Announcements

- Assignment 1 (**due next Wednesday**)

- Keep checking Discussion Board on Blackboard for Assignment **Q and A**
  - **Q1, Part 3**: No formal proof required but a description of the algorithm is needed.

- Tutorial this week
  - Hierarchical Models and continuation of OpenGL for Assignment 1

- **Office Hours today 1-2 pm**

Last week's review

- **2D Geometric Curves**
  - **Forms:** Explicit, Implicit, Parametric
  - **Primitives:** Lines, Circles, Ellipse, Super-ellipse
  - Normals and Tangents
  - Polygons

- **2D Transforms**
  - **Types:** Rigid, Conformal, Affine
  - **Examples:** Translation, Rotation, Scaling, Sheering
  - **Properties:** preserves parallelism, preserves linearity (for affine)
  - **Interpretations and Uses:**
    - changing of coordinate frames
    - hierarchical models

- **Homogeneous Coordinates**

Homogeneous Coordinates Review

Points

$$\bar{\mathbf{p}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \longrightarrow \hat{\bar{\mathbf{p}}} = \alpha \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha\,\mathbf{x} \\ \alpha\,\mathbf{y} \\ \alpha \end{bmatrix} \qquad \alpha \neq 0$$

Cartesian point **Homogeneous point**

$$\hat{\bar{\mathbf{p}}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{x}/\alpha \\ \mathbf{y}/\alpha \\ 1 \end{bmatrix} \qquad \alpha \neq 0 \longrightarrow \bar{\mathbf{p}} = \begin{bmatrix} \mathbf{x}/\alpha \\ \mathbf{y}/\alpha \end{bmatrix}$$

Homogeneous point **Cartesian point**

Vectors

$$\hat{\vec{\mathbf{v}}} = \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ 0 \end{bmatrix}$$

Homogeneous vector
(third component 0!)

$$\vec{\mathbf{v}} = \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{bmatrix}$$

Cartesian vector

Coordinate Free Geometry: Introduction to Basic Ideas

Computer Graphics, CSCD18

Fall 2008

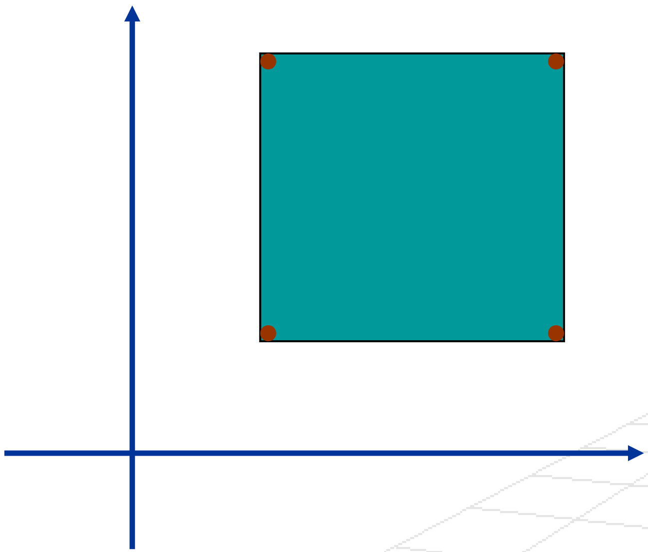
Instructor: Leonid Sigal

Coordinate Free Geometry

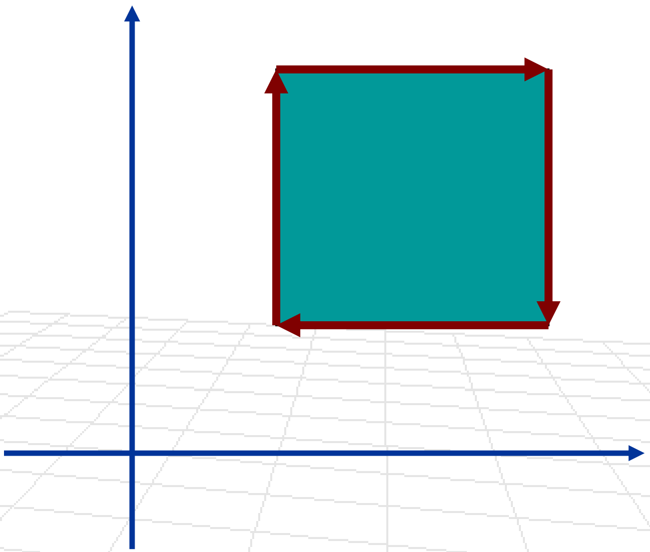
- **Coordinate Free Geometry** - style of expressing geometric objects and relations that avoids reliance on coordinate systems
- Useful in CG where many coordinate systems are in play
- Defines a restricted class of operations on points and vectors (provides a type of type checking)

Basic Idea

Coordinate Geometric Representation



Coordinate Free Geometric Representation



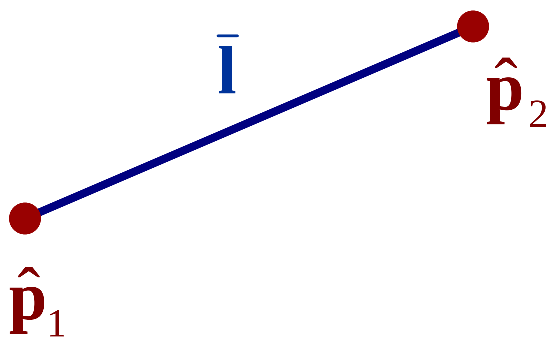
CFG Defines a set of Valid Operates on Basic Quantities

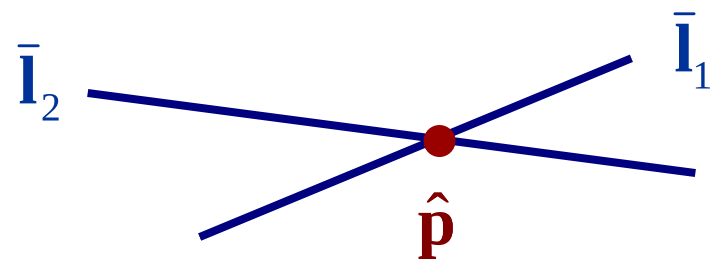
- **Scalar** – real number
- **Point** – location in space
- **Vector** – a direction and magnitude

- **Points and vectors may be represented the same but are not**
  - vector has no location in space, but point does
  - point has no magnitude, but vector does
  - we cannot add two points; we can add two vectors

CFG Style Operations with Lines

- Think back to last week





$$\bar{\mathbf{l}} = \hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_2$$

$$\hat{\mathbf{p}} = \bar{\mathbf{l}}_1 \times \bar{\mathbf{l}}_2$$

Valid CFG Operations

- point-vector addition

$$\bar{\mathbf{p}}_1 + \vec{\mathbf{v}}_1 = \bar{\mathbf{p}}_2$$

$$\vec{\mathbf{v}}_1 = \bar{\mathbf{p}}_2 - \bar{\mathbf{p}}_1$$

- vector-vector addition

$$\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_3$$

- vector scaling

$$\lambda\, \vec{\mathbf{v}}_1 = \vec{\mathbf{v}}_2$$

Valid CFG Operations

- magnitude of the vector

$$\mathbf{m} = \left\| \vec{\mathbf{v}} \right\|$$

- dot product

angle between vectors

$$\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 = \left\| \vec{\mathbf{v}}_1 \right\| \left\| \vec{\mathbf{v}}_2 \right\| \cos(\theta)$$

- cross product

$$\vec{\mathbf{v}}_1 \times \vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_3$$

where

$$\vec{\mathbf{v}}_3 \perp \vec{\mathbf{v}}_1$$

$$\vec{\mathbf{v}}_3 \perp \vec{\mathbf{v}}_2$$

$$\left\| \vec{\mathbf{v}}_3 \right\| = \left\| \vec{\mathbf{v}}_1 \right\| \left\| \vec{\mathbf{v}}_2 \right\| \cos(\theta)$$

Valid CFG Operations

- linear combination of vectors

$$\sum_{\mathbf{i}} \lambda_{\mathbf{i}} \vec{\mathbf{v}}_{\mathbf{i}} = \vec{\mathbf{v}}$$

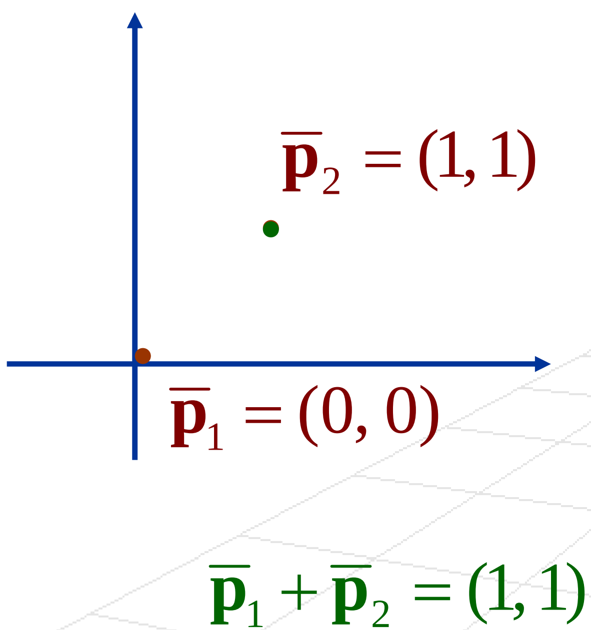
- affine combination of points

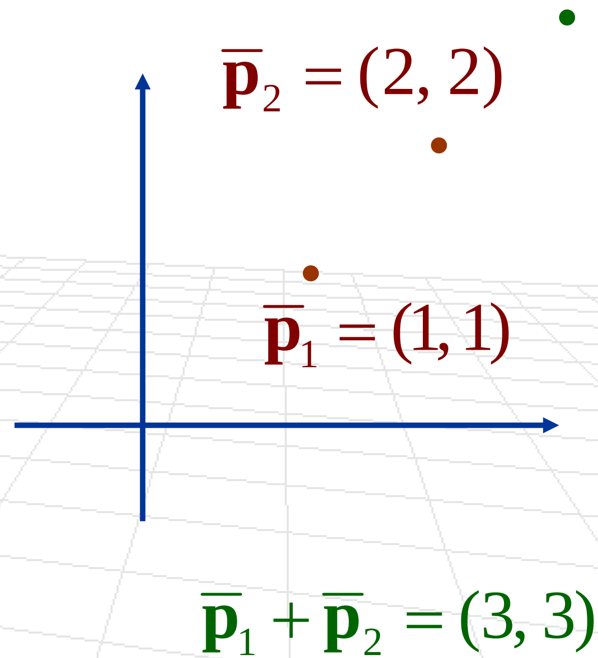
$$\sum_{\mathbf{i}} \lambda_{\mathbf{i}} \overline{\mathbf{p}}_{\mathbf{i}} = \overline{\mathbf{p}}, \quad \sum_{\mathbf{i}} \lambda_{\mathbf{i}} = 1$$

$$\sum_{\mathbf{i}} \lambda_{\mathbf{i}} \overline{\mathbf{p}}_{\mathbf{i}} = \vec{\mathbf{v}}, \quad \sum_{\mathbf{i}} \lambda_{\mathbf{i}} = 0$$

Valid CFG Operations

- These are the only **valid** operations in CFG
- All other operations are **undefined**





3D Surfaces

Computer Graphics, CSCD18

Fall 2008

Instructor: Leonid Sigal

3D surfaces

- Similar to 2D, 3D surfaces can be expressed in implicit and explicit form

Implicit Equation for the Plane

- Similar to 2D, 3D surfaces can be expressed in implicit and explicit form

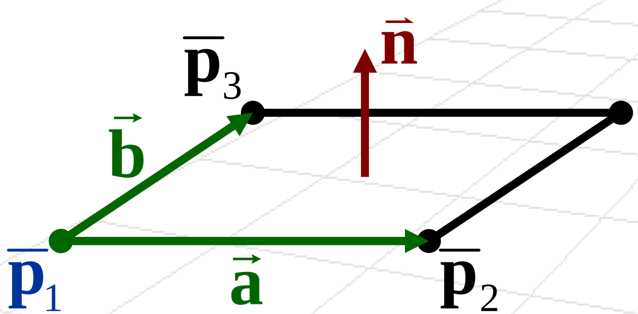
- **Implicit Form:**

normal to the plane

$$\left(\overline{\mathbf{p}} - \overline{\mathbf{p}}_1\right) \cdot \vec{\mathbf{n}} = 0$$

point on a plane

Plane can be defined uniquely from 3 non-collinear points $\overline{\mathbf{p}}_1, \overline{\mathbf{p}}_2, \overline{\mathbf{p}}_3$



$$\vec{\mathbf{a}} = \overline{\mathbf{p}}_2 - \overline{\mathbf{p}}_1$$

$$\vec{\mathbf{b}} = \overline{\mathbf{p}}_3 - \overline{\mathbf{p}}_1$$

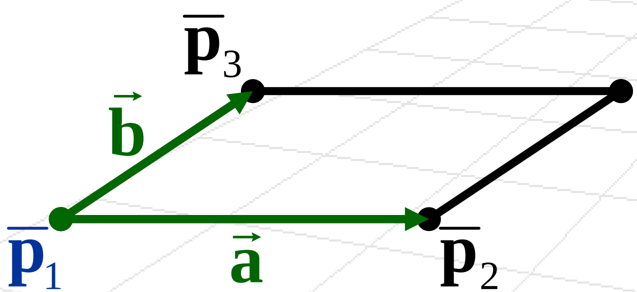
$$\vec{\mathbf{n}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$$

Parametric Equation for the Plane

- **Parametric Form:**

$$\overline{\mathbf{s}}(\alpha,\beta) = \overline{\mathbf{p}}_1 + \alpha\,\vec{\mathbf{a}} + \beta\,\vec{\mathbf{b}} \qquad \alpha,\beta \in \Re$$

Plane can be defined uniquely from 3 non-collinear points $\overline{\mathbf{p}}_1, \overline{\mathbf{p}}_2, \overline{\mathbf{p}}_3$



$$\vec{\mathbf{a}} = \overline{\mathbf{p}}_2 - \overline{\mathbf{p}}_1$$

$$\vec{\mathbf{b}} = \overline{\mathbf{p}}_3 - \overline{\mathbf{p}}_1$$

Are two equations consistent?

Parametric Form:

$$\overline{\mathbf{s}}(\alpha,\beta) = \overline{\mathbf{p}}_1 + \alpha\,\vec{\mathbf{a}} + \beta\,\vec{\mathbf{b}} \qquad \alpha,\beta \in \Re$$

Implicit Form:

$$\left(\overline{\mathbf{p}} - \overline{\mathbf{p}}_1\right)\cdot\vec{\mathbf{n}} = 0$$

Proof:

$$\left(\overline{\mathbf{p}}_1 + \alpha\,\vec{\mathbf{a}} + \beta\,\vec{\mathbf{b}} - \overline{\mathbf{p}}_1\right)\cdot\vec{\mathbf{n}} = 0$$

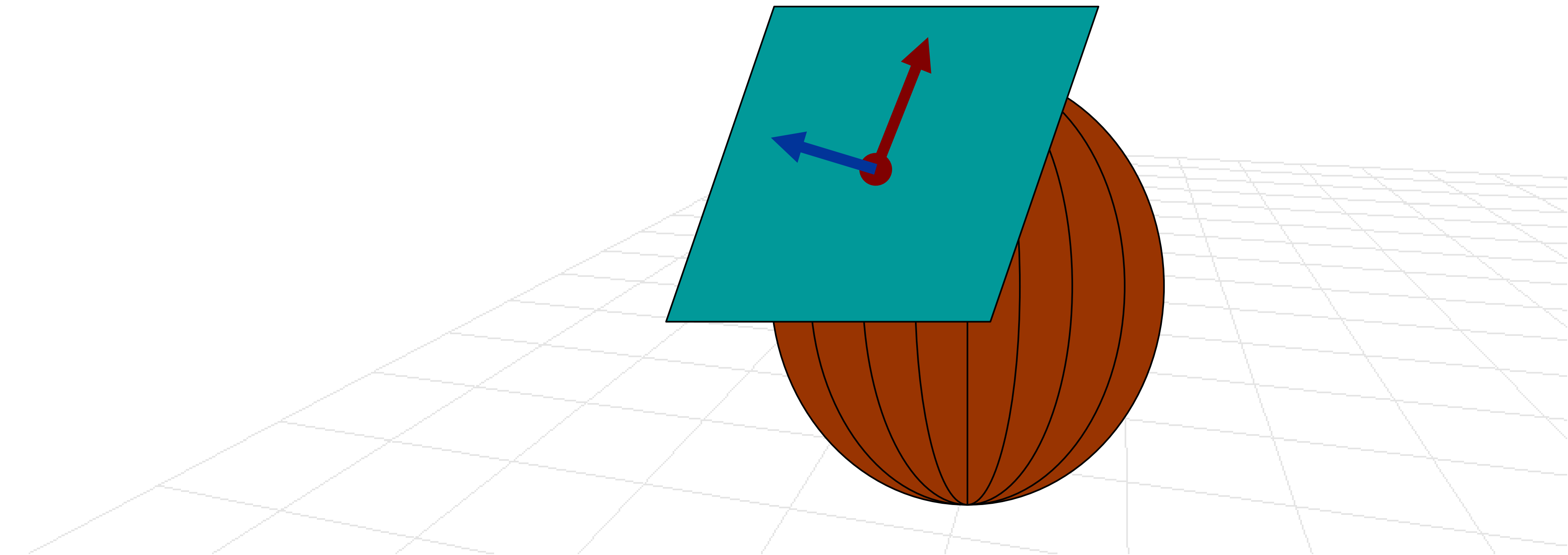
$$\left(\alpha\,\vec{\mathbf{a}} + \beta\,\vec{\mathbf{b}}\right)\cdot\vec{\mathbf{n}} = 0$$

$$\left(\alpha\,\vec{\mathbf{a}} + \beta\,\vec{\mathbf{b}}\right)\cdot(\vec{\mathbf{a}}\times\vec{\mathbf{b}}) = 0$$

$$\alpha\,\underbrace{\vec{\mathbf{a}}\cdot(\vec{\mathbf{a}}\times\vec{\mathbf{b}})}_{0} + \beta\,\underbrace{\vec{\mathbf{b}}\cdot(\vec{\mathbf{a}}\times\vec{\mathbf{b}})}_{0} = 0$$

Surface Tangents and Normals

- **Tangent** to a surface at a point is the instantaneous direction of the surface at that point
- **Tangent plane** to surface is plane containing all tangent vectors at that point
- **Normal** to a surface is a vector perpendicular to the tangent plane



Parametric Form

- **Tangent** to a curve at a point is given by:

$$\vec{\tau}(\lambda) = \frac{\mathbf{d}\overline{\mathbf{p}}(\lambda)}{\mathbf{d}\lambda} = \left(\frac{\mathbf{dx}(\lambda)}{\mathbf{d}\lambda}, \frac{\mathbf{dy}(\lambda)}{\mathbf{d}\lambda} \right)$$

evaluated at a given point

- The surface $\overline{\mathbf{s}}(\alpha,\beta) = \left(\mathbf{x}(\alpha,\beta), \mathbf{y}(\alpha,\beta), \mathbf{z}(\alpha,\beta)\right)$ has **two tangents in a tangent plane** at a point

$$\left.\frac{\partial\overline{\mathbf{s}}(\alpha,\beta)}{\partial\alpha}\right|_{\alpha_0,\beta_0} \qquad \left.\frac{\partial\overline{\mathbf{s}}(\alpha,\beta)}{\partial\beta}\right|_{\alpha_0,\beta_0}$$

- **Normal** to the surface at a point is then given by:

$$\vec{\mathbf{n}}(\alpha_0,\beta_0) = \left(\left.\frac{\partial\overline{\mathbf{s}}(\alpha,\beta)}{\partial\alpha}\right|_{\alpha_0,\beta_0} \right) \times \left(\left.\frac{\partial\overline{\mathbf{s}}(\alpha,\beta)}{\partial\beta}\right|_{\alpha_0,\beta_0} \right)$$

Parametric Form

- **Tangent plane** is then given by:

must intersect the surface at a given point

$$\vec{\mathbf{n}}(\alpha_0,\beta_0)\cdot\left(\overline{\mathbf{p}}-\overline{\mathbf{s}}(\alpha_0,\beta_0)\right)=0$$

- The surface $\overline{\mathbf{s}}(\alpha,\beta)=\left(\mathbf{x}(\alpha,\beta),\,\mathbf{y}(\alpha,\beta),\,\mathbf{z}(\alpha,\beta)\right)$ has **two tangents in a tangent plane** at a point

$$\left.\frac{\partial\overline{\mathbf{s}}(\alpha,\beta)}{\partial\alpha}\right|_{\alpha_0,\beta_0} \qquad \left.\frac{\partial\overline{\mathbf{s}}(\alpha,\beta)}{\partial\beta}\right|_{\alpha_0,\beta_0}$$

- **Normal** to the surface at a point is then given by:

$$\vec{\mathbf{n}}(\alpha_0,\beta_0)=\left(\left.\frac{\partial\overline{\mathbf{s}}(\alpha,\beta)}{\partial\alpha}\right|_{\alpha_0,\beta_0}\right)\times\left(\left.\frac{\partial\overline{\mathbf{s}}(\alpha,\beta)}{\partial\beta}\right|_{\alpha_0,\beta_0}\right)$$

Implicit Form

- **Normal** to a curve at a point is given by:

$$\vec{\mathbf{n}}(\overline{\mathbf{p}}_0) = \nabla \mathbf{f}(\overline{\mathbf{p}})\,|_{\overline{\mathbf{p}}_0} = \left(\left.\frac{\partial \mathbf{f}(\mathbf{x},\mathbf{y})}{\partial \mathbf{x}}\right|_{\overline{\mathbf{p}}_0}, \left.\frac{\partial \mathbf{f}(\mathbf{x},\mathbf{y})}{\partial \mathbf{y}}\right|_{\overline{\mathbf{p}}_0} \right)$$

- **Normal** to the surface at a point is then given by a gradient (same as in 2D)

$$\vec{\mathbf{n}}(\overline{\mathbf{p}}_0) = \nabla \mathbf{f}(\overline{\mathbf{p}})\,|_{\overline{\mathbf{p}}_0} = \left(\left.\frac{\partial \mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{z})}{\partial \mathbf{x}}\right|_{\overline{\mathbf{p}}_0}, \left.\frac{\partial \mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{z})}{\partial \mathbf{y}}\right|_{\overline{\mathbf{p}}_0}, \left.\frac{\partial \mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{z})}{\partial \mathbf{z}}\right|_{\overline{\mathbf{p}}_0} \right)$$

Example: Plane

Implicit Form:

$$\mathbf{f}(\overline{\mathbf{p}}) = (\overline{\mathbf{p}} - \overline{\mathbf{p}}_1) \cdot \vec{\mathbf{n}} = 0$$

$$\nabla \mathbf{f}(\overline{\mathbf{p}}) = \nabla(\overline{\mathbf{p}} \cdot \vec{\mathbf{n}} - \overline{\mathbf{p}}_1 \cdot \vec{\mathbf{n}})$$

not a function of p

$$\nabla \mathbf{f}(\overline{\mathbf{p}}) = \vec{\mathbf{n}}$$

Parametric Form:

$$\overline{\mathbf{s}}(\alpha, \beta) = \overline{\mathbf{p}}_1 + \alpha\,\vec{\mathbf{a}} + \beta\,\vec{\mathbf{b}} \qquad \alpha, \beta \in \Re$$

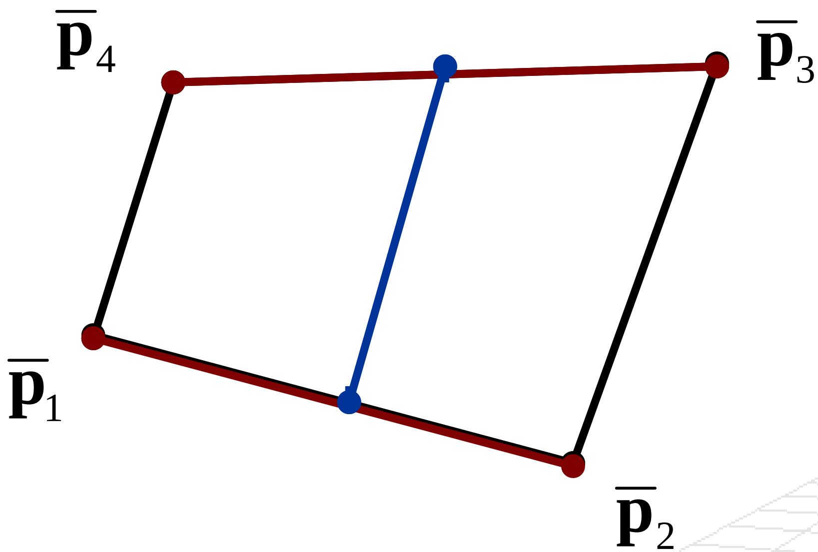
$$\frac{\partial \overline{\mathbf{s}}(\alpha, \beta)}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left(\overline{\mathbf{p}}_1 + \alpha\,\vec{\mathbf{a}} + \beta\,\vec{\mathbf{b}}\right) = \vec{\mathbf{a}}$$

$$\frac{\partial \overline{\mathbf{s}}(\alpha, \beta)}{\partial \beta} = \frac{\partial}{\partial \beta}\left(\overline{\mathbf{p}}_1 + \alpha\,\vec{\mathbf{a}} + \beta\,\vec{\mathbf{b}}\right) = \vec{\mathbf{b}}$$

$$\vec{\mathbf{n}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$$

Bilinear Patch

- Defined by 4 points, no 3 of which are co-linear



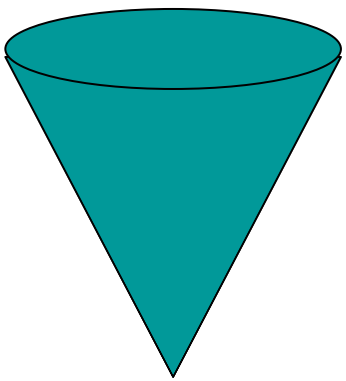
$$\overline{\mathbf{l}}_0(\alpha) = (1-\alpha)\overline{\mathbf{p}}_1 + \alpha\,\overline{\mathbf{p}}_2 \quad 0 \le \alpha \le 1$$

$$\overline{\mathbf{l}}_1(\alpha) = (1-\alpha)\overline{\mathbf{p}}_4 + \alpha\,\overline{\mathbf{p}}_3 \quad 0 \le \alpha \le 1$$

$$\overline{\mathbf{s}}(\alpha,\beta) = (1-\beta)\mathbf{l}_0(\alpha) + \beta\mathbf{l}_1(\alpha) \qquad 0 \le \alpha \le 1 \qquad 0 \le \beta \le 1$$

Cylinder

- Constructed by moving a point on a line along a planar curve (directrix) such that direction of line Is held constant



direction of the line

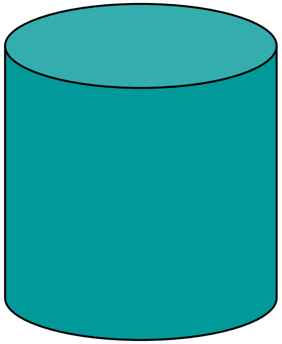
$$\overline{\mathbf{s}}(\alpha, \beta) = \overline{\mathbf{p}}_0(\alpha) + \beta \vec{\mathbf{d}}$$

planar curve (directrix)

- **Right cylinder:** direction of the line perpendicular to the plane containing the curve (directrix)
- **Circular cylinder:** directrix is a circle

Example: Right Circular Cylinder

$$\bar{\mathbf{s}}(\alpha,\beta) = \bar{\mathbf{p}}_0(\alpha) + \beta\vec{\mathbf{d}}$$



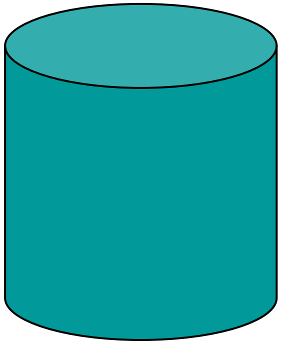
$$\bar{\mathbf{p}}_0(\alpha) = \left(\mathbf{r}\cos(\alpha), \mathbf{r}\sin(\alpha), 0\right) \qquad 0 \le \alpha \le 2\pi$$

$$\vec{\mathbf{d}} = (0, 0, 1)$$

$$\boxed{\bar{\mathbf{s}}(\alpha,\beta) = \left(\mathbf{r}\cos(\alpha), \mathbf{r}\sin(\alpha), \beta\right) \qquad 0 \le \beta \le 1}$$

Example: Right Circular Cylinder

$$\bar{\mathbf{s}}(\alpha,\beta) = \left(\mathbf{r}\cos(\alpha), \mathbf{r}\sin(\alpha), \beta\right) \qquad 0 \le \beta \le 1$$



How do we find a normal?

$$\frac{\partial \bar{\mathbf{s}}(\alpha,\beta)}{\partial \alpha} = \left(-\mathbf{r}\sin(\alpha), \mathbf{r}\cos(\alpha), 0\right)$$

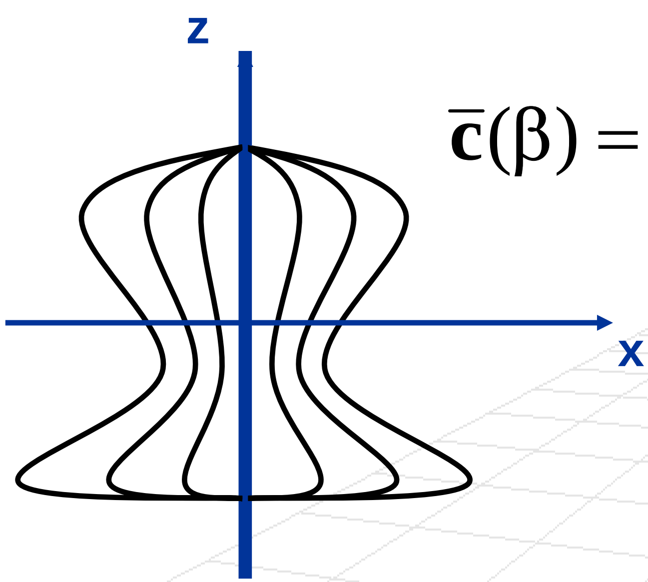
$$\frac{\partial \bar{\mathbf{s}}(\alpha,\beta)}{\partial \beta} = \left(0, 0, 1\right)$$

$$\vec{\mathbf{n}}(\alpha,\beta) = \frac{\partial \bar{\mathbf{s}}(\alpha,\beta)}{\partial \alpha} \times \frac{\partial \bar{\mathbf{s}}(\alpha,\beta)}{\partial \beta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\mathbf{r}\sin(\alpha) & \mathbf{r}\cos(\alpha) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \left(\mathbf{r}\cos(\alpha), \mathbf{r}\sin(\alpha), 0\right)$$

Surface of Revolution

- **Cylinder** is a special case of more general Surface of Revolution idea
- **Idea:** take any curve in x-z plane and revolve around z-axis



$$\bar{\mathbf{c}}(\beta) = \left(\mathbf{x}(\beta), 0, \mathbf{z}(\beta)\right)$$

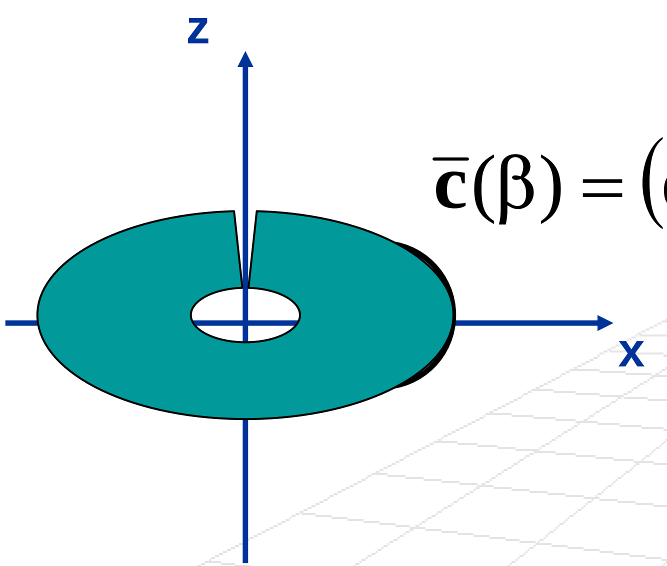
Parameter's of revolution

$$\left(\mathbf{r}\cos(\alpha), \mathbf{r}\sin(\alpha), 0\right)$$

$$\boxed{\bar{\mathbf{s}}(\alpha,\beta) = \left(\mathbf{x}(\beta)\cos(\alpha), \mathbf{x}(\beta)\sin(\alpha), \mathbf{z}(\beta)\right)}$$

Example: Torus

- **Cylinder** is a special case of more general Surface of Revolution idea
- **Idea:** take any curve in x-z plane and revolve around z-axis



$$\bar{\mathbf{c}}(\beta) = \left(\mathbf{d} + \mathbf{r}\cos(\beta), 0, \mathbf{r}\sin(\beta)\right)$$

Parameter's of revolution

$$\left(\mathbf{r}\cos(\alpha), \mathbf{r}\sin(\alpha), 0\right)$$

$$\bar{\mathbf{s}}(\alpha,\beta) = \left(\left(\mathbf{d} + \mathbf{r}\cos(\beta)\right)\cos(\alpha), \left(\mathbf{d} + \mathbf{r}\cos(\beta)\right)\sin(\alpha), \mathbf{r}\sin(\beta)\right)$$