14 Interpolation

14.1 Interpolation Basics

Goal: We would like to be able to define curves in a way that meets the following criteria:

1. Interaction should be natural and intuitive.
2. Smoothness should be controllable.
3. Analytic derivatives should exist and be easy to compute.
4. Representation should be compact.

Interpolation is when a curve passes through a set of "control points."

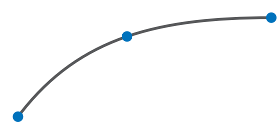


Figure 1: *
Interpolation

Approximation is when a curve approximates but doesn't necessarily contain its control points.

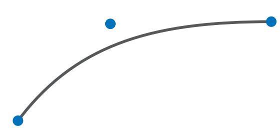


Figure 2: *
Approximation

Extrapolation is extending a curve beyond the domain of its control points.

Continuity - A curve is is C^n when it is continuous in up to its n^{th}-order derivatives. For example, a curve is in C^1 if it is continuous and its first derivative is also continuous.

Consider a cubic interpolant — a 2D curve, $\bar{c}(t) = \left[\begin{array}{cc} x(t) & y(t) \end{array}\right]$ where

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3, \tag{1}$$
$$y(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3, \tag{2}$$

Figure 3: *
Extrapolation

so

$$x(t) = \sum_{i=0}^{3} a_i t^i = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \vec{t}^T \vec{a}. \tag{3}$$

Here, $\vec{t}$ is the basis and $\vec{a}$ is the coefficient vector. Hence, $\bar{c}(t) = \vec{t}^T \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$. (Note: $T \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$ is a 4×2 matrix).

There are eight unknowns, four a_i values and four b_i values. The constraints are the values of $\bar{c}(t)$ at known values of t.

Example:
For $t \in (0, 1)$, suppose we know $\bar{c}_j \equiv \bar{c}(t_j)$ for $t_j = 0, \frac{1}{3}, \frac{2}{3}, 1$ as $j = 1, 2, 3, 4$. That is,

$$\bar{c}_1 = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \equiv \begin{bmatrix} x(0) & y(0) \end{bmatrix}, \tag{4}$$
$$\bar{c}_2 = \begin{bmatrix} x_2 & y_2 \end{bmatrix} \equiv \begin{bmatrix} x(1/3) & y(1/3) \end{bmatrix}, \tag{5}$$
$$\bar{c}_3 = \begin{bmatrix} x_3 & y_3 \end{bmatrix} \equiv \begin{bmatrix} x(2/3) & y(2/3) \end{bmatrix}, \tag{6}$$
$$\bar{c}_4 = \begin{bmatrix} x_4 & y_4 \end{bmatrix} \equiv \begin{bmatrix} x(1) & y(1) \end{bmatrix}. \tag{7}$$

So we have the following linear system,

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & (1/3)^2 & (1/3)^3 \\ 1 & 2/3 & (2/3)^2 & (2/3)^3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}, \tag{8}$$

or more compactly, $\begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} = C \begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}$. Then, $\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix} = C^{-1} \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix}$. From this we can find $\vec{a}$ and $\vec{b}$, to calculate the cubic curve that passes through the given points.

We can also place derivative constraints on interpolant curves. Let

$$\vec{\tau}(t) = \frac{d\bar{c}(t)}{dt} = \frac{d}{dt}\begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix}\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} 0 & 1 & t & t^2 \end{bmatrix}\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}, \tag{10}$$

that is, a different basis with the same coefficients.

Example:
Suppose we are given three points, $t_j = 0, \frac{1}{2}, 1$, and the derivative at a point, $\vec{\tau}_2(\frac{1}{2})$. So we can write this as

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x'_2 & y'_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/2 & (1/2)^2 & (1/2)^3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2(1/2) & 3(1/2)^2 \end{bmatrix}\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}, \tag{11}$$

and

$$\begin{bmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \\ \vec{\tau}_2 \end{bmatrix} = C\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix}, \tag{12}$$

which we can use to find $\vec{a}$ and $\vec{b}$:

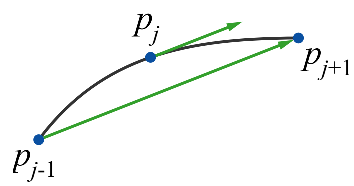
$$\begin{bmatrix} \vec{a} & \vec{b} \end{bmatrix} = C^{-1}\begin{bmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \\ \vec{\tau}_2 \end{bmatrix}. \tag{13}$$

Unfortunately, polynomial interpolation yields unintuitive results when interpolating large numbers of control points; you can easily get curves that pass through the control points, but oscillate in very unexpected ways. Hence, direct polynomial interpolation is rarely used except in combination with other techniques.

14.2 Catmull-Rom Splines

Catmull-Rom Splines interpolate degree-3 curves with C^1 continuity and are made up of cubic curves.

A user specifies only the points $[\bar{p}_1, ...\bar{p}_N]$ for interpolation, and the tangent at each point is set to be parallel to the vector between adjacent points. So the tangent at $\bar{p}_j$ is $\kappa(\bar{p}_{j+1} - \bar{p}_{j-1})$ (for endpoints, the tangent is instead parallel to the vector from the endpoint to its only neighbor). The value of κ is set by the user, determining the "tension" of the curve.



Between two points, $\bar{p}_j$ and $\bar{p}_{j+1}$, we draw a cubic curve using $\bar{p}_j$, $\bar{p}_{j+1}$, and two auxiliary points on the tangents, $\kappa(\bar{p}_{j+1} - \bar{p}_{j-1})$ and $\kappa(\bar{p}_{j+2} - \bar{p}_j)$.

We want to find the coefficients a_j when $x(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T$, where the curve is defined as $\bar{c}(t) = \begin{bmatrix} c(t) & y(t) \end{bmatrix}$ (similarly for $y(t)$ and b_j). For the curve between $\bar{p}_j$ and $\bar{p}_{j+1}$, assume we know two end points, $\bar{c}(0)$ and $\bar{c}(1)$ and their tangents, $\vec{c}\,'(0)$ and $\vec{c}\,'(1)$. That is,

$$
\begin{aligned}
x(0) &= x_j, && (14)\\
x(1) &= x_{j+1}, && (15)\\
x'(0) &= \kappa(x_{j+1} - x_{j-1}), && (16)\\
x'(1) &= \kappa(x_{j+2} - x_j). && (17)
\end{aligned}
$$

To solve for $\vec{a}$, set up the linear system,

$$
\begin{bmatrix} x(0) \\ x(1) \\ x'(0) \\ x'(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}. \tag{18}
$$

Then $\vec{x} = M\vec{a}$, so $\vec{a} = M^{-1}\vec{x}$. Substituting $\vec{a}$ in $x(t)$ yields

$$
\begin{aligned}
x(t) &= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_j \\ x_{j+1} \\ \kappa(x_{j+1} - x_{j-1}) \\ \kappa(x_{j+2} - x_j) \end{bmatrix} && (19)\\
&= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\kappa & 0 & \kappa & 0 \\ 2\kappa & \kappa - 3 & 3 - 2\kappa & -\kappa \\ -\kappa & 2 - \kappa & \kappa - 2 & \kappa \end{bmatrix} \begin{bmatrix} x_{j-1} \\ x_j \\ x_{j+1} \\ x_{j+2} \end{bmatrix}. && (20)
\end{aligned}
$$

For the first tangent in the curve, we cannot use the above formula. Instead, we use:

$$\vec{\tau}_1 = \kappa(\bar{p}_2 - \bar{p}_1) \tag{21}$$

and, for the last tangent:

$$\vec{\tau}_N = \kappa(\bar{p}_N - \bar{p}_{N-1}) \tag{22}$$