Announcements

- **Assignment 1 is out**

- Writing portion
  - 4 question
  - When we ask for “prove” something, we mean proof in a mathematical sense
  - Electronic submissions are preferred

- Programming
  - Start early
  - Starter code available for Linux and VC++

Last class review

	Line	Circle	Ellipse
Explicit:	$\mathbf{y} = \mathbf{mx} + \mathbf{b}$	**N/A**	**N/A**
Implicit:	$(\overline{\mathbf{x}} - \overline{\mathbf{x}}_0)\vec{\mathbf{n}} = 0$	$\lVert \overline{\mathbf{p}} - \overline{\mathbf{p}}_{\mathbf{c}} \rVert^2 - \mathbf{r}^2 = 0$	$\frac{\mathbf{x}^2}{\mathbf{a}^2} + \frac{\mathbf{y}^2}{\mathbf{b}^2} - 1 = 0$
Parametric:	$\overline{\mathbf{p}}(\lambda) = \overline{\mathbf{p}}_0 + \lambda\vec{\mathbf{d}}$	$\overline{\mathbf{p}}(\lambda) = \begin{bmatrix} \mathbf{r}\cos(2\pi\lambda) \\ \mathbf{r}\sin(2\pi\lambda) \end{bmatrix}$	$\overline{\mathbf{p}}(\lambda) = \begin{bmatrix} \mathbf{a}\cos(2\pi\lambda) \\ \mathbf{b}\sin(2\pi\lambda) \end{bmatrix}$

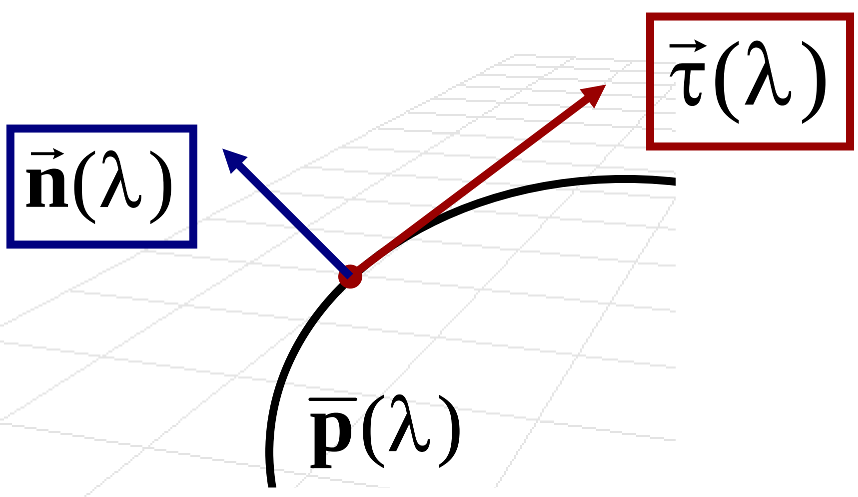
Tangents and Normals

- **Tangent from parametric form:** $\vec{\tau}(\lambda) = \left(\frac{dx(\lambda)}{d\lambda}, \frac{dy(\lambda)}{d\lambda} \right)$

  derivative

- **Normal from implicit form:** $\vec{n}(\lambda) = \left(\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y} \right)$

  gradient



2D Transformations (continuation)

Computer Graphics, CSCD18

Fall 2008

Instructor: Leonid Sigal

Transformations

- **Rigid transformations**
  - **Examples:** Translations, Rotations
  - **Properties:** preserve distance and angles

- **Conformal transformations**
  - **Examples:** translations, rotations, uniform scale
  - **Properties:** preserves angles (not distance)

- **Affine transformations**
  - **Examples:** translations, rotations, general scaling, reflections
  - **Properties:** preserves parallelism, preserves linearity (lines remain lines

Affine Transformation

$$\boxed{\overline{\mathbf{q}} = \mathbf{A}\,\overline{\mathbf{p}} + \overline{\mathbf{t}}}$$

- Any linear transformation $\mathbf{A}$ (can be rotation, scaling, reflection, etc.) followed by a translation $\mathbf{t}$
- Thereby translation, rotation, scaling, sheer are all special cases of affine transformation

- **Properties**
  - inverse of affine transformation is also affine
  - lines are preserved
  - given closed region (polygon) area under the affine transformation is scaled by $\det(\mathbf{A})$
  - compositions of affine transformations is still affine transformation

Proof: Inverse of Affine Transformation is also an Affine Transformation

$$\overline{\mathbf{q}} = \mathbf{A}\overline{\mathbf{p}} + \overline{\mathbf{t}}$$

$$\overline{\mathbf{q}} - \overline{\mathbf{t}} = \mathbf{A}\overline{\mathbf{p}}$$

$$\mathbf{A}^{-1}(\overline{\mathbf{q}} - \overline{\mathbf{t}}) = \overline{\mathbf{p}}$$

assime $\mathbf{A}^{-1}$ exists

$$\mathbf{A}^{-1}\overline{\mathbf{q}} - \mathbf{A}^{-1}\overline{\mathbf{t}} = \overline{\mathbf{p}}$$

$$\overline{\mathbf{p}} = \mathbf{B}\overline{\mathbf{q}} + \overline{\mathbf{v}}$$

where

$$\mathbf{B} = \mathbf{A}^{-1} \qquad \overline{\mathbf{v}} = \mathbf{A}^{-1}\overline{\mathbf{t}}$$

Proof: compositions of affine transformations is still affine transformation

$$\mathbf{F}_1(\bar{\mathbf{p}}) = \mathbf{A}_1\bar{\mathbf{p}} + \bar{\mathbf{t}}_1$$

$$\mathbf{F}_2(\bar{\mathbf{p}}) = \mathbf{A}_2\bar{\mathbf{p}} + \bar{\mathbf{t}}_2$$

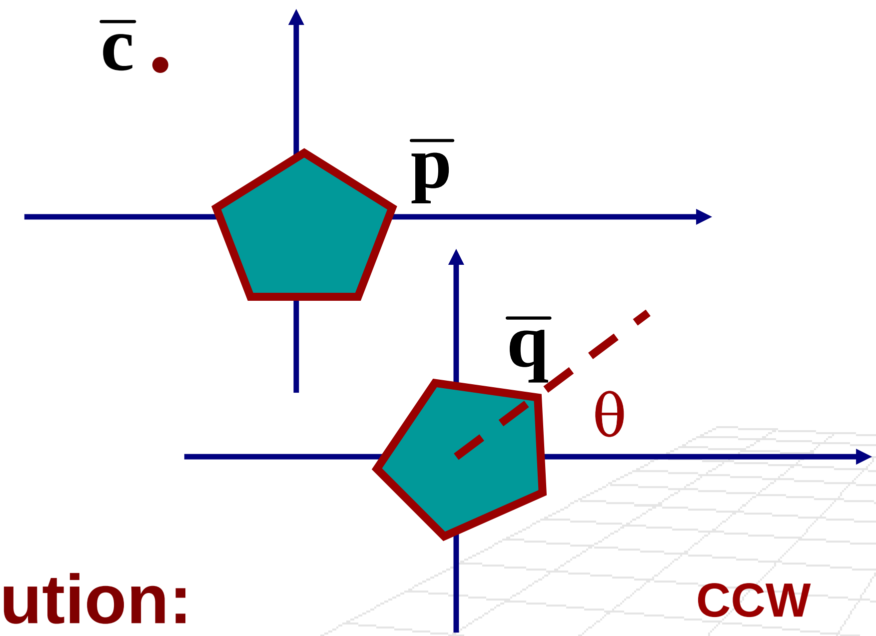
$$\begin{aligned}\mathbf{F}_2\left(\mathbf{F}_1(\bar{\mathbf{p}})\right) &= \mathbf{A}_2\left(\mathbf{A}_1\bar{\mathbf{p}} + \bar{\mathbf{t}}_1\right) + \bar{\mathbf{t}}_2 \\ &= \mathbf{A}_2\mathbf{A}_1\bar{\mathbf{p}} + \mathbf{A}_2\bar{\mathbf{t}}_1 + \bar{\mathbf{t}}_2 \\ &= \mathbf{A}\bar{\mathbf{p}} + \bar{\mathbf{t}}\end{aligned}$$

where

$$\mathbf{A} = \mathbf{A}_2\mathbf{A}_1 \qquad \bar{\mathbf{t}} = \mathbf{A}_2\bar{\mathbf{t}}_1 + \bar{\mathbf{t}}_2$$

Why composing transformations useful?

- Rotations as we have seen It in the last class rotate the object about the origin in CCW, what if we want to rotate about some other point $\mathbf{c}$?



$$
\begin{aligned}
&\mathbf{F}_1: \quad \mathbf{A}_1 = \mathbf{I} \quad \bar{\mathbf{t}}_1 = -\bar{\mathbf{c}} \\
&\mathbf{F}_2: \quad \mathbf{A}_2 = \mathbf{R}(\theta) \quad \bar{\mathbf{t}}_2 = 0 \\
&\mathbf{F}_3: \quad \mathbf{A}_3 = \mathbf{I} \quad \bar{\mathbf{t}}_3 = \bar{\mathbf{c}}
\end{aligned}
$$

$$\bar{\mathbf{q}} = \mathbf{F}_3\left(\mathbf{F}_2\left(\mathbf{F}_1(\bar{\mathbf{p}})\right)\right)$$

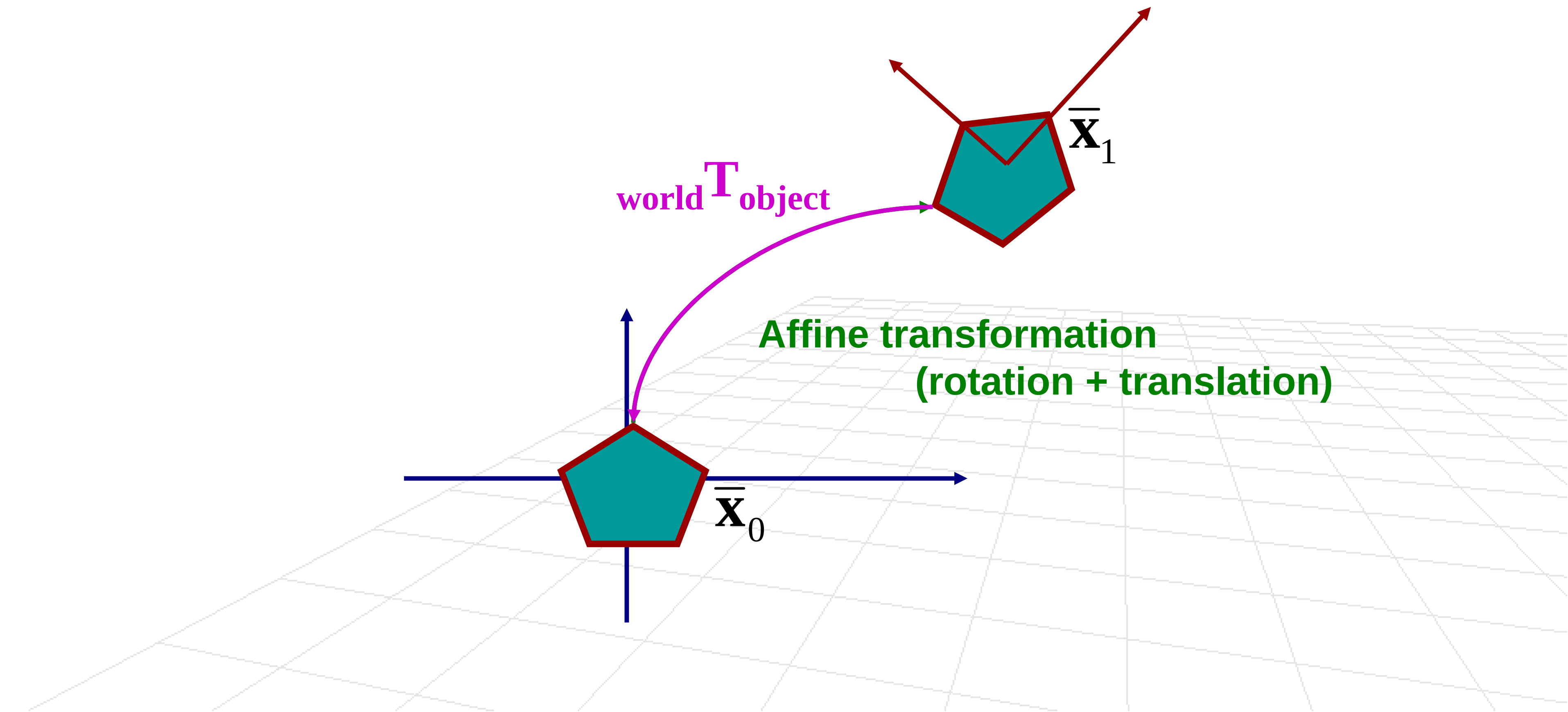
- **Solution:**
  - Translate by $-\bar{\mathbf{c}}$ (so that $\bar{\mathbf{c}}$ is the new origin)
  - Rotate
  - Translate back by $\bar{\mathbf{c}}$

Additional Affine Transformation Properties Proofs

In the Lecture Notes

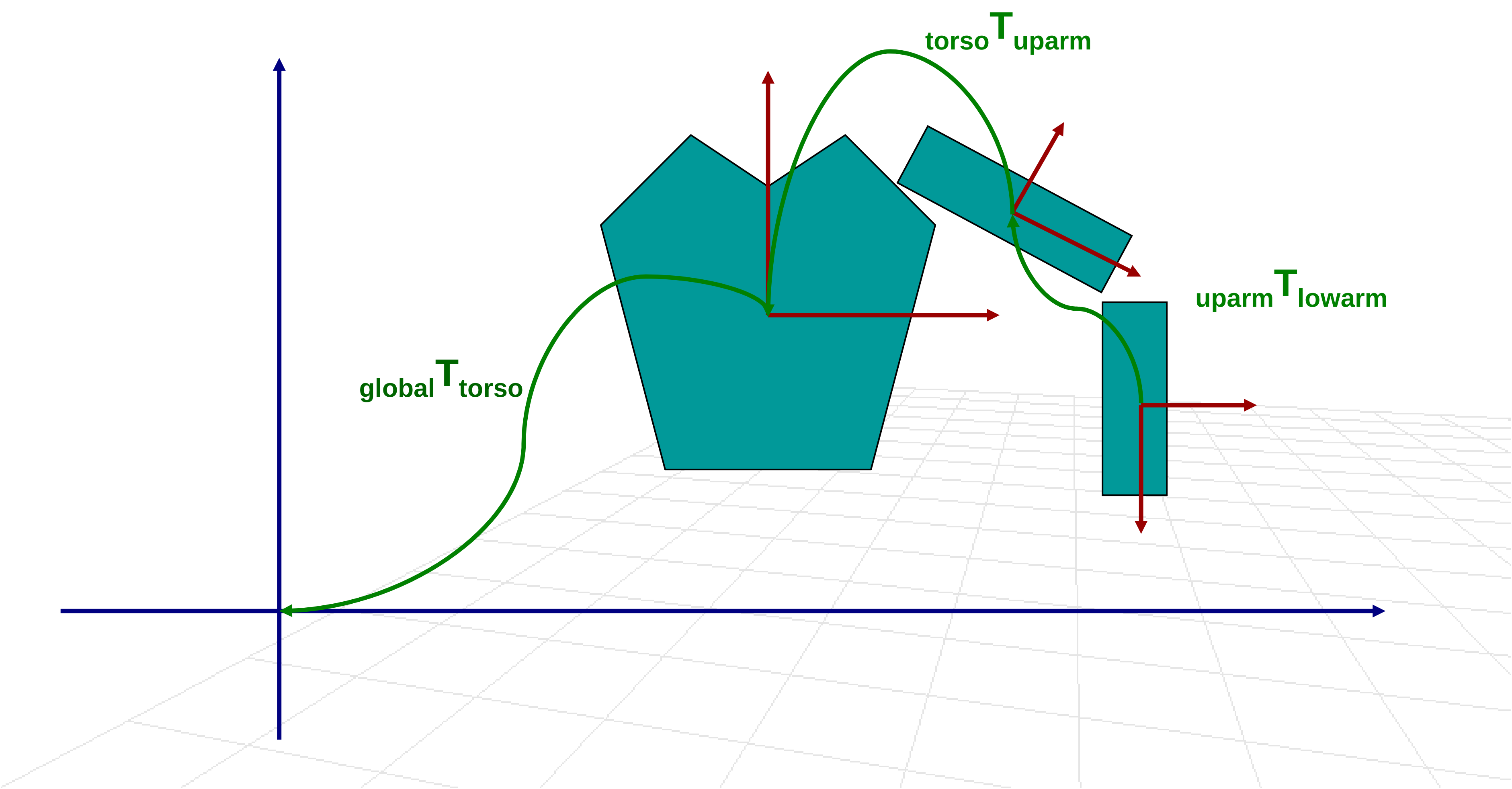
Changing Coordinate Frames

Can be interpreted as the transformation from object coordinate frame (red) to world coordinate frame (blue)



Hierarchical Models

$$\bar{p}_{global} = {}_{global}T_{torso} \times {}_{torso}T_{uparm} \times {}_{uparm}T_{lowarm} \times \bar{p}_{lowarm}$$



Homogeneous Coordinates

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Homogeneous Coordinates

- **Problem:** affine transformations often become complex and unwieldy to keep track of
- **Homogeneous coordinates** allow all the transformations to be specified by a single matrix multiply (OpenGL)
- How do we express a Cartesian point in homogeneous coordinates?

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \longrightarrow \alpha \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha\,\mathbf{x} \\ \alpha\,\mathbf{y} \\ \alpha \end{bmatrix} \qquad \alpha \neq 0$$

Cartesian point **Homogeneous point**

Homogeneous Coordinates

Example:

$$\begin{bmatrix}1\\2\end{bmatrix} \rightarrow \begin{bmatrix}1\\2\\1\end{bmatrix}, \quad \begin{bmatrix}1\\2\end{bmatrix} \rightarrow \begin{bmatrix}2\\4\\2\end{bmatrix}, \quad \begin{bmatrix}1\\2\end{bmatrix} \rightarrow \begin{bmatrix}5\\10\\5\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{x}\\\mathbf{y}\end{bmatrix} \longrightarrow \alpha\begin{bmatrix}\mathbf{x}\\\mathbf{y}\\1\end{bmatrix} = \begin{bmatrix}\alpha\,\mathbf{x}\\\alpha\,\mathbf{y}\\\alpha\end{bmatrix} \qquad \alpha \neq 0$$

Cartesian point **Homogeneous point**

Converting from Homogeneous Coordinates

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{x}/\alpha \\ \mathbf{y}/\alpha \\ 1 \end{bmatrix} \qquad \alpha \neq 0 \qquad \longrightarrow \qquad \begin{bmatrix} \mathbf{x}/\alpha \\ \mathbf{y}/\alpha \end{bmatrix}$$

Homogeneous point **Cartesian point**

- **Note:** two homogeneous points are not equal if they are not scalar multiples of one another

Homogeneous Transformations

- Turns out that many transformations become linear in homogeneous coordinates (mainly affine)

Affine in Cartesian Coordinates

$$\bar{\mathbf{q}} = \mathbf{A}\bar{\mathbf{p}} + \bar{\mathbf{t}}$$

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

Affine in Homogeneous Coordinates

$$\bar{\mathbf{q}} = \begin{bmatrix} \mathbf{A} & \bar{\mathbf{t}} \end{bmatrix} \hat{\mathbf{p}}$$

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} a & b & t_x \\ c & d & t_y \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

- But it's easier to always keep track of homogeneous representation, so

$$\boxed{\hat{\mathbf{q}} = \begin{bmatrix} \mathbf{A} & \bar{\mathbf{t}} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & 1 \end{bmatrix} \hat{\mathbf{p}}}$$

This is linear and easy to keep track of

Properties of Affine Transformation (cont.)

- With homogeneous representation for affine transformation, several additional properties of affine transformations become apparent
  - affine transformations are **associative**

$$(\mathbf{F}_3\ \mathbf{F}_2)\,\mathbf{F}_1 = \mathbf{F}_3\,(\mathbf{F}_2\ \mathbf{F}_1)$$

  - Affine transformations are **not** in general **commutative**

    (proof of this is a homework question)

$$\mathbf{F}_2\ \mathbf{F}_1 \neq \mathbf{F}_1\ \mathbf{F}_2$$

Vectors in Homogeneous Coordinates

$$\hat{\vec{\mathbf{v}}} = \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ 0 \end{bmatrix}$$

Homogeneous vector
(third component 0!)

Example:

$$\begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{p}_x \\ \mathbf{p}_y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{p}_x + \mathbf{v}_x \\ \mathbf{p}_y + \mathbf{v}_y \\ 1 \end{bmatrix}$$

Homogeneous vector **Homogeneous point** **Homogeneous point**

What else can we do with Homogeneous Coordinates?

- The equation of the line

$$\boxed{\begin{aligned}\mathbf{a} &= -\mathbf{bm}\\ \mathbf{c} &= -\mathbf{bd}\end{aligned}}$$

$$\mathbf{y} = \mathbf{mx} + \mathbf{d}$$

$$0 = \mathbf{ax} + \mathbf{by} + \mathbf{c}$$

- In homogeneous coordinates

$$\begin{bmatrix}\mathbf{a} & \mathbf{b} & \mathbf{c}\end{bmatrix}\begin{bmatrix}\mathbf{p_x}\\ \mathbf{p_y}\\ 1\end{bmatrix} = \bar{\mathbf{l}}^{\mathbf{T}}\,\hat{\mathbf{p}} = 0$$

Vector holding line parameters

Vector holding homogeneous coordinate of a point

Finding Line Passing Through 2 Points

- Equation of the line in homogeneous coordinates:

$$\begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix} \begin{bmatrix} \mathbf{p_x} \\ \mathbf{p_y} \\ 1 \end{bmatrix} = \bar{\mathbf{l}}^{\mathbf{T}} \, \hat{\mathbf{p}} = 0$$

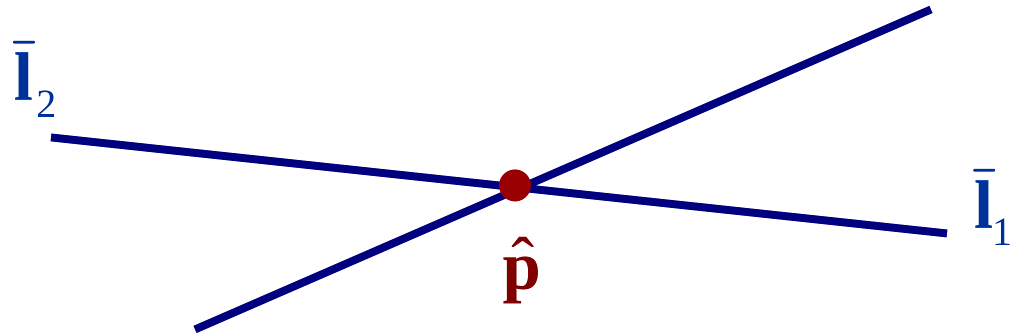
- If two homogeneous points $\hat{\mathbf{p}}_{\mathbf{1}}$ and $\hat{\mathbf{p}}_{\mathbf{2}}$ are on the line then

$$\bar{\mathbf{l}}^{\mathbf{T}} \, \hat{\mathbf{p}}_1 = 0 \qquad\qquad \bar{\mathbf{l}}^{\mathbf{T}} \, \hat{\mathbf{p}}_2 = 0$$

(vector $\bar{\mathbf{l}}$ must perpendicular to two 3D vectors)

$$\boxed{\bar{\mathbf{l}} = \hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_2}$$

Finding Intersection of Two Lines



- If two homogeneous points $\mathbf{p_0}$ and $\mathbf{p_1}$ are on the line then

$$\bar{\mathbf{l}}_1^{\mathbf{T}}\,\hat{\mathbf{p}} = 0 \qquad\qquad \bar{\mathbf{l}}_2^{\mathbf{T}}\,\hat{\mathbf{p}} = 0$$

(point $\hat{\mathbf{p}}$ must perpendicular to two 3D vectors holding the line parameters)

$$\boxed{\hat{\mathbf{p}} = \bar{\mathbf{l}}_1 \times \bar{\mathbf{l}}_2}$$