

Note the increase in dimensionality: $f_{12}(G,F,D,B)$ is defined over 4 variables

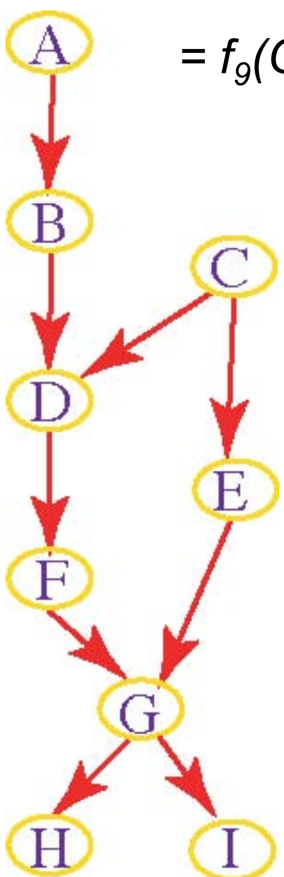
Elimination ordering: A, C, E, I, B, D, F

Variable elimination example: compute $P(G|H=h_1)$
Step 4: sum out non- query variables (one at a time)

$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C) \sum_A f_0(A)\, f_1(B,A)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C)$



$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E)\, f_{11}(D,B,E)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{12}(G,F,D,B) \sum_I f_8(I,G)$

$= f_9(G)\, f_{13}(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{12}(G,F,D,B)$

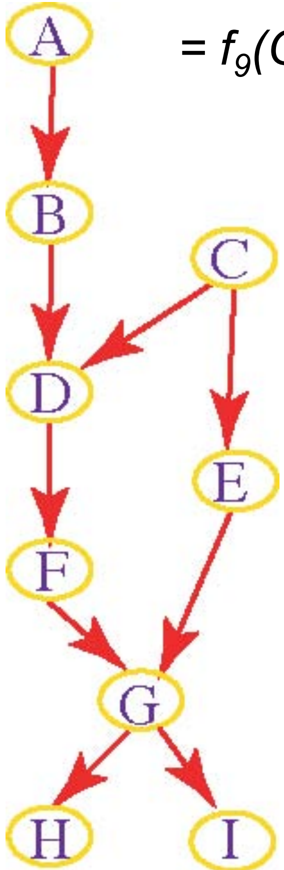
Elimination ordering: A, C, E, I, B, D, F

Variable elimination example: compute $P(G|H=h_1)$
Step 4: sum out non- query variables (one at a time)

$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C) \sum_A f_0(A)\, f_1(B,A)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C)$



$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E)\, f_{11}(D,B,E)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{12}(G,F,D,B) \sum_I f_8(I,G)$

$= f_9(G)\, f_{13}(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{12}(G,F,D,B)$

$= f_9(G)\, f_{13}(G) \sum_F \sum_D f_5(F, D)\, f_{14}(G,F,D)$

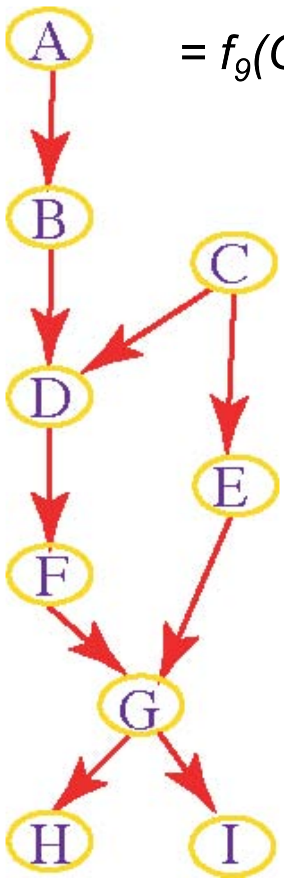
Elimination ordering: A, C, E, I, B, D, F

Variable elimination example: compute $P(G|H=h_1)$
Step 4: sum out non- query variables (one at a time)

$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C) \sum_A f_0(A)\, f_1(B,A)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C)$



$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E)\, f_{11}(D,B,E)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{12}(G,F,D,B) \sum_I f_8(I,G)$

$= f_9(G)\, f_{13}(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{12}(G,F,D,B)$

$= f_9(G)\, f_{13}(G) \sum_F \sum_D f_5(F, D)\, f_{14}(G,F,D)$

$= f_9(G)\, f_{13}(G) \sum_F f_{15}(G,F)$

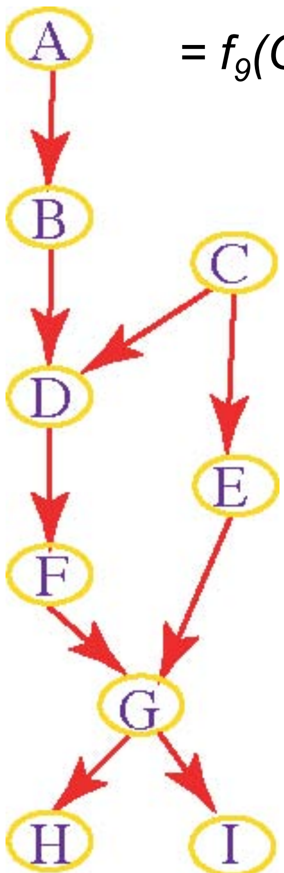
Elimination ordering: A, C, E, I, B, D, F

Variable elimination example: compute $P(G|H=h_1)$
Step 4: sum out non- query variables (one at a time)

$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C) \sum_A f_0(A)\, f_1(B,A)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C)$



$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E)\, f_{11}(D,B,E)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{12}(G,F,D,B) \sum_I f_8(I,G)$

$= f_9(G)\, f_{13}(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{12}(G,F,D,B)$

$= f_9(G)\, f_{13}(G) \sum_F \sum_D f_5(F, D)\, f_{14}(G,F,D)$

$= f_9(G)\, f_{13}(G) \sum_F f_{15}(G,F)$

$= f_9(G)\, f_{13}(G)\, f_{16}(G)$

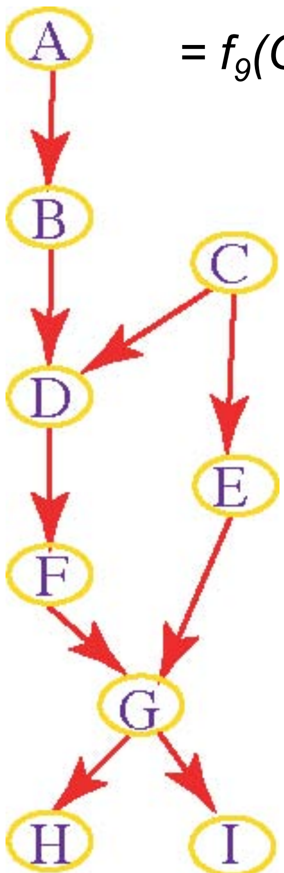
Elimination ordering: A, C, E, I, B, D, F

Variable elimination example: compute $P(G|H=h_1)$
Step 5: multiply the remaining factors

$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\ f_1(B,A)\ f_2(C)\ f_3(D,B,C)\ f_4(E,C)\ f_5(F, D)\ f_6(G,F,E)\ f_9(G)\ f_8(I,G)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\ f_3(D,B,C)\ f_4(E,C) \sum_A f_0(A)\ f_1(B,A)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\ f_3(D,B,C)\ f_4(E,C)$



$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E)\ f_{11}(D,B,E)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\ f_{12}(G,F,D,B) \sum_I f_8(I,G)$

$= f_9(G)\ f_{13}(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\ f_{12}(G,F,D,B)$

$= f_9(G)\ f_{13}(G) \sum_F \sum_D f_5(F, D)\ f_{14}(G,F,D)$

$= f_9(G)\ f_{13}(G) \sum_F f_{15}(G,F)$

$= f_9(G)\ f_{13}(G)\ f_{16}(G)$

$= f_{17}(G)$

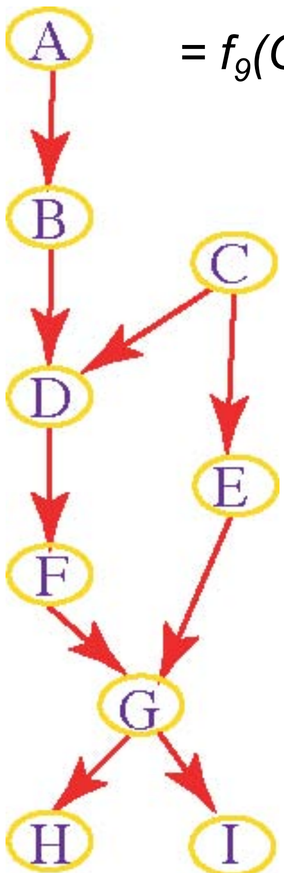
Elimination ordering: A, C, E, I, B, D, F

Variable elimination example: compute $P(G|H=h_1)$
Step 6: normalize

$P(G,H=h_1) = \sum_{A,B,C,D,E,F,I} f_0(A)\, f_1(B,A)\, f_2(C)\, f_3(D,B,C)\, f_4(E,C)\, f_5(F, D)\, f_6(G,F,E)\, f_9(G)\, f_8(I,G)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C) \sum_A f_0(A)\, f_1(B,A)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E) \sum_C f_2(C)\, f_3(D,B,C)\, f_4(E,C)$



$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B) \sum_I f_8(I,G) \sum_E f_6(G,F,E)\, f_{11}(D,B,E)$

$= f_9(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{12}(G,F,D,B) \sum_I f_8(I,G)$

$= f_9(G)\, f_{13}(G) \sum_F \sum_D f_5(F, D) \sum_B f_{10}(B)\, f_{12}(G,F,D,B)$

$= f_9(G)\, f_{13}(G) \sum_F \sum_D f_5(F, D)\, f_{14}(G,F,D)$

$= f_9(G)\, f_{13}(G) \sum_F f_{15}(G,F)$

$= f_9(G)\, f_{13}(G)\, f_{16}(G)$

$= f_{17}(G)$

$$P(G=g \mid H=h_1) = \frac{P(G=g, H=h_1)}{P(H=h_1)}$$

$$= \frac{P(G=g, H=h_1)}{\sum_{g' \in dom(G)} P(G=g', H=h_1)} = \frac{f_{17}(g)}{\sum_{g' \in dom(G)} f_{17}(g')}$$

Learning Goals For Today's Class

- Variable elimination
  - Carry out variable elimination by using factor representation and using the factor operations
  - Use techniques to simplify variable elimination

- Practice Exercises
  - Reminder: they are helpful for staying on top of the material, and for studying for the exam
  - Exercise 10 is on conditional independence.
  - Exercise 11 is on variable elimination
- Assignment 4 is due on Wednesday, April 3rd
  - You should now be able to solve all questions: 1, 2, 3, 4 and 5